OPTIMAL IMPLEMENTATION OF ENVIRONMENTAL IMPROVEMENT POLICY WITH IMPLEMENTATION COSTS *

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ABSTRACT. Many environmental problems result from human activities and are subjects of increasing concern in the world. Thus environmental improvement policies (EIPs) have become an issue of increasing importance. In this paper we consider a problem in which an agent implements the environmental improvement policy under uncertainty. If an emission level of a pollutant arrives at a critical level, the agent has to decrease the emission to a certain level in order to improve environment. The agent problem is to minimize the expected total discounted cost which includes a cost to implement the EIP and an associated damage from the pollutant under the assumption that a state process of the pollutant follows a geometric Brownian motion. Then we find critical emission levels of pollutant, an optimal implementation times, optimal sizes of implementation and evaluate the optimal EIP (OEIP) by using an impulse control method. Some numerical examples are practiced to illustrate our results.

1 Introduction The problem of environmental pollution results from human activities. Human activities discharge harmful matter and pollutants; waste, greenhouse gases, and so on. For example, increasing atmospheric greenhouse gases are contributing to climate change. According to IPCC [12], the scientific assessment of climate changes estimated that the global mean surface atmospheric temperature will increase by 1 to 3.5 degrees centigrade by the year 2100. It leads to a number of potentially serious consequences. One could have an increase in the incidence of heat waves, floods, and droughts as the global climate changes. These events significantly affect on human welfare as well as natural ecosystems. They are subjects of increasing concern in the world. To prevent damage from pollutants, we have to employ the EIPs; environmental taxes, marketable permits, and subsidies. See, for example, Bertram, Stephens, and Wallace [2] and Jenkins and Lamech [14]. They examine these market based policies.

In economic theory, one usually assumes free disposal condition and solves some problems. See Mas-Colell, Whinston and Green [19]; Chapter 7 and Part 4. However by recognizing environmental problems, one have to consider disposal cost. Hence, it costs to improve environment in order to reduce damage from worsening environment. In this paper we consider the following EIP: If an emission level of a pollutant arrives at a critical level, an agent has to decrease the emission to a certain level in order to improve environment. If not, he receives higher damage. Thus he has to decide times to implement the EIP (or the levels of a pollutant) and sizes of implementation of the EIP. Therefore we examine optimal implementation times, optimal emission levels of pollutant, and optimal sizes of enforcing the EIP. We also evaluate the OEIP. To this end, we use an impulse control method. Related works are as follows: Neuman and Costanza [22] study the management of renewable resources by using an impulse contorl method. However they study that the state of system is determicistic. On the contray, our anlyasis assume the dynamics of the pollutant is stochatic. Willassen [24] studies the optimal cutting strategy for an ongoing forest by using an impulse control method. Cortazor, Schwartz and Salinas [8], Zepapadeas [25] and Tsujimura [23] also study environmental problems by using optimal stopping methods.

Other important fields of applications of impulse control methods are portfolio management and the problem of exchange rate. Portfolio managers may intervene to rebalance their portfolios. At that time trading securities demands the transaction costs. This corresponds to disposal costs of environment

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problems. Constantinides and Richard [7] examine cash management problem by degenerating to the inventory model when the cumulative demand for cash follows by a Brownian motion. They prove the existence of an optimal policy. Eastham and Hastings [10] examine the problem of portfolio management as the problem of maximizing total utility of consumption. Korn [17] discusses the portfolio optimization with transaction costs by using an impulse control method in Chapter 5 and 6. Mizuta [20] studies constraints, necessary conditions and optimality conditions by comparing with Constantinides and Richard [7], Harrrison et al [11] and Jeanblanc-Picqué [13]. Cadenillas [5] presents a survey of this field with transaction costs. On the other hand, in the problem of exchange rate, authorities, i.e., central banks, intervene the fundamental like money supply and try to maintain that the exchange rate keeps in given target zone. Jeanblanc-Picqué [13] proves the existence of an optimal control with the constant and proportional intervention cost. However he does not treat a running cost. Korn [16] examines the case that intervention size is random. Cadenillas and Zapatero [6] treat a running cost and assume that exchange rate follows geometric Brownian motion. And they show the numerical examples. But they don't show the existence of an optimal solution. While this paper shows the existence of an optimal impulse control. When they prove that a solution of QVI is the value function, they assume that conditions which include determined parameters. On the other hand, we assume a condition expressed with given parameters when we prove that. We also present the numerical examples. Korn [18] presents some applications of impulse control.

This paper is structured as follows: The next section shows the description of the problem. Section 3 analyzes the problem. Section 4 examines numerical examples and descirbe the comparative statics analysis. Section 5 contains the conclusion of this paper.

2 The Model Consider a problem of an agent who implements the EIP. If he doesn't implement the EIP, he receives damage from increasing a pollutant. So he must implement the EIP in order to reduce damage. Since implementing the EIP makes him the irreversible expenses, he must find the optimal times to implement the EIP, the optimal emission levels of pollutant and the optimal sizes of the EIP so as to minimize the expected total discounted damage with expending the irreversible cost.

Let $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ denote a filtered probability space satisfying the usual conditions, i.e., $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$. Here $\mathcal{F}_t$ is generated by a Brownian motion, W_t, in $\mathbb{R}$, i.e., $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Let $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_i \leq \cdots$ be $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times such that $\tau_i \to +\infty$ as $i \to +\infty$ a.s.. For each i, τ_i assigns an impulse $\zeta_i \in \mathbb{R}_+$, where ζ_i is $\mathcal{F}_{\tau_i}$-measurable. Suppose that the result of giving the impulse is that the state of the pollutant jumps immediately from x to a new state $\eta(x, \zeta)$, where $\eta : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ is a given function. In this paper $\eta(x, \zeta)$ is given by $x - \zeta$.

Let X_t be the state of the pollutant defined by the following stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dW_t, \quad \tau_i \leq t < \tau_{i+1}, \ \forall i \geq 0; \\ X_{\tau_i} = \eta(X_{\tau_i^-}, \zeta_i) = X_{\tau_i^-} - \zeta_i; \\ X_0 = x \in \mathbb{R}_{++}, \end{cases} \tag{2.1}$$

where $\mu(> 0)$ and $\sigma(> 0)$ are constants and

$$\tau_0 := 0 \quad \text{and} \quad \tau_{i+1}^- = \tau_i \quad \text{if} \quad \tau_{i+1} = \tau_i. \tag{2.2}$$

An impulse control for the system is defined as a double sequence

$$v := (\tau_1, \tau_2, \cdots, \tau_i, \cdots; \zeta_1, \zeta_2, \cdots, \zeta_i, \cdots). \tag{2.3}$$

We interpret $\tau_1, \tau_2, \cdots$ as the implementation times, i.e., the times when the agent decides to implement the EIP. Furthermore, for each implementation times he also decides the magnitude of the EIP, $\zeta_1, \zeta_2, \cdots$.

Definition 2.1 (Admissible Impulse Control). *An impulse control, v, is called an admissible impulse control, if the followings are satisfied:*

$$0 \leq \tau_i \leq \tau_{i+1}, \quad P-a.s. \quad \forall i \geq 0,$$

$$\tau_i \textit{ is a } \{\mathcal{F}_t\}_{t\geq 0} - \textit{stopping time}, \quad \forall i \geq 0,$$

$$\zeta_i \ is \ \mathcal{F}_{\tau_i} - measurable, \quad \forall i \geq 0,$$

$$P\left[\lim_{i\to\infty} \tau_i \leq T\right] = 0, \quad \forall T \in [0,\infty).$$

Let $\mathcal{V}$ denote the set of admissible impulse controls. If the impulse control v is given by (2.3), then the system, $\mathcal{X}^{x,v} := \{X_t^{x,v}\}_{t\geq 0}$ is given by

$$\begin{cases} dX_t^{x,v} = \mu X_t^{x,v} ds + \sigma X_t^{x,v} dW_t; & \tau_{i-1} \leq t < \tau_i < \infty; \\ X_{\tau_i}^{x,v} = \eta(X_{\tau_i^-}^{x,v}, \zeta_i) = X_{\tau_i^-}^{x,v} - \zeta_i; & i = 1, 2, \cdots; \\ X_{\tau_0}^{x,v} = x. \end{cases} \tag{2.4}$$

Let $D : \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function satisfying

$$E\left[\int_0^\infty e^{-rt} D(X_t^{x,v}) dt\right] < \infty, \quad x \in \mathbb{R}_{++}, v \in \mathcal{V}, \tag{2.5}$$

where E is the expectation w.r.t. P and r is a constant discount rate. We interpret that $D(x)$ is the damage function associated with the state of the pollutant and given by

$$D(x) := ax^2, \quad x \in \mathbb{R}_{++}, \tag{2.6}$$

where $a > 0$ is a constant. Let v_0 represent a control which the agent does not implement forever. In this case the expected present value of the flow of $D(x)$ is written as

$$E\left[\int_0^\infty e^{-rt} D(X_t^{x,v_0}) dt\right] = \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in \mathbb{R}_{++}, \tag{2.7}$$

By (2.5), we assume that

(A.1)

$$r - 2\mu - \sigma^2 > 0.$$

Let $K : \mathbb{R}_+ \to \mathbb{R}_{++}$ represent the cost to implement the EIP and is given by

$$K(\zeta) := c + b\zeta, \quad \zeta \in \mathbb{R}_+, \tag{2.8}$$

where $b(> 0)$ and $c(> 0)$ are constants. Note that

$$K(\zeta + \zeta') \leq K(\zeta) + K(\zeta'), \quad \text{for } \zeta \leq \zeta'. \tag{2.9}$$

This leads that $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times hold strictly increasing sequences, i.e., $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots$. The damage function, $D(x)$, consists of a part of proportional to x^2 and does not contain constant. Because in general damage from pollutants occurs when one discharges pollutants. Therefore we define the damage function as eq. (2.6). On the other hand, the implementation cost, $K(\zeta)$, involves the constant cost parameter, c. For example, it represents the cost to decide to implement the EIP or not. The implementation cost contains the constant cost, since making decision requires researches on the magnitude of damage, forecast of future environmental conditions and so on. Øksendal [21] studies the effect of the constant cost in impulse control problems and shows it has big effects. Needless to say, the characteristics of the damage function and the state of pollutant can vary with the type of pollutant in question. Then the expected total discounted cost function associated with the control v is defined by

$$J^v(x) = E\left[\int_0^\infty e^{-rt} D(X_t^{x,v}) dt + \sum_{i=1}^\infty e^{-r\tau_i} K(X_{\tau_i^-}^{x,v}, X_{\tau_i}^{x,v}) 1_{\{\tau_i < \infty\}}\right], \quad x \in \mathbb{R}_{++}. \tag{2.10}$$

Therefore the agent's problem is to find the value function J^* defined by

$$J^*(x) = \inf_{v\in\mathcal{V}} J^v(x), \quad x \in \mathbb{R}_{++}, \tag{2.11}$$

and finds the OEIP, i.e., an optimal admissible impulse control $v^* \in \mathcal{V}$ such that

$$J^*(x) = J^{v^*}(x), \quad x \in \mathbb{R}_{++}. \tag{2.12}$$

3 Analysis In this section we prove that a QVI-control which is introduced later is an optimal impulse control for the problem (2.11) and (2.12). Then we show the existence of an optimal impulse control. Lastly we verify a solution of QVI is the value function for the agent problem. To this end we first introduce some notations.

Let M denote the implementation operator on the space of functions $\phi : \mathbb{R}_{++} \to \mathbb{R}$ defined by

$$M\phi(x) = \inf_{\zeta \in [0,x)} \{\phi(\eta(x,\zeta)) + K(\zeta)\}, \quad x \in \mathbb{R}_{++}. \tag{3.1}$$

Suppose that for each $x \in \mathbb{R}_{++}$ there exists at least one $\hat{\zeta} \in \mathbb{R}_+$ such that the infimum in (3.1) is attained and that a measurable selection $\hat{\zeta} = \psi_\phi(x)$ of such minimum points $\hat{\zeta}$ exists. Then we have

$$M\phi(x) = \phi(\eta(x, \psi_\phi(x))) + K(\psi_\phi(x)), \quad x \in \mathbb{R}_{++}. \tag{3.2}$$

Furthermore assume that ϕ is a twice continuously differentiable function on $\mathbb{R}$. Let us define the infinitesimal generator L of the $\mathcal{X}^v$ as follows:

$$[L\phi](x) := \lim_{t \downarrow 0} \frac{e^{-rt} E[\phi(X_t^{x,v})] - \phi(x)}{t}, \quad x \in \mathbb{R}_{++}.$$

Then, by Ito formula, we obtain

$$[L\phi](x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + \mu x \phi'(x) - r\phi(x). \tag{3.3}$$

We cannot apply the classical Dynkin formula for ϕ, since ϕ is not C^2 on the boundary of the continuous region. However we can apply a generalized Dynkin formula for ϕ if ϕ is stochastically C^2. That is, we use a generalized Dynkin formura on the set which has a Green measure of $\mathcal{X}^v$ zero. Here the Green measure of $\mathcal{X}^v$ is the expected total occupation measure $G(\cdot, x)$ defined by

$$G(F, x) = E\left[\int_0^\infty X_t^{x,v} 1_F dt\right], \quad F \subset \mathbb{R}_{++},\ x \in \mathbb{R}_{++}, \tag{3.4}$$

where 1_F is the indicator of a Borel set F. A continuous function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ is called stochastically C^2 w.r.t. $\mathcal{X}^{x,v}$ if $[L\phi](x)$ is well defined pointwise for almost all x with respect to the Green measure $G(\cdot, x)$. The following equality which is called the generalized Dynkin formula will be used in the proof of Theorem 3.1.

$$E[e^{-r\theta^-} \phi(X_{\theta^-}^{x,v})] = E[e^{-r\tau_i} \phi(X_{\tau_i}^{x,v})] + E\left[\int_{\tau_i}^{\theta^-} e^{-rt}[L\phi](X_t^{x,v}) dt\right], \quad x \in \mathbb{R}_{++}, \tag{3.5}$$

for all i, all bounded stopping times θ such that $\tau_i \le \theta \le \tau_{i+1}$. See Brekke and Øksendal [3] for more details.

Definition 3.1 (QVI). *The following relations are called the quasi-variational inequalities for the problem (2.11) and (2.12);(See Bensoussan and Lions [1] for more details.)*

$$[L\phi](x) + D(x) \ge 0, \quad \text{for a.a. } x \text{ w.r.t. } G(\cdot, x), \quad x \in \mathbb{R}_{++}. \tag{3.6}$$

$$\phi(x) \le M\phi(x), \quad x \in \mathbb{R}_{++}. \tag{3.7}$$

$$[[L\phi](x) + D(x)][\phi(x) - M\phi(x)] = 0, \quad \text{for a.a. } x \text{ w.r.t. } G(\cdot, x), \quad x \in \mathbb{R}_{++}. \tag{3.8}$$

Definition 3.2 (QVI-control). *Let ϕ be a solution of the QVI. Then the impulse control $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \cdots; \hat{\zeta}_1, \hat{\zeta}_2, \cdots)$ is called a QVI-control:*

$$(\hat{\tau}_0, \hat{\zeta}_0) = (0, 0), \tag{3.9}$$

$$\hat{\tau}_{i+1} = \inf\{t \geq \hat{\tau}_i ; X_t^{x,\hat{v}} \notin H\}, \tag{3.10}$$

$$\hat{\zeta}_{i+1} = \psi_\phi(X_{\hat{\tau}_{i+1}^-}^{x,\hat{v}}). \tag{3.11}$$

Here H is the continuous region defined by

$$H := \{x ; \phi(x) < M\phi(x)\}, \tag{3.12}$$

and $X_t^{x,\hat{v}}$ is the result of applying the impulse control $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_i ; \hat{\zeta}_1, \hat{\zeta}_2, \cdots, \hat{\zeta}_i)$, $i = 1, 2, \cdots$ to $\mathcal{X}^{x,v}$.

The following Theorem 3.1 is a minor modification of Theorem 3.1. in Brekke and Øksendal [4]. Since we use it to prove Theorem 3.3, we present it.

Theorem 3.1. *(I) Let a continuous function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ be a solution of the QVI and satisfy the followings:*

$$\phi \text{ is stochastically } C^2 \text{ w.r.t. } \mathcal{X}^{x,v}; \tag{3.13}$$

$$\lim_{t\to\infty} e^{-rt}\phi(X_t^{x,v}) = 0, \quad P-a.s., \quad x \in \mathbb{R}_{++},\ v \in \mathcal{V}; \tag{3.14}$$

$$\text{the family } \{\phi(X_\tau^{x,v})\}_{\tau<\infty} \text{ is uniformly integrable w.r.t. } P, \quad x \in \mathbb{R}_{++},\ v \in \mathcal{V}. \tag{3.15}$$

Then we obtain

$$\phi(x) \leq J^v(x) \quad x \in \mathbb{R}_{++},\ v \in \mathcal{V}. \tag{3.16}$$

(II) Suppse that, in addition to (3.6)–(3.8) and (3.13)–(3.15), we have

$$[L\phi](x) + D(x) = 0, \quad x \in H. \tag{3.17}$$

Furthermore, suppose $\hat{v} \in \mathcal{V}$, i.e., the impulse control is a QVI-control. Then we obtain

$$\phi(x) = J^{\hat{v}}(x), \quad x \in \mathbb{R}_{++}. \tag{3.18}$$

Hence we have

$$\phi(x) = J^*(x) = J^{\hat{v}}(x), \tag{3.19}$$

and therefore $\hat{v}$ is an optimal impulse control.

Proof. (I) Assume that ϕ satisfies (3.13)–(3.15). Choose $v \in \mathcal{V}$. Let $\theta_{i+1} := \tau_i \vee (\tau_{i+1} \wedge s)$ for any $s \in \mathbb{R}_+$. Then by the generalized Dynkin formula, (3.5), we obtain

$$E[e^{-r\theta_{i+1}^-}\phi(X_{\theta_{i+1}^-}^{x,v})] = E[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] + E\left[\int_{\tau_i}^{\theta_{i+1}} e^{-rt}[L\phi](X_t^{x,v})dt\right]. \tag{3.20}$$

Hence from (3.6) we obtain

$$E[e^{-r\theta_{i+1}^-}\phi(X_{\theta_{i+1}^-}^{x,v})] \geq E[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] - E\left[\int_{\tau_i}^{\theta_{i+1}} e^{-rt}D(X_t^{x,v})dt\right]. \tag{3.21}$$

Taking $\lim_{s\to\infty}$ we have by the dominated convergence theorem

$$E[e^{-r\tau_{i+1}^-}\phi(X_{\tau_{i+1}^-}^{x,v})] - E[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] \geq -E\left[\int_{\tau_i}^{\tau_{i+1}} e^{-rt}D(X_t^{x,v})dt\right]. \tag{3.22}$$

Summing from $i = 0$ to $i = m$ yields

$$\phi(x) + \sum_{i=1}^{m} E[e^{-r\tau_i}\phi(X^{x,v}_{\tau_i}) - e^{-r\tau_i^-}\phi(X^{x,v}_{\tau_i^-})] \tag{3.23}$$
$$- E[e^{-r\tau_{m+1}^-}\phi(X^{x,v}_{\tau_{m+1}^-})] \le E\left[\int_0^{\tau_{m+1}} e^{-rt}D(X^{x,v}_t)dt\right].$$

Since after giving the impulse the state of the system jumps immediately from $X^{x,v}_{\tau_i^-}$ to a new state $\eta(X^{x,v}_{\tau_i^-}, \zeta_i)$ for all $\tau_i < \infty$, by eq. (3.1) and $\eta(X^{x,v}_{\tau_i^-}, \zeta_i) = X^{x,v}_{\tau_i}$ we obtain

$$\phi(\eta(X^{x,v}_{\tau_i^-}, \zeta_i)) \ge M\phi(X^{x,v}_{\tau_i^-}) - K(\zeta_i), \quad \tau_i < \infty. \tag{3.24}$$

Therefore we have

$$\phi(x) + \sum_{i=1}^{m} E[[e^{-r\tau_i}M\phi(X^{x,v}_{\tau_i^-}) - e^{-r\tau_i^-}\phi(X^{x,v}_{\tau_i^-})]1_{\{\tau_i<\infty\}}] \tag{3.25}$$
$$\le E\left[\int_0^{\tau_{m+1}} e^{-rt}D(X^{x,v}_t)dt + e^{-r\tau_{m+1}^-}\phi(X^{x,v}_{\tau_{m+1}^-}) + \sum_{i=1}^{m} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<\infty\}}\right].$$

By eq. (3.7) we have

$$M\phi(X^{x,v}_{\tau_i^-}) - \phi(X^{x,v}_{\tau_i^-}) \ge 0. \tag{3.26}$$

Hence we obtain

$$\phi(x) \le E\left[\int_0^{\tau_{m+1}} e^{-rt}D(X^{x,v}_t)dt + e^{-r\tau_{m+1}^-}\phi(X^{x,v}_{\tau_{m+1}^-}) + \sum_{i=1}^{m} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<\infty\}}\right]. \tag{3.27}$$

Taking $\lim_{m\to\infty}$ we obtain by using (3.14), (3.15) and the dominated convergence theorem

$$\phi(x) \le E\left[\int_0^{\infty} e^{-rt}D(X^{x,v}_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<\infty\}}\right]. \tag{3.28}$$

Therefore eq. (3.16) is proved.

(II) Assume that eq. (3.17) holds and $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \cdots; \hat{\zeta}_1, \hat{\zeta}_2, \cdots)$ is the QVI-control. Then repeat the argument in part (i) for $v = \hat{v}$. Then all the ineqs. (3.21)-(3.28) become equalities. Thus we obtain

$$\phi(x) = E\left[\int_0^{\infty} e^{-rt}D(X^{x,\hat{v}}_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i}K(X^{x,\hat{v}}_{\tau_i^-}, X^{x,\hat{v}}_{\tau_i})1_{\{\tau_i<\infty\}}\right]. \tag{3.29}$$

Hence we get by eq. (3.18). Combining eq. (3.18) with ineq. (3.16), we obtain

$$\phi(x) \le \inf_{v\in\mathcal{V}} J^v(x) \le J^{\hat{v}}(x) = \phi(x). \tag{3.30}$$

Therefore $\phi(x) = J^*(x)$ and $v^* = \hat{v}$ is optimal.

□

Let us return to our problem. We consider the following EIP. When the pollutant level reaches at a threshold, $\overline{x}$, the agent implements the EIP. As a result of it, the pollutant level, $\overline{x}$, immediately decreases to β. Then we conjecture that there exists an optimal solution $v^*(\tau^*, \zeta^*)$ characterized by parameters $(\beta, \overline{x})$ with $0 < \beta < \overline{x} < \infty$ such that

$$\tau_i^* := \inf\{t > \tau_{i-1}^*; X^{x,v^*}_{t^-} \notin (0, \overline{x})\}, \tag{3.31}$$

$$X^{x,v^*}_{\tau_i^*} := X^{x,v^*}_{\tau_i^*-} - \zeta_i^* = \beta, \tag{3.32}$$

i.e., eq. (2.4) becomes to

$$\begin{cases} dX^{x,v^*}_t = \mu X^{x,v^*}_t dt + \sigma X^{xv^*}_t dW_t; \quad \tau^*_{i-1} \le t < \tau^*_i < \infty; \\ X^{x,v^*}_{\tau_i^*} = \eta(X^{x,v^*}_{\tau_i^*-}, \zeta_i^*); \quad i = 1, 2, \cdots; \\ X^{x,v^*}_{\tau_0} = x, \end{cases} \tag{3.33}$$

Therefore in this case the value function seems to satisfy

$$\begin{aligned} J^*(x) &= J^*(\eta(x, \zeta^*)) + K(\zeta^*) \\ &= J^*(\beta) + c + b(x - \beta), \quad x \in [\overline{x}, \infty). \end{aligned} \tag{3.34}$$

Assume that J^* is a stochastically $C^2(\mathbb{R})$-function w.r.t. $\mathcal{X}^{x,v^*}$. Furthermore assume that an impulse control is as follows. Once the pollutant level within the continuous region $(0, \overline{x})$, it remains in that region thereafter. Howerver if the initial level of the pollutant x is $x = \overline{x} + \varepsilon$, where $\varepsilon > 0$, then the optimal impulse control is $\zeta = (\overline{x} + \varepsilon) - \beta$. Thus we have

$$J^*(\overline{x} + \varepsilon) = J^*(\beta) + c + b(\overline{x} + \varepsilon - \beta). \tag{3.35}$$

Replacing x into $\overline{x}$ in eq. (3.34) and subtracting from eq. (3.35), we obtain

$$J^*(\overline{x} + \varepsilon) - J^*(\overline{x}) = b\varepsilon. \tag{3.36}$$

Dividing eq. (3.36) by ε and taking $\lim_{\varepsilon \to 0}$ in eq. (3.36), we get

$$J^{*'}(\overline{x}) = b. \tag{3.37}$$

By (3.31) and (3.32), eq (3.34) is minimized at $\zeta = \overline{x} - \beta$. Hence by the first order condition for the minimization $\partial[J^*(\eta(\overline{x}, \zeta)) + K(\zeta)]/\partial\zeta \mid_{\zeta = \overline{x} - \beta} = 0$ we obtain

$$J^{*'}(\beta) = b, \tag{3.38}$$

since J^* is a stochastically $C^2(\mathbb{R})$-function. Dixit [9] and Constantinides and Rechard [7] discussed similar equations of eqs. (3.37) and (3.38) for more details. Furthermore, we can conjecture that eq. (3.17) holds in the continuous region $(0, \overline{x})$. Following the standard methods of ordinary differential equations we have the general solution of eq (3.17) given by

$$\phi(x) = A_1 x^{\lambda_1} + A_2 x^{\lambda_2} + \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in (0, \overline{x}), \tag{3.39}$$

where A_1 and A_2 are constants to be determined, and λ_1 and λ_2 are the solutions to the characteristic equation,

$$\gamma(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\lambda - r = 0, \quad \lambda \in \mathbb{R}. \tag{3.40}$$

Hence we get

$$\lambda_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left[\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}, \tag{3.41}$$

$$\lambda_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \left[\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}. \tag{3.42}$$

Since $\lambda_2 < 0$, to prevent the value from diverging, we set the coefficient $A_2 = 0$. Thus we have

$$\phi(x) = A_1 x^{\lambda_1} + \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in (0, \overline{x}). \tag{3.43}$$

Remark 3.1. *The first term on the right-hand side of eq. (3.43) is the value of the option to implement the EIP for some times in the future. In other words, when the state process of the pollutant $\mathcal{X}^{x,v^*}$ hits the threshold $\overline{x}$, the agent exercises the option and implements the EIP. Then we can evaluate the value of the EIP by calculating the first term on the right-hand side of eq. (3.43). Since our problem is the cost minimization problem, we evaluate the value of the EIP by changing the sign of the first term on the right-hand side of eq. (3.43). Hence we calculate the value of the EIP by the form*

$$EIP = -A_1 x^{\lambda_1}. \tag{3.44}$$

The second term, $ax^2/(r-2\mu-\sigma^2)$, is the expected present value of the flow of $D(x)$ when we ignore the barrier at $\overline{x}$ by eq. (2.7).

Let us define $\phi(x)$ by

$$\phi(x) := \begin{cases} A_1 x^{\lambda_1} + \frac{ax^2}{r-2\mu-\sigma^2}, & x \in (0, \overline{x}), \\ \phi(\beta) + c + b(x-\beta), & x \in [\overline{x}, \infty). \end{cases} \tag{3.45}$$

Here A_1, $\overline{x}$, β are parameters which are uniquely determined by following simultaneous equations:

$$\phi(\overline{x}) = \phi(\beta) + c + b(\overline{x} - \beta), \tag{3.46}$$

$$\phi'(\overline{x}) = b, \tag{3.47}$$

$$\phi'(\beta) = b. \tag{3.48}$$

In order to study how $\overline{x}$ and β depends on A_1 later, let us define $\Phi(x)$ as

$$\Phi(x) := A_1[x^{\lambda_1} - \beta^{\lambda_1}] + \frac{a(x^2-\beta^2)}{r-2\mu-\sigma^2} - c - b(x-\beta), \tag{3.49}$$

where we fix $A_1 < 0$. See Figure 1. The first, second and third derivative of eq. (3.49) are given by

$$\Phi'(x) = \lambda_1 A_1 x^{\lambda_1 - 1} + \frac{2ax}{r-2\mu-\sigma^2} - b, \tag{3.50}$$

$$\Phi''(x) = \lambda_1(\lambda_1 - 1) A_1 x^{\lambda_1 - 2} + \frac{2a}{r-2\mu-\sigma^2}, \tag{3.51}$$

$$\Phi'''(x) = \lambda_1(\lambda_1-1)(\lambda_1-2) A_1 x^{\lambda_1 - 3}. \tag{3.52}$$

See Figure 2, 3 and 4, respectively. Since $\lambda_1 > 2$ and $A_1 < 0$, $\Phi'''(x)$ is negative. Hence $\Phi'(x)$ has its unique maximum point, $\breve{x}(A_1)$. From eq. (3.51) $\Phi''(x) = 0$ iff

$$A_1 = -\frac{1}{\lambda_1(\lambda_1-1)} x^{2-\lambda_1} \frac{2a}{r-2\mu-\sigma^2}. \tag{3.53}$$

From eq. (3.50), $\Phi'(\breve{x}(A_1)) > 0$ iff

$$\begin{aligned} \breve{x} &> \frac{r-2\mu-\sigma^2}{2a} \frac{\lambda_1 - 1}{\lambda_1 - 2} b \\ &=: \tilde{x}. \end{aligned} \tag{3.54}$$

Thus $\Phi'(\breve{x}(A_1)) > 0$ iff $\breve{x} > \tilde{x}$. This implies that

$$\lambda_1 A_1 \breve{x}^{\lambda_1 - 1} > \lambda_1 \tilde{A}_1 \tilde{x}^{\lambda_1 - 1} \quad \text{or} \quad A_1 > \tilde{A}_1,$$

where by (3.53) and (3.54) $\tilde{A}_1$ is given by

$$\tilde{A}_1 = -\frac{1}{\lambda_1(\lambda_1 - 1)} \left(\frac{r - 2\mu - \sigma^2}{2a} \right)^{1-\lambda_1} \left(\frac{\lambda_1 - 1}{\lambda_1 - 2} b \right)^{2-\lambda_1} . \tag{3.55}$$

Note that $\Phi'(0) = -b < 0$. Therefore for any $\tilde{A}_1 < A_1 < 0$, eqs. (3.47) and (3.48) have two solutions $\overline{x}(A_1)$ and $\beta(A_1)$ such that $0 < \beta(A_1) < \check{x}(A_1) < \overline{x}(A_1)$.

Hereafter we assume that $\tilde{A}_1 < A_1 < 0$. In oder to show existence of an optimal impulse control, we study how $\overline{x}$ and β depend on A_1. We refer to Øksendal [21] Lemma 2.3. To this end we first differentiate eq. (3.47) w.r.t. A_1 and obtain

$$\overline{x}'(A_1) = -\left[\lambda_1(\lambda_1 - 1)A_1\overline{x}^{\lambda_1 - 2} + \frac{2a}{r - 2\mu - \sigma^2} \right]^{-1} \lambda_1 \overline{x}(A_1)^{\lambda_1 - 1} > 0. \tag{3.56}$$

Since $\lambda_1(\lambda_1 - 1)A_1\overline{x}^{\lambda_1 - 2} + 2a/(r - 2\mu - \sigma^2) = \Phi''(\overline{x}(A_1)) < 0$, inequality of (3.56) holds. (3.56) means $\overline{x}(A_1)$ increases in A_1. Thus by eq. (3.47) we have

$$\lim_{A_1 \to 0} \lambda_1 A_1 \overline{x}(A_1)^{\lambda_1 - 1} + \frac{2a\overline{x}(A_1)}{r - 2\mu - \sigma^2} - b = \lim_{A_1 \to 0} \frac{2a\overline{x}(A_1)}{r - 2\mu - \sigma^2} - b. \tag{3.57}$$

It follows that

$$\lim_{A_1 \to 0} \overline{x}(A_1) = +\infty. \tag{3.58}$$

Note that $\Phi'(\check{x}(A_1)) > 0$ iff $\check{x}(A_1) > \tilde{x}$ or $A_1 > \tilde{A}_1$ and $\overline{x}(A_1) > \check{x}(A_1)$. If A_1 decreases to $\tilde{A}_1$, $\overline{x}(A_1)$ decreases to $\tilde{x}$, i.e.,

$$\lim_{A_1 \to \tilde{A}_1} \overline{x}(A_1) = \tilde{x}. \tag{3.59}$$

Similarly we differentiate eq. (3.48) w.r.t. A_1 and by $\Phi''(\beta(A_1)) > 0$ obtain

$$\beta'(A_1) < 0. \tag{3.60}$$

(3.60) implies $\beta(A_1)$ decreases in A_1. Thus by eq. (3.48) we have

$$\lim_{A_1 \to 0} \lambda_1 A_1 \beta(A_1)^{\lambda_1 - 1} + \frac{2a\beta(A_1)}{r - 2\mu - \sigma^2} - b = \lim_{A_1 \to 0} \frac{2a\beta(A_1)}{r - 2\mu - \sigma^2} - b. \tag{3.61}$$

From (3.61) we obtain

$$\lim_{A_1 \to 0} \beta(A_1) = \frac{r - 2\mu - \sigma^2}{2a} b. \tag{3.62}$$

Furthermore we have

$$\lim_{A_1 \to \tilde{A}_1} \beta(A_1) = \tilde{x}. \tag{3.63}$$

Now we are ready to show the existence of an optimal impulse control. Here we assume that

(A.2)

$$\frac{a}{r - 2\mu - \sigma^2} > b,$$

where $a/(r - 2\mu - \sigma^2)$ is the present value of damege per the state of pullutant. If the above inequality does not hold, it will never be optimal to implement the EIP as far as $c > 0$.

Theorem 3.2. *Assume that (A.1) and (A.2) hold. ϕ satisfies eqs. (3.46) – (3.48). Then there exists an optimal impulse control $v^*(\tau^*, \zeta^*)$ characterized by $(\beta(A_1), \overline{x}(A_1))$ with $0 < \beta(A_1) < \overline{x}(A_1) < +\infty$ such that (3.31) and (3.32).*

Proof. By eq. (3.46) we obtain

$$g(\overline{x}(A_1), \beta(A_1), A_1) = c, \tag{3.64}$$

where $g(\overline{x}(A_1), \beta(A_1), A_1) := A_1[\overline{x}(A_1)^{\lambda_1} - \beta(A_1)^{\lambda_1}] + a[\overline{x}(A_1)^2 - \beta(A_1)^2]/(r - 2\mu - \sigma^2) - b[\overline{x}(A_1) - \beta(A_1)]$. The derivative of g w.r.t. A_1 is

$$\begin{aligned} \frac{\partial g}{\partial A_1} &= [\overline{x}(A_1)^{\lambda_1} - \beta(A_1)^{\lambda_1}] + A_1[\lambda_1 \overline{x}(A_1)^{\lambda_1 - 1}\overline{x}'(A_1) - \lambda_1 \beta(A_1)^{\lambda_1 - 1}\beta'(A_1)] \\ &\quad + \frac{2a}{r - 2\mu - \sigma^2}[\overline{x}(A_1)\overline{x}'(A_1) - \beta(A_1)\beta'(A_1)] - b[\overline{x}'(A_1) - \beta'(A_1)]. \end{aligned} \tag{3.65}$$

From (3.56) and (3.60), the first and third terms are positive, while the second and fourth terms are negative. The sign of (3.65) depends on the relation to these terms. To investigate the relation, firstly, we suppose that

$$\overline{x}(A_1)^{\lambda_1} - \beta(A_1)^{\lambda_1} - b[\overline{x}'(A_1) - \beta'(A_1)] > 0. \tag{3.66}$$

(3.58) and (3.62) reveal that the left-hand side of (3.66) is positive. From (3.59) and (3.63), it follows that (3.66) is zero. Hence ineq. (3.66) holds. Secondly suppose that

$$A_1[\lambda_1 \overline{x}(A_1)^{\lambda_1 - 1}\overline{x}'(A_1) - \lambda_1 \beta(A_1)^{\lambda_1 - 1}\beta'(A_1)] + \frac{2a}{r - 2\mu - \sigma^2}[\overline{x}(A_1)\overline{x}'(A_1) - \beta(A_1)\beta'(A_1)] > 0. \tag{3.67}$$

To show ineq. (3.67) we require that

$$\frac{2a}{r - 2\mu - \sigma^2}\overline{x}(A_1)^{2-\lambda_1} > -A_1\lambda_1, \tag{3.68}$$

$$\frac{2a}{r - 2\mu - \sigma^2}\beta(A_1)^{2-\lambda_1} > A_1\lambda_1. \tag{3.69}$$

By (3.58) and (3.59), we have

$$1 > \frac{1}{\lambda_1(\lambda_1 - 1)}. \tag{3.70}$$

Since it is obvious that ineq. (3.70) holds, we obtain ineq. (3.68). On the other hand, eqs. (3.62) and (3.63) implies tha the minmum of $\beta(A_1)$ is attained when A_1 goes to 0. Then we obtain

$$\left(\frac{r - 2\mu - \sigma^2}{2a}\right)^{1-\lambda_1} b^{2-\lambda_1} > 0. \tag{3.71}$$

Since it is clear that ineq. (3.71) holds, we have ineq. (3.69). Thus it follows ineq. (3.67). Both ineqs. (3.66) and (3.67) imply that ineq. (3.65) is positive. Therefore it follows that

$$\lim_{A_1 \to 0} g(\overline{x}(A_1), \beta(A_1), A_1) = +\infty, \tag{3.72}$$

$$\lim_{A_1 \to \hat{A}_1} g(\overline{x}(A_1), \beta(A_1), A_1) = 0. \tag{3.73}$$

From eqs. (3.72) and (3.73) there exists A_1 such that $g(\overline{x}(A_1), \beta(A_1), A_1) = c$ by using the mean value theorem. Therefore we conclude that there exists an optimal impulse control $v^*(\tau^*, \zeta^*)$ defined by (3.31) and (3.32). □

For simplicity we put $\overline{x} = \overline{x}(A_1)$ and $\beta = \beta(A_1)$. We show ϕ is the value function of the agent problem eqs. (2.11) and (2.12). To this end we first show that the followings. For $x \in (0, \overline{x})$, by (3.45) the first and second derivative of $\phi(x)$ are respectively

$$\phi'(x) = \lambda_1 A_1 \overline{x}^{\lambda_1 - 1} + \frac{2a\overline{x}}{r - 2\mu - \sigma^2}, \tag{3.74}$$

$$\phi''(x) = \lambda_1(\lambda_1 - 1)A_1 \overline{x}^{\lambda_1 - 2} + \frac{2a}{r - 2\mu - \sigma^2}. \tag{3.75}$$

Note that eq. (3.75) equals to eq. (3.51). It is obvious that there exist A_1 such that $\Phi'(\overline{x}(A_1)) = \Phi'(\beta(A_1)) = 0$ from Theorem 3.2. It follows that $\phi'(\overline{x}) = \phi'(\beta) = b$. Furthemore, since $\Phi'(x(A_1))$ has a unique maximum point,

$$\Phi'(x) \begin{cases} < 0, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = 0, & x = \beta \text{ or } \overline{x}, \\ > 0, & x \in (\beta, \overline{x}). \end{cases} \tag{3.76}$$

Therefore it follows that

$$\phi'(x) \begin{cases} < b, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = b, & x = \beta \text{ or } \overline{x}, \\ > b, & x \in (\beta, \overline{x}). \end{cases} \tag{3.77}$$

Theorem 3.3. *Assume that (A.1) and (A.2) hold. Let A_1, $\overline{x}$, and β with $0 < \beta < \overline{x} < \infty$ be solutions of the simultaneous equations (3.46)–(3.48). Here we assume that the implementation cost satisfies the following:*

(A.3)

$$\begin{aligned} &(r - 2\mu - \sigma^2)\left(\frac{\lambda_1 - 1}{\lambda_1 - 2}\right) b \\ &\quad > (r - \mu)b + \left[(r - \mu)^2 b^2 + 4ar\left[\frac{r - 2\mu - \sigma^2}{2a}\left(-\frac{(\lambda_1 - 1)^2}{2\lambda_1(\lambda_1 - 2)}\right) b^2 + c\right]\right]^{\frac{1}{2}} \end{aligned}$$

ϕ defined by (3.45) is a solution of the QVI. Then ϕ is the value function of the agent problem eqs. (2.11) and (2.12). Furthermore, an optimal impulse control is given by (3.31) and (3.32).

Proof. First we show ϕ is a solution of the QVI. To accomplish this we confirm that ϕ satisfies eqs. (3.6) – (3.8).

(I) Consider eq. (3.6) under two distinct cases, $x \in (0, \overline{x})$ or $x \in [\overline{x}, \infty)$.

(i) If $x \in (0, \overline{x})$, by (3.45) and the derivation of eq. (3.43) it is clear that

$$[L\phi](x) + D(x) = 0.$$

(ii) If $x \in [\overline{x}, \infty)$, by (3.45) we have

$$[L\phi](x) + D(x) = \mu x b - r[\phi(\beta) + c + b(x - \beta)] + ax^2. \tag{3.78}$$

If (3.78) is positive, we have

$$\overline{x}(A_1) > \frac{1}{2a}\left\{(r - \mu)b + \left[(r - \mu)^2 b^2 + 4ar\left[\phi(\beta(A_1)) + c - b\beta(A_1)\right]\right]^{\frac{1}{2}}\right\} \tag{3.79}$$

Cadenillas and Zapatero [6] assume that similar inequalities to ineq. (3.79) hold to prove Theorem 4.1 in Cadenillas and Zapatero [6]. While we express ineq. (3.79) with given parameters. Note that $\Phi'(\tilde{x}(A_1)) > 0$ iff $\check{x} > \tilde{x}$ implies $A_1 > \tilde{A}_1$. Taking $\lim_{A_1 \to \tilde{A}_1}$ both sides of ineq. (3.79), by (3.54), (3.55), (3.59), (3.72) and (A.3) we obtain (3.78) is positive.

Therefore ϕ satisfies ineq. (3.6).

(II) Next we show ineq. (3.7). We divide the region into $(0,\beta)$, $[\beta,\overline{x})$ and $[\overline{x},\infty)$.

(i) For $x \in (0,\beta)$, by (3.77) $\zeta = 0$ is optimal. Then we have

$$\begin{aligned} M\phi(x) &= \inf_{\zeta\in(0,\beta)}\{\phi(\eta(x,\zeta)) + K(\zeta)\} \\ &= [\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=0} \\ &= \phi(x) + c \\ &> \phi(x). \end{aligned}$$

(ii) For $x \in [\beta,\overline{x})$, since equality in (3.77) holds at $x = \beta$, $\zeta = x - \beta$ is optimal. Thus we obtain

$$\begin{aligned} M\phi(x) &= \inf_{\zeta\in(0,x-\beta]}\{\phi(\eta(x,\zeta)) + K(\zeta)\} \\ &= [\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\beta} \\ &= \phi(\beta) + c + b(x-\beta) \\ &> \phi(x). \end{aligned}$$

Inequality holds by eq. (3.46) and (3.76).

(iii) For $x \in [\overline{x},\infty)$, since equality in (3.77) holds at $x = \overline{x}$ or β. Hence either $\zeta = x-\overline{x}$ or $\zeta = x-\beta$ is optimal. Therefore we have

$$\begin{aligned} M\phi(x) &= \min\left[\inf_{\zeta\in(0,x-\overline{x}]}\{\phi(\eta(x,\zeta)) + K(\zeta)\},\ \inf_{\zeta\in(x-\overline{x},x)}\{\phi(\eta(x,\zeta)) + K(\zeta)\}\right] \\ &= \min\left[[\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\overline{x}},\ [\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\beta}\right] \\ &= \phi(\beta) + c + b(x-\beta) \\ &= \phi(x). \end{aligned}$$

Here third equality holds by eq. (3.46).

Therefore ϕ satisfies ineq. (3.7).

(III) It follows immediately from foregoing consideration that ϕ also satisfies eq.(3.8).

Hence ϕ is a solution of the QVI. Thus, by Theorem 3.1, ϕ is the value function of the agent problem eqs. (2.11) and (2.12). Furthermore an optimal impulse control is given by (3.31) and (3.32). The proof completes. □

4 Numerical Examples and Comparative Statics In this section we calculate A_1, $\overline{x}$ and β by using a numerical method and evaluate the optimal implementation size, ζ^* and the OEIP. Furthermore we present comparative statics for ζ^*, β and the OEIP by changing parameters. Because to evaluate them gives us economic intuitions. When the agent decides to implement the EIP, these results are useful.

The base case parameters used are listed in Table 1. The results of the numerical examples are presented in Table 2. There we vary parameters by $\pm 10\%$. Furthermore Table 2 illustrates the comparative statics. First, we find the results from comparative statics to ζ^*: *The optimal implementation size ζ^* is increasing in the discount rate, r, the diffusion parameter, σ, the proportional cost parameter, b and the constant cost parameter, c. On the other hand, ζ^* is decreasing in the drift parameter, μ and the proportional damage parameter, a.* They mean as follows. When the discount rate is high, the future damage from the pollutant is more serious. Hence the optimal implementation size increases in the discount rate. Since the diffusion parameter means uncertainty on damage, an increase in uncertainty raises the optimal implementation size. The optimal implementation size also increases in the cost to implement the OEIP. On the other hand, since the drift parameter means the growth rate of the pollutant, the higher growth rate of pollutant decreases the optimal implementation size. Similarly the proportional damage parameter decreases the optimal implementation size.

Secondly the comparative statics to the OEIP give us the following results: *An increase in the growth rate of the pollutant, uncertainty, the proportional cost and the constant cost raises the value of the OEIP. While an increase in the discount rate and the proportional damage parameter decrease the value of the OEIP.* From the results we have the following implications. Since λ_1, A_1 and $\overline{x}$ affect the OEIP, the comparative statics for the OEIP are complicated. Although λ_1, A_1 and $\overline{x}$ don't have the same effect of a change in the parameters, we have plausible results. The OEIP increases in the growth rate of the pollutant, uncertainty, the proportional implementation cost and the constant implementation cost. While the OEIP decreases in the discount rate and the proportional damage parameter. Since the OEIP leads to flexibility to the agent's decision from Remark 3.1, higher uncertainty rises the value of the OEIP. The proportional and the constant implementation cost lead to the same direction effect.

5 Conclusion In this paper, we study the general environmental improvement policy by using an impulse control method. Furthermore, by using numerical results we describe the comparative statics. They give us some economic implications. One of the main implications of this paper is that the threshold of $\mathcal{X}^v$, implementation size and the value of the OEIP increase in uncertainty. The paper considers one pollutant for simplicity. However there are a lot of pollutants in the world. Thus it is important to extend to a multi pollutants model. Other interesting extension of this paper are to consider technological progressive, which improve environment, generalization of damege function like $D(x) := ax^\rho,\ \rho > 0$ and nonlinear implementation cost function. Nevertheless the model of this paper provides a useful first step for the future works.

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Table 1: Base Case Parameters

r	μ	σ	a	b	c
0.06	0.01	0.15	0.25	5	5

Table 2: The results of numerical examples.

	λ_1	$-A_1$	$\overline{x}$	β	ζ^*	OEIP
base case	2.3656	8.3385	2.1946	0.3217	1.8729	53.533
$r: +10\%$	2.4783	5.1543	2.3255	0.3769	1.9486	41.7354
$: -10\%$	2.2472	15.2801	2.0599	0.2675	1.7924	77.5109
$\mu: +10\%$	2.3205	10.0666	2.1727	0.3077	1.8650	60.9423
$: -10\%$	2.4116	6.9882	2.2167	0.3360	1.8808	47.65
$\sigma: +10\%$	2.2363	13.6144	2.2634	0.3122	1.9513	84.602
$: -10\%$	2.5178	5.5399	2.1256	0.3314	1.7942	36.9834
$a: +10\%$	–	9.4114	2.0611	0.2893	1.7717	52.0798
$: -10\%$	–	7.2921	2.3538	0.3618	1.9920	55.2498
$b: +10\%$	–	8.1951	2.2657	0.3617	1.9039	56.73
$: -10\%$	–	8.4902	2.1238	0.2827	1.8411	50.4375
$c: +10\%$	–	8.2628	2.2672	0.3183	1.9489	57.2877
$: -10\%$	–	8.4206	2.1184	0.3256	1.7928	49.7248

Notes: λ_1 is independent of parameters a, b, and c.
OEIP is calculated by $-A_1\overline{x}^{\lambda_1}$.

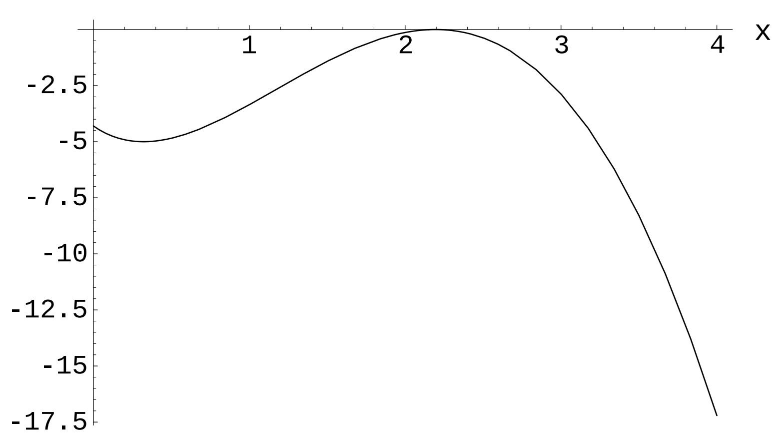

Figure 1: $\Phi(x)$

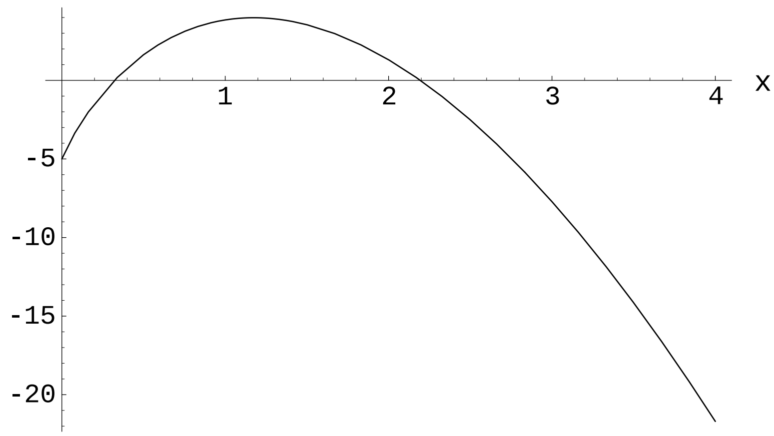

Figure 2: $\Phi'(x)$

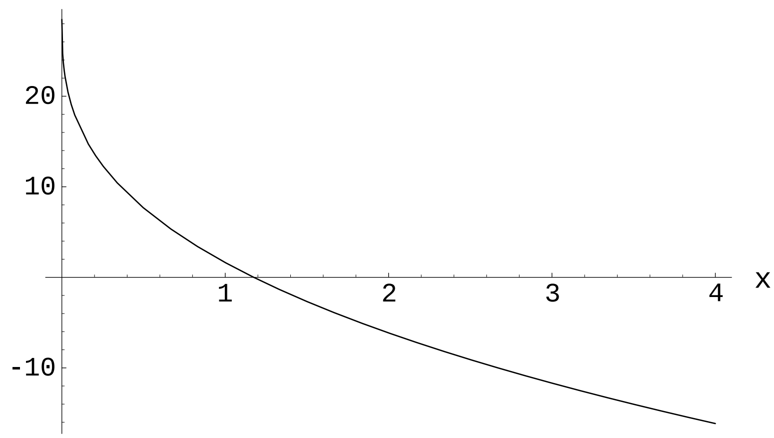

Figure 3: $\Phi''(x)$

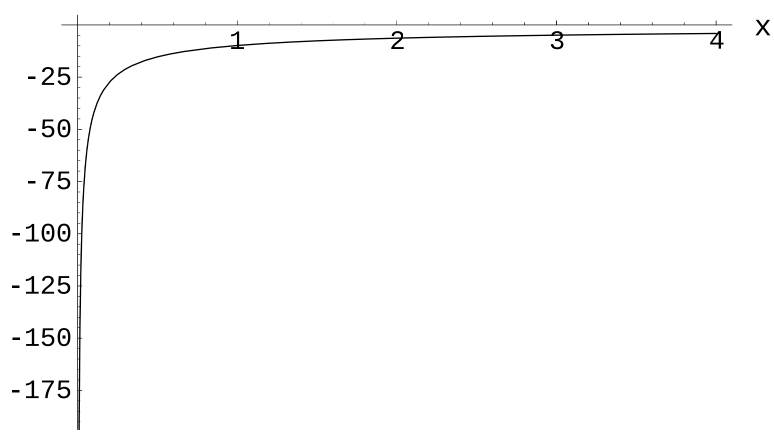

Figure 4: $\Phi'''(x)$