EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we afford some sufficient conditions to guarantee the existence of positive solutions for the nonlinear boundary value problem

$$(BVP) \qquad \begin{cases} (E) \quad \big(\phi(u')\big)' + f(t,u) = 0, \qquad 0 < t < 1, \\ (BC) \begin{cases} u(0) - B_0\big(u'(0)\big) = 0, \\ u(1) + B_1\big(u'(1)\big) = 0. \end{cases} \end{cases}$$

1. Introduction

In this paper, we consider the boundary value problem of the form

$$(BVP) \qquad \begin{cases} (E) \quad \big(\phi(u')\big)' + f(t,u) = 0, \qquad 0 < t < 1, \\ (BC) \begin{cases} u(0) - B_0\big(u'(0)\big) = 0, \\ u(1) + B_1\big(u'(1)\big) = 0, \end{cases} \end{cases}$$

where $f,\ \phi,\ B_0,\ B_1$ satisfy

(H_1) $f \in C([0,1] \times [0,\infty); [0,\infty))$;

(H_2) $\phi \in C^1(\mathbb{R};\mathbb{R})$ is an odd and strictly increasing function on $\mathbb{R}$;

(H_3) $B_0(v)$ and $B_1(v)$ are both increasing, continuous, odd functions defined on $(-\infty,\infty)$ and at least one of them satisfies the following condition: there exists a $\theta > 0$ such that

$$0 \le B_i(v) \le \theta v \qquad \text{for all } v \ge 0,\ i = 1 \text{ or } 2.$$

The boundary value problem (BVP) aries in many different areas of applied mathematics and physics; see [1-5, 8-9] and the references therein.

2. Main Results

Throughout this paper, we let Ψ be the inverse function of φ. In order to obtain our main results (Theorems 2.1 and 2.2), we need the following two lemmas:

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Lemma 2.A (Kransnosel'skii [7], see, also Guo and Lakshmikantham [6]). *Let X be a Banach space, and $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $||Tu|| \le ||u||$, $u \in K \cap \partial\Omega_1$ and $||Tu|| \ge ||u||$, $u \in K \cap \partial\Omega_2$, or

(ii) $||Tu|| \ge ||u||$, $u \in K \cap \partial\Omega_1$ and $||Tu|| \le ||u||$, $u \in K \cap \partial\Omega_2$,

then T has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Here $|| \cdot ||$ stands for the sup norm.

Lemma 2.B. *Assume that*

(H) *ϕ is convex on $[0, \infty)$.*

Then, we have the following results:

(A_1) *$\phi(xy) \le x\phi(y)$ for $0 \le x \le 1$ and $y \ge 0$,*

(A_2) *$\phi(xy) \ge x\phi(y)$ for $x \ge 1$ and $y \ge 0$,*

(A_3) *$\Psi(xy) \ge x\Psi(y)$ for $0 \le x \le 1$ and $y \ge 0$,*

(A_4) *$\Psi(xy) \le x\Psi(y)$ for $x \ge 1$ and $y \ge 0$.*

Proof. Let $F(t) := \phi(ct) - c\phi(t)$, where $c,\ t \in [0, \infty)$. Since ϕ is a convex function on $[0, \infty)$, $\phi'(t)$ is increasing on $[0, \infty)$. Thus, we have

$$F'(t) = c\phi'(ct) - c\phi'(t) = c\big(\phi'(ct) - \phi'(t)\big).$$

Hence

$$F'(t) \begin{cases} \le 0, & \text{if } 0 \le c \le 1, \\ \ge 0, & \text{if } c \ge 1 \end{cases}$$

on $[0, \infty)$. It follows from $\phi(0) = F(0) = 0$ that $F(t) \le 0$ for $0 \le c \le 1$ and $F(t) \ge 0$ for $c \ge 1$ on $[0, \infty)$. Therefore, (A_1) and (A_2) hold.

Since ϕ is convex on $[0, \infty)$ and satisfies $\phi(0) = 0$, Ψ is concave on $[0, \infty)$ and satisfies $\Psi(0) = 0$. Hence, $\Psi'(t)$ is decreasing on $[0, \infty)$. Using the same argument, we can obtain the desired results (A_3) and (A_4).

We now are in a position to state and prove the following two main results.

Theorem 2.1. *Assume that (H) holds and there exist two distinct positive constants R_1 and R_2 such that*

$$f(t, u) \le \phi\Big(\frac{R_1}{\theta + 1}\Big) \quad \text{on } [0, 1] \times [0, R_1] \tag{1}$$

and

$$f(t, u) \ge \phi(32R_2) \quad \text{on } \Big[\frac{1}{4}, \frac{3}{4}\Big] \times \Big[\frac{1}{4}R_2, R_2\Big]. \tag{2}$$

Then (BVP) has at least one positive solution $u(t)$ such that $||u||$ between R_1 and R_2.

Proof. Without loss of generality, we may assume that $R_1 < R_2$. First, we define a set K as follows:

$$K := \{u \in C[0, 1] |\ u(t) \text{ is a nonnegative concave function}\}.$$

Clearly, K is a cone. It follows from Lemma 1 of Wang [9] that

$$u(t) \geq \frac{1}{4}||u|| \quad \text{on } [\frac{1}{4}, \frac{3}{4}], \text{ for all } u \in K. \tag{3}$$

Suppose that $u(t)$ is a solution of (BVP), then it follows from the boundary condition (BC) that $u'(0) \geq 0$ and $u'(1) \leq 0$. Therefore, for each given solution $u(t)$ of (BVP), there exists a $\sigma \in [0,1]$ such that $u'(\sigma) = 0$.

Next, we define an operator $T : K \to C[0,1]$ by

$$Tu(t) := \begin{cases} B_0 \circ \Psi\Big(\int_0^\sigma f\big(s, u(s)\big)ds\Big) + \int_0^t \Psi\Big(\int_s^\sigma f\big(r, u(r)\big)dr\Big)ds, & 0 \leq t \leq \sigma, \\ B_1 \circ \Psi\Big(\int_\sigma^1 f\big(s, u(s)\big)ds\Big) + \int_t^1 \Psi\Big(\int_\sigma^s f\big(r, u(r)\big)dr\Big)ds, & \sigma \leq t \leq 1. \end{cases}$$

It is clear that T is well-defined (see Wang [9]). By the definition of T, we see that

$$(Tu(t))' = \begin{cases} \Psi\Big(\int_t^\sigma f\big(r, u(r)\big)dr\Big) \geq 0, & 0 \leq t \leq \sigma, \\ -\Psi\Big(\int_\sigma^t f\big(r, u(r)\big)dr\Big) \leq 0, & \sigma \leq t \leq 1 \end{cases}$$

is continuous, decreasing on $[0,1]$ and satisfies $(Tu(t))'|_{t=\sigma} = 0$. Thus, $Tu \in K$ for each $u \in K$ and $Tu(\sigma)$ is the maximum of $Tu(t)$ on $[0,1]$. This shows that $TK \subset K$. Furthermore, it is easy to check that $T : K \to K$ is completely continuous.

Finally, we show that Tu has at least one fixed point on K by applying Theorem 2.A. Without loss of generality, we may assume that

$$0 \leq B_0(v) \leq \theta v \qquad \text{for all } v \geq 0.$$

Let

$$\Omega_1 := \{u \in K |\ ||u|| < R_1\},$$
$$\Omega_2 := \{u \in K |\ ||u|| < R_2\}.$$

Thus, for $u \in K$ with $||u|| = R_1$, it follows from (1) that

$$\begin{aligned} ||Tu|| = Tu(\sigma) &\leq B_0 \circ \Psi\Big(\int_0^1 f\big(s, u(s)\big)ds\Big) + \Psi\Big(\int_0^1 f\big(r, u(r)\big)dr\Big)ds \\ &\leq B_0\big(\frac{R_1}{\theta+1}\big) + \frac{R_1}{\theta+1} \\ &\leq \frac{\theta R_1}{\theta+1} + \frac{R_1}{\theta+1} \\ &= R_1 = ||u||. \end{aligned}$$

Hence,

$$||Tu|| \leq ||u|| \qquad \text{for } u \in K \cap \partial\Omega_1. \tag{4}$$

On the other hand, if $\frac{1}{2} \le \sigma$, then it follows from (A_3) of Lemma 2.1 and (2) that, for $u \in K$ with $||u|| = R_2$,

$$\begin{aligned} Tu(\frac{1}{2}) &\ge \int_0^{\frac{1}{2}} \Psi\Big(\int_s^{\frac{1}{2}} f\big(r,u(r)\big)dr\Big)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{1}{2}} \Psi\Big(\int_s^{\frac{1}{2}} f\big(r,u(r)\big)dr\Big)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{1}{2}} \Psi\big(\phi(32R_2)(\frac{1}{2}-s)\big)ds \\ &\ge \int_{\frac{1}{4}}^{\frac{1}{2}} (\frac{1}{2}-s)\Psi\big(\phi(32R_2)\big)ds \\ &= R_2 = ||u||. \end{aligned}$$

Similarly, if $\sigma \le \frac{1}{2}$, then, for $u \in K$ with $||u|| = R_2$,

$$\begin{aligned} Tu(\frac{1}{2}) &\ge \int_{\frac{1}{2}}^{1} \Psi\Big(\int_{\frac{1}{2}}^{s} f\big(r,u(r)\big)dr\Big)ds \\ &\ge \int_{\frac{1}{2}}^{\frac{3}{4}} \Psi\big(\phi(32R_2)(s-\frac{1}{2})\big)ds \\ &\ge \int_{\frac{1}{2}}^{\frac{3}{4}} (s-\frac{1}{2})\Psi\big(\phi(32R_2)\big)ds \\ &= R_2 = ||u||. \end{aligned}$$

Thus,

$$(5) \qquad ||Tu|| \ge ||u|| \qquad \text{for } u \in K \cap \partial\Omega_2.$$

Therefore, it follows from (4), (5) and Theorem 2.A that T has a fixed point $u \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. This shows that the fixed point u is a positive solution of (BVP).

Lemma 2.C. *Assume that*

(H^*) *ϕ is sub-multiplicative on $[0,\infty)$, i.e., ϕ satisfies*

$$\phi(xy) \le \phi(x)\phi(y) \qquad \textit{for all } x,y \ge 0.$$

Then,

(R_5) *Ψ is supermultiplicative on $[0,\infty)$, i.e., Ψ satisfies*

$$\Psi(xy) \ge \Psi(x)\Psi(y) \qquad \textit{for all } x,y \ge 0.$$

Proof. Suppose to the contrary that there exist $x_0, y_0 \ge 0$ such that

$$\Psi(x_0y_0) < \Psi(x_0)\Psi(y_0).$$

Since Ψ is the inverse function of ϕ and ϕ is strictly increasing,

$$x_0y_0 < \phi\big(\Psi(x_0)\Psi(y_0)\big) \le \phi\big(\Psi(x_0)\big)\phi\big(\Psi(y_0)\big) = x_0y_0,$$

which gives a contradiction.

Theorem 2.2.

Assume that (H^*) holds and there exist two distinct positive constants R_1 and R_2 such that conditions (1) and (2) hold. Then (BVP) has at least one positive solution $u(t)$ such that $||u||$ between R_1 and R_2.

Proof. The proof is quite similar to that of Theorem 2.1, therefore, we omit the details.

Now, for the convenience, we introduce the following notations. Let

$$\max f_0 := \lim_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{\phi(u)},$$

$$\min f_0 := \lim_{u\to 0^+} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{\phi(u)},$$

$$\max f_\infty := \lim_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{\phi(u)},$$

$$\min f_\infty := \lim_{u\to \infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{\phi(u)}.$$

From the above definitions, we have the following

Remark 2.3. Let $\phi(u) := |u|^{p-2}u$ for $u \in \mathbb{R}$, where $p > 1$ is a constant.

(i) Suppose that $\max f_0 = C_1 \in \big[0, \phi(\frac{1}{\theta+1})\big)$. Taking $\varepsilon = \phi(\frac{1}{\theta+1}) - C_1 > 0$, there exists a $R_1 > 0$ (R_1 can be chosen small arbitrarily) such that

$$\max_{t\in[0,1]} \frac{f(t,u)}{\phi(u)} \le \varepsilon + C_1 = \phi(\frac{1}{\theta+1}) \qquad \text{for } u \in [0, R_1].$$

Hence,

$$f(t,u) \le \phi(\frac{1}{\theta+1})\phi(u) \le \phi(\frac{1}{\theta+1})\phi(R_1) = \phi(\frac{R_1}{\theta+1}) \quad \text{on } [0,1]\times[0,R_1],$$

which satisfies the hypothesis (1) of Theorems 2.1 and 2.2.

(ii) Suppose that $\min f_\infty = C_2 \in \big(\phi(128), \infty\big]$. Taking $\varepsilon = C_2 - \phi(128) > 0$, there exists $R_2 > 0$ (R_2 can be chosen large arbitrarily) such that

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{\phi(u)} \ge -\varepsilon + C_2 = \phi(128) \qquad \text{for } u \in [\frac{1}{4}R_2, \infty).$$

Hence,

$$\begin{aligned} f(t,u) &\ge \phi(128)\phi(u) \ge \phi(128)\phi(\frac{1}{4}R_2) \\ &= \phi(32R_2) \quad \text{on } [\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4}R_2, R_2] \subset [0,1]\times[\frac{1}{4}R_2,\infty), \end{aligned}$$

which satisfies the hypothesis (2) of Theorems 2.1 and 2.2.

(iii) Suppose that $\min f_0 = C_3 \in \big(\phi(128), \infty\big]$. Taking $\varepsilon = C_3 - \phi(128) > 0$, there exists $R_2 > 0$ (R_2 can be chosen large arbitrarily) such that

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u)}{\phi(u)} \ge -\varepsilon + C_3 = \phi(128) \qquad \text{for } u \in (0, R_2].$$

Hence,

$$\begin{aligned} f(t,u) &\geq \phi(128)\phi(u) \geq \phi(128)\phi\big(\frac{1}{4}R_2\big) \\ &= \phi(32R_2) \quad \text{on } [\frac{1}{4},\frac{3}{4}] \times [\frac{1}{4}R_2, R_2] \subset [0,1]\times[0,R_2], \end{aligned}$$

which satisfies the hypothesis (2) of Theorems 2.1 and 2.2.

(iv) Suppose that $\max f_\infty = C_4 \in \big[0, \phi(\frac{1}{\theta+1})\big)$. Taking $\varepsilon = \phi(\frac{1}{\theta+1}) - C_4 > 0$, there exists a $\delta > 0$ (δ can be chosen large arbitrarily) such that

$$\max_{t\in[0,1]} \frac{f(t,u)}{\phi(u)} \leq \varepsilon + C_4 = \phi\big(\frac{1}{\theta+1}\big) \qquad \text{for } u \in [\delta, \infty). \tag{6}$$

Thus, we have the following two cases:

Case (a). Assume that $\max\limits_{t\in[0,1]} f(t,u)$ is bounded, say

$$f(t,u) \leq M \qquad \text{on } [0,1]\times[0,\infty),$$

where $M \geq \phi(\frac{1}{\theta+1})$ is a constant. Taking $R_1 = \Big(\frac{M}{\phi(\frac{1}{\theta+1})}\Big)^{\frac{1}{p-1}}$ (since M can be chosen large arbitrarily, R_1 can be chosen large arbitrarily, too). Then,

$$\begin{aligned} f(t,u) \leq M &= R_1^{p-1}\phi\big(\frac{1}{\theta+1}\big) \\ &= \phi\big(\frac{R_1}{\theta+1}\big). \end{aligned}$$

Case (b). Assume that $\max\limits_{t\in[0,1]} f(t,u)$ is unbounded. Thus, there exists a $R_1 > \delta$ (R_1 can be chosen large arbitrarily) and $t_0 \in [0,1]$ such that

$$f(t,u) \leq f(t_0, R_1) \qquad \text{on } [0,1]\times[0,R_1].$$

It follows from $R_1 > \delta$ and (6) that

$$f(t,u) \leq f(t_0,R_1) \leq R_1^{p-1}\phi\big(\frac{1}{\theta+1}\big) = \phi\big(\frac{R_1}{\theta+1}\big) \quad \text{on } [0,1]\times[0,R_1].$$

By cases (a) and (b), the hypothesis (1) of Theorems 2.1 and 2.2 is satisfied.

Now, we consider the following $p-$Laplacian boundary value problem

$$(BVP^*) \qquad \begin{cases} (E^*) & \big(|u'|^{p-2}u'\big)' + f(t,u) = 0, \qquad 0 < t < 1,\ p > 1, \\ (BC) & \begin{cases} u(0) - B_0\big(u'(0)\big) = 0, \\ u(1) + B_1\big(u'(1)\big) = 0. \end{cases} \end{cases}$$

It follows from Remark 2.3 that we obtain the following corollaries hold, immediately.

Corollary 2.4. *Let $A := (\theta+1)^{1-p}$ and $B := (128)^{p-1}$. Then, (BVP^*) has at least one positive solution if*
(a) $max f_0 = C_1 \in [0, A)$ *and* $min f_\infty = C_2 \in (B, \infty]$, *or*
(b) $min f_0 = C_3 \in (B, \infty]$ *and* $max f_\infty = C_4 \in [0, A)$.

Corollary 2.5. *Let $A := (\theta+1)^{1-p}$ and $B := (128)^{p-1}$. Then, (BVP^*) has at least two positive solutions u_1 and u_2 such that*

$$0 < ||u_1|| < R < ||u_2||,$$

if the following hypotheses hold:
(a) $min f_\infty = C_2,\ min f_0 = C_3 \in (B, \infty]$,
(b) *there exists a $R > 0$ such that*

$$f(t,u) \le \left(\frac{R}{\theta+1}\right)^{p-1} \quad on\ [0,1] \times [0, R]$$

Corollary 2.6. *Let $A := (\theta+1)^{1-p}$ and $B := (128)^{p-1}$. Then, (BVP^*) has at least two positive solutions u_1 and u_2 such that*

$$0 < ||u_1|| < R < ||u_2||,$$

if the following hypotheses hold:
(a) $max f_0 = C_1,\ max f_\infty = C_4 \in [0, A)$,
(b) *there exists a $R > 0$ such that*

$$f(t,u) \ge (32R)^{p-1} \quad on\ [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}R, R].$$

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