CONGRUENCES ON HYPER BCK-ALGEBRAS

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Abstract. In this paper we introduce the concept of congruences on hyper BCK-algebras and investigate the relationship between hyper BCK-ideals and those congruences. We show that

1. If θ is a regular congruence on a hyper BCK-algebra H then so H/θ is;
2. There is a one-to-one correspondence between the set of all closed hyper BCK-ideals with condition (I*) and the set of all regular conguences with condition (C*);
3. For any regular congruence φ on H, φ satisfies (C*) if and only if H/φ is a BCK-algebra.

1 Introduction Since the theory of BCK-algebras was produced by K. Iséki in 1966, a number of papers are published. BCK-algebras are algebraic systems with an operation * which is an abstraction of one representing "set difference" in set theory or "implication" in logics. Some of the variants of BCK-algebras, especially BCI-algebras, are also considered and many results are obtained. Recently, the hyper-structure theory (or mutialgebras) of BCK-algebras is proposed by Jun, Zahedi, Xin, and Borzoei in [4] and some fundamental results are proved. They also define hyper BCK-ideals and investigated their properties. But they have few results concerning with those obtained in the theory of BCK-algebras and BCI-algebras. Because it is not defined a concept of congruence on a hyper BCK-algebra. In this paper we define congruences on hyper BCK-algebras with taking into consideration of kernels of homomorphisms. As is well-known, the quotient structures of BCK(BCI)-algebras by congruences are not BCK(BCI)-algebras. It is similar in the case of hyper BCK-algebras. Now we have natural questions:

(Q1) What congruences do quotient structures make into hyper BCK-algebras?

In the theory BCI-algebras we know that the set of all closed ideals and the set of all regular congruences are isomorphic as lattices ([2]).

(Q2) Are there such correpondences in the theory of hyper BCK-algebras?

We consider these questions and show that

1. If φ is a regular congruence (defined below) then H/φ is a hyper BCK-algebra;
2. There is a one-to-one correspondence between the set $Con_R^*(H)$ of all regular congruences with (C*) and the set $\mathcal{I}_c^*(H)$ of all closed hyper BCK-ideals with (I*);
3. $\varphi \in Con_R^*(H)$ if and only if H/φ is a BCK-algebra.

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2 hyper BCK-algebra and hyper congruence First of all, we define hyper BCK-algebras and hyper congruences on them. By a *hyper BCK-algebra* we mean a hyper algebra $(H;\circ,0)$, where 0 is a constant element in H, and it satisfies the following axioms: For all $x,y,z\in H$,

(HK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(HK2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(HK3) $x\circ H\ll\{x\}$,
(HK4) $x\ll y$ and $y\ll x$ imply $x=y$,

,where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A$ $\exists b\in B$; $a\ll b$. By $\mathcal{H}$ we mean the class of all hyper BCK-algebras.

Example 1 (cf. [3])

(1) Let $(H;*,0)$ be a BCK-algebra and define a hyper operation "$\circ$" on H by $x\circ y=\{x*y\}$ for all $x,y\in H$. Clearly it is a hyper BCK-algebra. Hence the class of all BCK-algebras is a subclass of $\mathcal{H}$

(2) Define a hyper operation "$\circ$" on $H:=[0,\infty)$ by

$$x\circ y:=\begin{cases}[0,x] & \text{if } x\le y\\ (0,y] & \text{if } x>y\neq 0\\ \{x\} & \text{if } y=0\end{cases}$$

for all $x,y\in H$.

(3) Let $H=\{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We denote a hyper BCK-algebra $(H;\circ,0)$ simply by H if no confusion arises. An equivalence relation φ is called a congruence on a hyper BCK-algebra H if it satisfies the condition, for any $x,x',y,y'\in H$,

$$(x,y),(x',y')\in\varphi\Longrightarrow(x\circ x',y\circ y')\in\varphi$$

, where $(x\circ x',y\circ y')\in\varphi$ is defined by $(a,b)\in\varphi$ for some $a\in x\circ x'$ and $b\in y\circ y'$. By $Con(H)$ we mean the set of all congruences on H.

We define a quotient structure H/φ for some congruence φ on H as follows:

$$H/\varphi=\{[x]\mid x\in H\},$$

where

$$[x]=\{y\in H\mid(x,y)\in\varphi\}.$$

For every $[x]$ and $[y]$ we define $[x]\circ[y]=\bigcup\{[a]\mid a\in x\circ y\}$ and

$$[x]\ll[y]\Longleftrightarrow[0]\in[x]\circ[y]$$

Since the class $\mathcal{H}$ of all hyper BCK-algebras does not form a variety, we see that H/φ is not always a hyper BCK-algebra for $\varphi\in Con(H)$. Well,

> Is there a congruence φ for which H/φ is in $\mathcal{H}$? What kind of congruences φ do H/φ make hyper BCK-algebras?

By a *regular* congruence we mean a congruence φ such that

$$(x \circ y, \{0\}), (y \circ x, \{0\}) \in \varphi \text{ implies } (x, y) \in \varphi$$

and denote by $Con_R(H)$ the set of all regular congruences on H. It is easy to see that our definition of regular congruences is a generalization of that of BCI-algebras (cf.[2]).

Proposition 1. *Let φ a congruence on a hyper BCK-algebra H. Then,*
$\varphi \in Con_R(H) \iff H/\varphi \in \mathcal{H}$

Proof. Suppose that $\varphi \in Con_R(H)$ is regular. In order to show $H/\varphi \in \mathcal{H}$, we only show that it satisfies (HK1) and (HK4).

• (HK1) : It is sufficient to show tht $[(x\circ z)\circ(y\circ z)] \ll [x\circ y]$. For every $[u] \in [(x\circ z)\circ(y\circ z)]$, there is an element $a \in (x \circ z) \circ (y \circ z)$ such that $[u] = [a]$. Since $H \in \mathcal{H}$ and it satisfies the condition (HK1), for the element a there exists $b \in x \circ y$ such that $a \ll b$. This implies $0 \in a \circ b$ and $[0] \in [a] \circ [b] = [u] \circ [b]$, that is, $[u] \ll [b]$. It follows from $[b] \in [x \circ y]$ that $[(x \circ z) \circ (y \circ z)] \ll [x \circ y]$.

• (HK4) : Suppose that $[x] \ll [y]$ and $[y] \ll [x]$. Since $[0] \in [x \circ y]$ and $[0] \in [y \circ x]$, there are elements $a \in x \circ y$ and $b \in y \circ x$ such that $[0] = [a]$ and $[0] = [b]$. Thus we have $(a, 0), (b, 0) \in \varphi$ and $(x \circ y, \{0\}), (y \circ x, \{0\}) \in \varphi$. Since φ is regular, it follows $(x, y) \in \varphi$, that is, $[x] = [y]$.

Conversely, let $H/\varphi \in \mathcal{H}$. Assume that $(x\circ y, \{0\}), (y\circ x, \{0\}) \in \varphi$. Since $[x]\circ[y] = \{[0]\}$ and $[y]\circ[x] = \{[0]\}$, we have $[x] \ll [y]$ and $[y] \ll [x]$ by definition. This implies that $[x] = [y]$ and $(x, y) \in \varphi$. □

Example 2

Let $f : H \to H'$ be a homomorphism, that is,

(1) $f(0) = 0$ and

(2) $f(x \circ y) = f(x) \circ f(y)$,

where $f(x \circ y) = \bigcup\{f(a) \mid a \in x \circ y\}$.

We define Θ to be the set $\{(x, y) \mid f(x) = f(y)\}$. We are able to show that Θ is a regular congruence on H. Since it is trivial that Θ is an equivalence relation, we only have to prove

1. $(x, y), (x', y') \in \Theta \Longrightarrow (x \circ x', y \circ y') \in \Theta$;
2. $(x \circ y, \{0\}), (y \circ x, \{0\}) \in \Theta \Longrightarrow (x, y) \in \Theta$.

1. Suppose that $(x, y), (x', y') \in \Theta$. Since $x \circ x' \neq \emptyset$, there is an element $a \in x \circ x'$. By $f(a) \in f(x \circ x') = f(x) \circ f(x') = f(y) \circ f(y') = f(y \circ y')$, there exists an element $b \in y \circ y'$ such that $f(a) = f(b)$. It follows from $a \in x \circ x'$ and $b \in y \circ y'$ that $(a, b) \in \Theta$ and $(x \circ x', y \circ y') \in \Theta$.

2. Suppose that $(x\circ y, \{0\}), (y\circ x, \{0\}) \in \Theta$. From definition there are elements $a \in x\circ y$ and $b \in y \circ x$ such that $(a, 0), (b, 0) \in \Theta$. Since $0 = f(a) \in f(x \circ y) = f(x) \circ f(y)$, we have $f(x) \ll f(y)$. Similarly we get $f(y) \ll f(x)$. This implies that $f(x) = f(y)$ and hence $(x, y) \in \Theta$.

Therefore Θ is the regular congruence on H.

For the regular congruence Θ, we can show the "Homomorphism Theorem".

Theorem 1. *Let $f : H \to H'$ be a homomorphism and $\Theta = \{(x, y) \mid f(x) = f(y)\}$. Then we have $H/\Theta \cong f(H)$.*

Proof. In the example above, we show that Θ is the regular congruence. So we can conclude that H/Θ is a hyper BCK-algebra. Now we define a map ν from H/Θ to $f(H)$ by $\nu([x]) = f(x)$. It is easy to show that the map ν gives an isomorphism from H/Θ onto $f(H)$. □

3 Closed hyper BCK-ideal Let I be a non-empty subset of a hyper BCK-algebra $(H;\circ,0)$. Then I is said to be a *hyper BCK-ideal* of H if

(HI1) $0 \in I$,

(HI2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

A hyper BCK-ideal I is said to be closed if $x \ll y$ and $y \in I$ imply $x \in I$. By $\mathcal{I}(H)$ and $\mathcal{I}_c(H)$ we mean the set of all hyper BCK-ideals of H and the set of all closed hyper BCK-ideals of H, respectively.

Example 3 (cf.[3])

(1) Let $(H;\circ,0)$ be the hyper BCK-algebra in Example 1(1). Then every ideal I of the BCK-algebra $(H;*,0)$ is a hyper BCK-ideal of H.

(2) Let $(H;\circ,0)$ be the hyper BCK-algebra in Example 1(2). Then $(H;\circ,0)$ have no proper hyper BCK-ideals, i.e., there are only two hyper BCK-ideals $\{0\}$ and H itself. In fact, if I is a hyper BCK-ideal of H and $I \neq \{0\}$, then there is $a \in I$ such that $a \neq 0$. For any $b \in [0,a]$, we have $b \circ a = [0,b] \ll \{a\}$ and so $b \circ a \ll I$. Since $a \in I$, it follows from (HI2) that $b \in I$ so that $[0,a] \subseteq I$. Moreover for every $a < c$, we get $c \circ a = (0,a] \ll I$ and so $c \in I$. Therefore $(a,\infty) \subseteq I$, i.e., $I = H$.

(3) Let $(H;\circ,0)$ be the hyper BCK-algebra in Example 1(3). Then $I_1 = \{0,1\}$ is a closed hyper BCK-ideal of H, but $I_2 = \{0,2\}$ is not a hyper BCK-ideal of H because $1 \circ 2 = \{0,1\} \ll I_2$ and $2 \in I_2$, but $1 \notin I_2$.

As is well-known, if $f : X \to Y$ is a homomorphism for BCK-algebras X, Y then *ker* $f = \{x \in X \mid f(x) = 0\}$ is a BCK-ideal. For hyper BCK-algebras H, H', we have a similar result.

Proposition 2. *Let $H, H' \in \mathcal{H}$ and $f : H \to H'$ be a homomorphism. Then ker $f = \{x \in X \mid f(x) = 0\}$ is a closed hyper BCK-ideal.*

Proof. It sufficies to prove that

1. $x \ll y$ and $y \in ker\, f \Longrightarrow x \in ker\, f$;

2. $x \circ y \ll ker\, f$, $y \in ker\, f \Longrightarrow x \in ker\, f$.

1. Suppose that $x \ll y$ and $y \in ker\, f$. Since $0 \in x \circ y$, we have $f(0) = 0 \in f(x) \circ f(y) = f(x) \circ 0$ and hence $f(x) \ll 0$. This yields to $f(x) = 0$, that is, $x \in ker\, f$.

2. We assume that $x \circ y \ll ker\, f$ and $y \in ker\, f$. By definition, for any element $a \in x \circ y$ there exists $t \in ker\, f$ such that $a \ll t$. Hence $a \in ker\, f$. Since $0 = f(a) \in f(x) \circ f(y) = f(x) \circ 0$, it follows that $f(x) \ll 0$.This means that $f(x) = 0$ and $x \in ker\, f$. $\square$

Since $ker\, f$ can be considered as a subset $[0]_\Theta$ which is an equivalence class of 0 in H/Θ, we may expect that $[0]_\varphi$ is the closed hyper BCK-ideal for every $\varphi \in Con_R(H)$.

Proposition 3. *For every $\varphi \in Con_R(H)$, $[0]_\varphi$ is the closed hyper BCK-ideal, that is, $[0]_\varphi \in \mathcal{I}_c(H)$.*

Proof. For brevity, we set $J = [0]_\varphi$ where $\varphi \in Con_R(H)$. As above, we shall show that

1. $x \ll y$ and $y \in J \Longrightarrow x \in J$;

2. $x \circ y \ll J$ and $y \in J \Longrightarrow x \in J$.

Firstly, suppose that $x \ll y$ and $y \in J$. It follows from $0 \in x \circ y$ that $(x \circ y, \{0\}) \in \varphi$. On the other hand, since φ is a congruence, we have $(y \circ x, 0 \circ x) \in \varphi$ by $(y, 0) \in \varphi$. From $0 \circ x = \{0\}$, $(y \circ x, \{0\}) \in \varphi$. Since φ is regular we get $(x, y) \in \varphi$ and hence $(x, 0) \in \varphi$. This implies $x \in J$.

Secondly, we assume that $x \circ y \ll J$ and $y \in J$. It follows from $y \in J$ that $(y, 0) \in \varphi$ and $(x \circ y, x \circ 0) \in \varphi$. There are elements $a \in x \circ y$ and $b \in x \circ 0$ such that $(a, b) \in \varphi$ by definition. For this element a, since $x \circ y \ll J$, there exists $t \in J$ such that $a \ll t$. We have $a \in J$ from the above. This implies that $(a, 0) \in \varphi$ and thus $(b, 0) \in \varphi$. Therefore $(x \circ 0, \{0\}) \in \varphi$ and clearly $(0 \circ x, \{0\}) \in \varphi$. We see from the regularity of φ that $(x, 0) \in \varphi$ and $x \in J$. □

The proposition means that the map ξ from $Con_R(H)$ to $\mathcal{I}_c(H)$ defined by $\xi(\varphi) = [0]_\varphi$ is well-defined. Now it is natural to ask whether the map is injective or surjective. It is easy to show that ξ is not injective. In the example 1(3) above, there are only three regular congruences $\omega = \{(0,0), (1,1), (2,2)\}$, $\iota = H \times H$, and $\varphi = \{(0,0), (1,1), (2,2), (1,2), (2,1)\}$. For these congruences, it is clear $[0]_\omega = [0]_\varphi = \{0\}$ but $\omega \neq \varphi$. Hence ξ is not injective.

This means that there is no one-to-one correspondence between $Con_R(H)$ and $\mathcal{I}_c(H)$. If there were a one-to-one correspondence η from $Con_R(H)$ onto $\mathcal{I}_c(H)$, then the map ξ had to be injective. Because, that $\xi(\varphi) = \xi(\psi)$ for $\varphi, \psi \in Con_R(H)$ implies $\varphi = (\eta \circ \xi)(\varphi) = \eta(\xi(\varphi)) = \eta(\xi(\psi)) = \psi$. But this contradicts to the example above.

Therefore we have no ways to construct regular congruences from closed hyper BCK-ideals such that $Con_R(H)$ and $\mathcal{I}_c(H)$ are equipotent.

Now we have a question.

> Can we construct regular congruences from closed hyper closed BCK-ideals with some conditions? What are those conditions if we could?

We shall think about the question. Let us consider the following condition (C*) about a regular congruence φ.

(C*) For every $a \in x \circ x$, we have $(a, 0) \in \varphi$ for any $x \in H$

We denote the set of all regular congruences with (C*) by $Con^*_R(H)$. Correponding to (C*), we also think of a following condition (I*) about a closed hyper BCK-ideals $I \in \mathcal{I}_c(H)$.

(I*) $x \circ x \ll I$ for every $x \in H$

Let $\mathcal{I}^*_c(H)$ be the set of all closed hyper BCK-ideals with (I*). We are able to prove the existence of one-to-one correpondence betwee $Con^*_R(H)$ and $\mathcal{I}^*_c(H)$. As preliminaries, we have some results.

Proposition 4. *Let $J \in \mathcal{I}(H)$. Then we have*
J is closed $\Longleftrightarrow$ "$A \ll J$ implies $A \subseteq J$" for every non-empty subset A of H.

Proof. Suppose that J is closed and $A \ll J$. For every element $x \in A$ there exists $y \in J$ such that $x \ll y$. It follows from closedness of J that $x \in J$. This means $A \subseteq J$.

Conversely if $x \ll y$ and $y \in J$, since $\{x\} \ll J$, then we have $\{x\} \subseteq J$ by assumption. This yields that $x \in J$ and J is closed. □

Proposition 5. *If $J \in \mathcal{I}^*_c(H)$, then we have, for every non-empty subset A, B, C of H,*
$A \circ B \ll C$, $C \subseteq J \Longrightarrow A \circ B \subseteq J$.

Proof. We suppose that $A \circ B \ll C$ and $C \subseteq J$ for some closed hyper BCK-ideal J. For any element $x \in A \circ B$ there exists $y \in C \subseteq J$ such that $x \ll y$. Since J is closed, we have $x \in J$ and hence $A \circ B \subseteq J$. □

We denote $[0]_\varphi = \{x \in H \mid (x, 0) \in \varphi\}$ by simply I_φ for any $\varphi \in Con^*_R(H)$.

Proposition 6. $\varphi \in Con^*_R(H) \Longrightarrow I_\varphi \in \mathcal{I}^*_c(H)$

Proof. We only show that I_φ satisfies the condition $x \circ x \ll I_\varphi$. For every element $a \in x \circ x$, we have $(a, 0) \in \varphi$ from the assumption. This implies that $a \in [0]_\varphi = I_\varphi$ and $x \circ x \subseteq I_\varphi$. Thus $x \circ x \ll I_\varphi$. □

Conversely we can construct congruences from closed hyper BCK-ideals with (I*). Let I be an closed hyper BCK-ideal with (I*), that is, $I \in \mathcal{I}^*_c(H)$. We define φ_I as $\{(x, y) \mid x \circ y, y \circ x \ll I\}$.

Proposition 7. $I \in \mathcal{I}^*_c(H) \Longrightarrow \varphi_I \in Con^*_R(H)$

Proof. First of all we shall show that φ_I is an equivalence relation. It is easy to prove that φ_I is reflexive and symmetric. For transitivity, suppose that $(x, y), (y, z) \in \varphi$. We have $x \circ y \subseteq I$ from $x \circ y \ll I$ and Proposition 4. Since $(x \circ z) \circ (y \circ z) \ll x \circ y$ and $x \circ y \subseteq I$, we have $(x \circ z) \circ (y \circ z) \subseteq I$ from Proposition 5. We also have $x \circ z \subseteq I$ from $y \circ z \subseteq I$ and Proposition 5. Thus we see $x \circ z \ll I$. Similarly $z \circ x \ll I$. This means that $(x, z) \in \varphi$ and hence that φ_I is transitive.

Next we assume that $(x, y), (x', y') \in \varphi_I$. There are elements $a \in x \circ x'$ and $b \in y \circ y'$. So we have $a \circ b \subseteq (x \circ x') \circ (y \circ y')$. Now we obtain that

$$\begin{aligned} ((x \circ x') \circ (y \circ y')) \circ (y' \circ x') &= ((x \circ (y \circ y')) \circ x') \circ (y' \circ x') \\ &\ll (x \circ (y \circ y')) \circ y' \\ &= (x \circ y') \circ (y \circ y') \\ &\ll x \circ y \subseteq I. \end{aligned}$$

It follows from Proposition 5 that $(x \circ y') \circ (y \circ y') \subseteq I$. On the other hand, since $((x \circ x') \circ (y \circ y')) \circ (y' \circ x') \ll (x \circ y') \circ (y \circ y')$ and $(x \circ y') \circ (y \circ y') \subseteq I$, we also have $((x \circ x') \circ (y \circ y')) \circ (y' \circ x') \subseteq I$. By assumption, $y' \circ x' \subseteq I$. This implies $(x \circ x') \circ (y \circ y') \subseteq I$. Thus we have $a \circ b \subseteq I$ and hence $a \circ b \ll I$. Similarly $b \circ a \ll I$ and $(a, b) \in \varphi_I$. This means $(x \circ x', y \circ y') \in \varphi_I$. That is, φ_I is the congruence.

Lastly we prove that φ_I is regular. If $(x \circ y, \{0\}), (y \circ x, \{0\}) \in \varphi_I$, then we have $(a, 0), (b, 0) \in \varphi_I$ for some $a \in x \circ y$ and $b \in y \circ x$ by definition. For any $u \in x \circ y$, it follows from $a \circ u \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x \subseteq I$ that $a \circ u \subseteq I$ and thus $a \circ u \ll I$. Similarly, we have $u \circ a \ll I$ and thus $(a, u) \in \varphi_I$. This yields to $(u, 0) \in \varphi_I$. Since $u \in u \circ 0 \ll I$, we get $u \in I$. This means $x \circ y \subseteq I$ and $x \circ y \ll I$. Similarly $y \circ x \ll I$ and therefore $(x, y) \in \varphi_I$. □

From the above, we get that

- $I \in \mathcal{I}^*_c(H) \Longrightarrow \varphi_I \in Con^*_R(H) \Longrightarrow I_{\varphi_I} \in \mathcal{I}^*_c(H)$;
- $\varphi \in Con^*_R(H) \Longrightarrow I_\varphi \in \mathcal{I}^*_c(H) \Longrightarrow \varphi_{I_\varphi} \in Con^*_R(H)$.

We ask whether I (or φ) is identical with I_{φ_I} (or φ_{I_φ}, respectively). Concerning to the question, we have the following.

Theorem 2. *(1)* $I \in \mathcal{I}_c^*(H) \Longrightarrow I = I_{\varphi_I}$

(2) $\varphi \in Con_R^*(H) \Longrightarrow \varphi = \varphi_{I_\varphi}$

Proof. (1) Let I be in $\mathcal{I}_c^*(H)$ and $x \in I$. Since $x \circ 0 \ll \{x\} \subseteq I$, we have $x \circ 0 \subseteq I$ by assumption. On the other hand, $0 \circ x = \{0\}$. That is, $x \circ 0 \ll I$ and $0 \circ x \ll I$. This means that $(x, 0) \in \varphi_I$. It follows that $x \in I_{\varphi_I}$ and hence $I \subseteq I_{\varphi_I}$.

Conversely, if $x \in I_{\varphi_I}$, then we have $(x, 0) \in \varphi_I$ by definition. This implies that $x \circ 0 \ll I$ and $0 \circ x \ll I$. Since $x \in x \circ 0 \subseteq I$, we get $x \in I$ and $I_{\varphi_I} \subseteq I$. Thus we have $I = I_{\varphi_I}$ if $I \in \mathcal{I}_c^*(H)$.

(2) It can be proved similarly. □

From the above we can conclude that

Theorem 3. *There is a one-to-one correpondence between* $Con_R^*(H)$ *and* $\mathcal{I}_c^*(H)$ *for each hyper BCK-algebra* H.

Proof. The map $\varphi \mapsto \varphi_I$ gives an desired result. □

Since each $\varphi \in Con_R^*(H)$ is of course a regular congruence, we always have a quotient hyper BCK-algebra H/φ. For the quotient structure we have a stronger result.

Theorem 4. *Let* φ *be a regular congrunce. Then*

$\varphi \in Con_R^*(H) \Longleftrightarrow H/\varphi$ *: BCK-algebra*

Proof. At first we suppose $\varphi \in Con_R^*(H)$. For every element $[x] \in H/\varphi$, it is sufficient to show that $[x] \circ [x] = \{[0]\}$ from Cor.3.13 in [3]. For any element $a \in x \circ x$, since $(a, 0) \in \varphi$ by (C*), we have $[a] = [0]$. Then $[x] \circ [x] = \bigcup\{[a] \mid a \in x \circ x\} = \bigcup\{[0] \mid a \in x \circ x\} = \{[0]\}$. Thus H/φ is the BCK-algebra.

The converse holds clearly. □

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