# ON CERTAIN GENERALIZATIONS OF COUNTABLY COMPACT SPACES AND LINDELÖF SPACES 

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#### Abstract

The purpose of this paper is to correct some results in Pareek [P] and Kočinac $[\mathrm{K}]$ concerning certain generalizations of countably compact spaces and Lindelöf spaces and also to present several additional results and pose a number of interesting open questions on this topic.


## 1 Introduction and Definitions

A very interesting and useful characterization of countable compactness in Hausdorff spaces is the following: A $T_{2}$ space $X$ is countably compact iff for every open cover $\mathcal{G}$ of $X$ there exists a finite subset $F$ of $X$ such that

$$
X=\operatorname{st}(F, \mathcal{G})=\bigcup\{G \in \mathcal{G}: G \cap F \neq \emptyset\}
$$

The latter covering property has been called starcompactness by Fleischmann [F]. In [F] Fleischmann proved that every countably compact space is starcompact, and that every regular starcompact Hausdorff space is countably compact. He also mentioned the result of R.S. Houston that in fact every Hausdorff starcompact space is countably compact. Another weaker form of countable compactness is feeble compactness, which has been introduced by Bagley, Connell and McKnight Jr [BCK] in 1958 under the name of light compactness. A topological space $X$ is feebly compact [PW2] iff every locally finite family of non-empty open subsets of $X$ is finite. One checks easily that the result in [PW ${ }_{2}$ ], p. 50, is also true without the assumption of Hausdorffness so that a space $X$ is feebly compact if and only if every countable open cover of $X$ has a finite subfamily whose union is dense, i.e., $X$ is almost countably compact. Thus Theorem 2.9 of [P] actually repeats in part this known fact. A theorem of Mardesic and Mrowka says that pseudocompactness and feeble compactness are equivalent in Tikhonov spaces. On the other hand, countable compactness and pseudocompactness coincide in weakly normal $T_{2}$ spaces. A space is called weakly normal if two disjoint closed subsets, one of them being countable, have disjoint neighbourhoods (see [D]). Several other generalizations of countably compact and Lindelöf spaces have been presented in the literature, see e.g. [PW1], [P], [K] and the papers mentioned in the references of [P] and $[\mathrm{K}]$. Other interesting and related papers are [vDRRT] and [Sa].

The purpose of this paper is to correct some results in $[\mathrm{P}]$ and $[\mathrm{K}]$ concerning certain generalizations of countably compact spaces and Lindelöf spaces. We will also present several additional results and pose a number of interesting open questions.

We will now recall the definitions of the concepts we need. A topological space $X$ is called $n$-starcompact (resp. $\omega$-starcompact) [ P$]$ if for every open cover $\mathcal{G}$ of $X$ there exists a finite subset $F$ of $X$ (resp. there is a finite subset $F$ and $n \in \omega_{0}$ ) such that $X=s t^{n}(F, \mathcal{G})$ holds. Observe that $X=s t^{n}(F, \mathcal{G})=s t\left(s t^{n-1}(F, \mathcal{G}), \mathcal{G}\right)$ for each $n>1$. $X$ is called almost starcompact (resp. almost n-starcompact) if for each open covering $\mathcal{G}$ of $X$ there exists a
finite subset $F \subseteq X$ such that $X=\overline{s t(F, \mathcal{G})}$ (resp. $X=\overline{s t^{n}(F, \mathcal{G})}$ ). Almost Lindelöf, almost starlindelöf, starlindelöf, n-starlindelöf and $\omega$-starlindelöf spaces [P] are defined in an analoguous way. As is well known, a $T_{1}$ space $X$ is countably compact (resp. $\omega_{1}-$ compact) if every countable infinite (resp. every uncountable) subset has a limit point. On the other hand, a space $X$ is said to be $\omega_{1}$-Lindelöf (resp. almost $\omega_{1}$-Lindelöf [P] if every open cover of cardinality, at most $\aleph_{1}$, has a countable subcover (resp. has a countable subfamily whose union is dense). Pseudo $\omega_{1}$-compact (resp. pseudo $\omega_{1}$-Lindelöf ) spaces are those spaces in which locally finite (resp. locally countable) families of non-empty open subsets are countable. A space $X$ is called quasi-regular (see e.g. [EGW]) (resp. $\aleph_{0}-$ collectionwise Hausdorff (briefly $\left.\aleph_{0}-\mathrm{cwH}\right)$ ) iff every non-empty open set contains the closure of an appropriate non-empty open set (resp. points of any countable closed-and-discrete subset have pairwise disjoint neighbourhoods). We will denote the set of all finite (resp. countably infinite) subsets of a subset $A$ by $[A]^{<\omega}$ (resp. $[A]^{\omega}$ ). Also, the first and second transfinite cardinals will be denoted by $\omega_{0}$ (or $\omega$ ) and $\omega_{1}$, respectively.

No separation axioms are assumed in this paper. Moreover, regular spaces and completely regular spaces need not be Hausdorff.

Remark 1.1 We consider the Alexandroff deleted point topology on [ 0,1 ] where the basic open neighbourhoods of $x=0$ are of the form $G_{\epsilon}=[0, \epsilon)-\left\{\frac{1}{2}, \frac{1}{3}, \ldots\right\}$ with $0<\epsilon<1$ and all other points have their usual (euclidean) neighbourhoods. This is a well known quasiregular Hausdorff space but it is not $\aleph_{0}-\mathrm{cwH}$ since the points of the closed-and-discrete set $\left\{0, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ evidently have no pairwise disjoint neighbourhoods.

Remark 1.2 Consider the compact Hausdorff space $X=(\omega+1) \times(\omega+1)$. If $G_{n}=\{(n, n)\}$ for each $n \in \omega$, then $\left\{G_{n}: n \in \omega\right\}$ is a cellular family having no nondegenerate discrete open refinement. This is so because every neighbourhood of $(\omega, \omega)$ intersects infinitely many sets $G_{n}$.

Remark 1.3 In both of the proofs of Theorem 2.2 in $[\mathrm{P}]$ and Theorem $1.5 \mathrm{in}[\mathrm{K}]$ it is unfortunately assumed that every Hausdorff space is $\aleph_{0^{-}}$collectionwise Hausdorff. Therefore, these statements and their proofs should be regiven in a correct way. This will be done in our Propotion 2.1. Also, as we observed in Remark 1.2, a cellular family, i.e., a pairwise disjoint family of nonempty open sets, does not necessarily have a discrete open refinement in regular Hausdorff spaces so we shall present a correct version of Theorem 2.5 in [K]. This will be our Proposition 3.5. In addition, a revised form of Theorem 3.10 in [P] will be given as our Proposition 3.5.

Remark 1.4 It is quite easy to observe by utilizing the starcompactness property of countably compact spaces that every point-finite (even every point-countable) open cover of such spaces has a finite subcover. Furthermore the well known Iseki\&Kasahara Theorem says that if every point-finite open cover of a regular $T_{2}$ space $X$ has a finite subcover then $X$ is countably compact. C.M. Pareek has falsely asserted on the other hand in Theorem 2.2 of this paper [P] that any $T_{2}$ space in which every point-finite open cover has a finite subcover is countably compact. Z. Frolik has actually defined a counterexample in 1960 for emphasizing the essentiality of the regularity condition in above theorem. The following
which is given by Engelking is a slightly simplified version of Frolik's example, see page 241 of $[\mathrm{E}]$. Let $A_{n}=\left\{(2 k-1) 2^{-n}: k=1,2, \ldots, n\right\}$ for each positive integer $n$ and let

$$
X_{0}=[0,1]-\bigcup_{n=1}^{\infty} A_{n} \quad, \quad X=X_{0} \cup\left\{x_{1}, x_{2}, x_{3}, \ldots\right\}
$$

where $x_{n} \notin[0,1]$ for each $n$. Let $X_{0}$ be equipped with the usual subspace topology of $[0,1]$ and let the basic nbhd's of $x_{n}$ in $X$ be the sets of the form $\left\{x_{n}\right\} \cup\left(U \cap X_{0}\right)$ whereas $U$ is an open sets of $[0,1]$ containing $A_{n}$. This is a non-regular $T_{2}$ space which is not countably compact since $K=\left\{x_{1}, x_{2}, \ldots\right\}$ is an infinite closed-discrete set in $X$. Yet every point-finite open cover of $X$ has a finite subcover since by the well known Baire theorem it is certainly impossible to define a point-finite open cover of $X$ having infinitely many members; for the details reader must consult [E], page $241-242$. One should easily notice in here by utilizing the reasoning of the proof of this fact that $X$ is actually not $\aleph_{0}$-collectionwise Hausdorff and thus not regular since the points of $K$ have not pairwise disjoint open nbhd's in $X$ (but $X$ is evidently quasi-regular); see the Proposition 2.1 coming next.

## $\underline{2 \text { Results on certain generalizations of countably compact spaces }}$

Proposition 2.1 Let $X$ be an $\aleph_{0}$-collectionwise Hausdorff space. Then the following are equivalent :
(1) $X$ is countably compact,
(2) Every point finite open over has a finite subcover,
(3) Every countable point finite open cover has a finite subcover.

Proof. (1) $\Rightarrow(2)$ : Let $\mathcal{G}$ be a point finite open cover of the countably compact space $X$. By the well known theorem of Fleischmann $[\mathrm{F}]$ there exists a finite subset $A \subseteq X$ such that $X=\operatorname{st}(A, \mathcal{G})$. Since each point of $A$ lies only in finitely many members of $\mathcal{G}$ it is clear that $\mathcal{G}$ has a finite subcover.
$(2) \Rightarrow(3)$ : This is obvious.
$(3) \Rightarrow(1)$ : Suppose that $X$ is not countably compact. By Theorem 3.10.3. in [E] there exists a countably infinite closed-and-discrete subset of $X$, say $A=\left\{x_{n}: n<\omega\right\}$. Since $X$ is $\aleph_{0}-\mathrm{cwH}$, there exists a cellular family $\left\{G_{n}: n<\omega\right\}$ with $x_{n} \in G_{n}$ for each $n<\omega$. If $\mathcal{G}=\{X-A\} \cup\left\{G_{n}: n<\omega\right\}$ then $\mathcal{G}$ is a countable point finite open cover having evidently no finite subcover, a contradiction.

Corollary 2.2 Since every regular Hausdorff space is $\aleph_{0}-c w H$, the three conditions in Proposition 2.1 are equivalent in regular Hausdorff spaces.

Recall that a space $X$ is said to be countably S-closed [DEG] if every countable cover of $X$ by regular closed subsets has a finite subcover.

Proposition 2.3 A countably S-closed space is almost countably compact, and an almost
countably compact space is almost 2-starcompact.

Proof. The first assertion is straightforward (see also [DEG]). Now suppose that $\mathcal{G}$ is an open cover of an almost countably compact space $X$ such that $X \neq \overline{s t^{2}(A, \mathcal{G})}$ for each $A \in[X]^{<\omega}$. By induction there exists a sequence $\left\{x_{n}\right\}$ of points satisfying $x_{n+1} \notin \overline{s t^{2}\left(A_{n}, \mathcal{G}\right)}$ where $A_{n}=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. For each $n \in \quad$ choose $G_{n+1} \in \mathcal{G}$ with $x_{n+1} \in G_{n+1}$ and let us define $V_{n+1}=G_{n+1}-\overline{s t^{2}\left(A_{n}, \mathcal{G}\right)}$. In addition, choose $V_{1} \in \mathcal{G}$ with $x_{1} \in V_{1}$. Now let $x \in X$ and take $G \in \mathcal{G}$ such that $x \in G$. If $G \cap V_{n}$ is nonempty, we have $G \subseteq s t^{2}\left(A_{n}, \mathcal{G}\right)$ and so $G \cap V_{m}$ is empty for each $m>n$. This shows that $\left\{V_{n}: n \in \quad\right\}$ is a locally finite family of nonempty open sets contradicting our assumption that $X$ is almost countably compact.

Proposition 2.4 Every $T_{1}$ weakly normal almost countably compact space is almost starcompact.

Proof. Let $X$ be $T_{1}$, wealy normal and almost countably compact, and let $\mathcal{G}$ be an open cover of $X$ with $X \neq \overline{s t(A, \mathcal{G})}$ for each $A \in[X]^{<\omega}$. As in the proof of the previous result, by induction there exists a sequence $\left\{x_{n}\right\}$ of points satisfying $x_{n+1} \notin \overline{\operatorname{st}\left(A_{n}, \mathcal{G}\right)}$ where $A_{n}=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. For each $n \in \quad$ choose now $G_{n+1} \in \mathcal{G}$ with $x_{n+1} \in G_{n+1}$ and let $V_{n+1}=G_{n+1}-\overline{\operatorname{st}\left(A_{n}, \mathcal{G}\right)}$. Moreover, define $V_{1} \in \mathcal{G}$ with $x_{1} \in V_{1}$. Clearly, $\left\{V_{n}: n \in \quad\right\}$ is a cellular family. Since $X$ is $T_{1}$, it is easily checked that the set $A=\left\{x_{n}: n \in \omega\right\}$ is closed and discrete, and clearly contained in the union set, say $W$, of $\left\{V_{n}: n \in\right\}$. Since $X$ is weakly normal, there exists an open set $U$ such that $A \subseteq U \subseteq \bar{U} \subseteq W$. Now it is easily verified that the cellular family $\left\{V_{n} \cap U: n \in \quad\right\}$ is a discrete family, hence locally finite, thus contradicting our assumption that $X$ is almost countably compact.

Pareek has proved that $[\mathrm{P}]$ every regular almost starcompact space is almost countably compact. We have, on the other hand, the following results. In here one should remember once again that neither metacompact nor $\aleph_{1}$-collectionwise normal (in particularly normal) spaces in this paper are $T_{1}$.

Proposition 2.5 A metacompact almost starcompact space is almost countably compact.

Proof. Let $\mathcal{G}=\left\{G_{n}\right\}_{n \in \omega}$ be an open covering of the metacompact space $X$. Then we have

$$
X=\bigcup_{n \in \omega}\left(\overline{G_{n}}-\overline{\bigcup_{k<n} G_{k}}\right)=\bigcup_{n}\left(G_{n}-\overline{\bigcup_{k<n} G_{k}}\right) \cup \bigcup_{n}\left(\partial G_{n}-\overline{\bigcup_{k<n} G_{k}}\right)
$$

Since $\mathcal{G}$ is an open covering, the families

$$
\mathcal{G}_{1}=\left\{G_{n}-\overline{\bigcup_{k<n} G_{k}}\right\}_{n \in \omega} \quad, \quad \mathcal{K}=\left\{\overline{\overline{G_{n}}-\overline{\bigcup G_{k}}}\left(=\overline{G_{n}-\overline{\bigcup_{k<n} G_{k}}}\right)\right\}_{n \in \omega}
$$

are both locally finite. If infinite number of members of $\mathcal{G}_{1}$ are non-empty then we may suppose without losing the generality that all members of $\mathcal{G}_{1}$ are non-empty and thus $\mathcal{G}_{1}$ would be a cellular family. We also have

$$
\overline{\overline{G_{n}}-\overline{\bigcup_{k<n} G_{k}}} \cap\left(\partial G_{m}-\overline{\bigcup_{k<m} G_{k}}\right)=\emptyset \quad(n<m)
$$

Since $\mathcal{K}$ is a locally finite closed covering, then, for each $n \in \omega$ there exists an open set $U_{n}$ satisfying $\partial G_{n}-\overline{\bigcup G_{k}} \subseteq U_{n}$ and furthermore the family $\mathcal{U}=\left\{U_{n}\right\}_{n \in \omega}$ is locally finite (see [D], Theorem 1.5 of chpt VIII). Thus $\mathcal{G}_{1} \cup \mathcal{U}$ is a locally finite open (countable) covering of $X$. Then for each $x \in X$, we have an open nbhd $W_{x}$ of $x$ such that

$$
\operatorname{ord}\left(W_{x}, \mathcal{G}_{1} \cup \mathcal{U}\right)=\operatorname{card}\left\{G \in \mathcal{G}_{1} \cup \mathcal{U}: W_{x} \cap G \neq \emptyset\right\}
$$

is finite. Let now $\mathcal{W}$ be a precise point finite open refinement of $\left\{W_{x}: x \in X\right\}$. Then we evidently have $X \neq \overline{s t(F, \mathcal{W})}$ for any finite $F \subseteq X$. Thus all but a finite number of members of $\mathcal{G}_{1}$ are necessarily empty. Therefore there exists an $n_{0} \in \omega$ such that $G_{n} \subseteq \bigcup_{k \leq n_{0}} G_{k}$ for each $n>n_{0}$.

Proposition 2.6 A quasi-regular almost starcompact space $X$ is almost countably compact.

Proof. Let $\mathcal{G}=\left\{G_{n}\right\}_{n \in \omega}$ be an infinite open covering of $X$. If infinite number of members of the cellular and locally finite family $\left\{G_{n}-\overline{\bigcup_{k<n} G_{k}}: n \in \omega\right\}$ are non-empty then one can suppose without losing the generality that each of them is non-empty. Then define non-empty open set $W_{n}$ for each $n \in \omega$ so that $\overline{W_{n}} \subseteq G_{n}^{*}=G_{n}-\overline{\bigcup_{k<n} G_{k}}$. Since $G_{n}^{*}$ sets are pairwise disjoint

$$
W_{n} \cap \overline{\bigcup_{k<n} G_{k}^{*} \cup\left(X-\overline{\left.\bigcup_{n \in \omega} W_{n}\right)}\right.}=\emptyset \quad(n \in \omega)
$$

Then there is no finite subset $F \subseteq X$ satisfying $X=\overline{\operatorname{st}\left(F, \mathcal{G}^{*}\right)}$ whereas $\mathcal{G}^{*}$ is the locally finite open covering $\mathcal{G}^{*}=\left\{G_{n}^{*}\right\}_{n \in \omega} \cup\left\{X-\overline{\bigcup W_{n}}\right\}$. Therefore there exists an $n_{0} \in \omega$ such that $G_{n}^{*}=\emptyset$ for each $n>n_{0}$, i.e. $X=\frac{n \in \omega}{\bigcup_{n \leq n_{0}} G_{n}}$ holds.

Corollary 2.7 (Pareek) A regular almost starcompact space is almost countably compact.

Now we want to proove in here the following related result which is a slight generalization of a fact from [vDRRT, Theorem 2.1.8.]: Open discrete families in any regular $\omega$-starcompact space are necessarily finite. We follow almost the same arguments.

Proposition 2.8 Open discrete families in a quasi-regular $\omega$-starcompact space are finite.

Proof. Suppose $\mathcal{G}=\left\{G_{n}: n \in \omega\right\}$ is an infinite discrete family of non-empty open sets in a quasi-regular space $X$. Then quasi regularity of $X$ yields for each $n \in \omega$, the existence of a monotonically non-increasing sequence of open sets $U_{n, k}$ such that

$$
\emptyset \neq U_{n, 0} \subseteq \overline{U_{n, 0}} \subseteq \ldots \subseteq U_{n, n-1} \subseteq \overline{U_{n, n-1}} \subseteq U_{n, n} \subseteq \overline{U_{n, n}} \subseteq G_{n}
$$

Let us pick the point $x_{n} \in U_{n, 0}$ and define the open sets

$$
\begin{gathered}
W=X-\overline{\bigcup_{n \in \omega} U_{n, n}}, \\
W_{n, k}=U_{n, k+1}-\overline{U_{n, k-1}} \quad(k=1,2, \ldots, n-1)
\end{gathered}
$$

and finally $W_{n, 0}=U_{n, 1}, W_{n, n}=G_{n}-\overline{U_{n, n-1}}$ for each $n \in \omega$. It is easy to observe that i) $x_{n} \in W_{n, k}$ if and only if $k=0$, ii) $G_{n}=\bigcup_{k=0}^{n} W_{n, k}$, iii) $W_{n, i} \cap W_{m, k}=\emptyset$ if $n \neq m ; W_{n, k} \cap W_{n, i} \subseteq U_{n, k+1}-\overline{U_{n, i-1}} \subseteq U_{n, k+1}-\overline{U_{n, k+1}}=\emptyset$ if $k+2 \leq i$ and therefore $W_{n, i} \cap W_{n, k} \neq \emptyset$ necessarily implies that $|i-k| \leq 1$. Furthermore we have $X-\bigcup_{n \in \omega} G_{n} \subseteq$ $X-\bigcup_{n \in \omega} \overline{U_{n, n}}=X-\overline{\bigcup_{n \in \omega} U_{n, n}}=W$ and thus $\mathcal{W}=\{W\} \cup\left\{W_{n, k}: n \in \omega, k \leq n\right\}$ is an open cover for $X$ after ii). Let us finally prove that $X-s t^{n}(F, \mathcal{W}) \neq \emptyset$ holds for any $F \in[X]^{<\omega}$ and any fixed $n \in \omega$. In fact since members of $\mathcal{G}$ are pairwise disjoint and $F$ finite we evidently have an $N>n$ such that $G_{N} \cap F=\emptyset$. Besides $\operatorname{st}\left(x_{N}, \mathcal{W}\right)=W_{N, 0}$ by i) and $s t^{m}\left(x_{N}, \mathcal{W}\right) \subseteq \bigcup_{k \leq m} W_{N, k}$ for $m \leq N$ after iii). All these facts imply that $s t^{n}\left(x_{N}, \mathcal{W}\right) \subseteq G_{N}$ and $F \cap \overline{s t}^{n}\left(x_{N}, \mathcal{W}\right)=\emptyset$. Thus $x_{N} \in X-s t^{n}(F, \mathcal{W})$ follows as required, i.e. any quasi-regular space having an infinite open-discrete family would be non $\omega$-starcompact.

A similar result will be obtained in Proposition 3.2. Now we want to prove that every normal $\omega$-starcompact space is $\omega_{0}$-compact. For establishing this we utilize a known lemma, coming next. We give here, in this context, a direct and short proof of this lemma by slightly modifying a method which can be found in classical texts (see [D], p. 255 or [B], p.369). It is well known that a space $X$ is called $\aleph_{n}$-collectionwise normal iff for every discrete family $\left\{K_{\alpha}: \alpha<\omega_{n}\right\}$ of closed sets in $X$ there exists an open (discrete) family $\left\{G_{\alpha}: \alpha<\omega_{n}\right\}$ of open sets of $X$ satisfying $K_{\alpha} \subseteq G_{\alpha}$ for each $\alpha<\omega_{n}$. As is well known $\aleph_{0}$-collectionwise normal space are precisely normal spaces.

Lemma 2.9 Every point finite countable open covering of normal space has a countable open star-finite refinement.

Proof. Let $\mathcal{U}=\left\{U_{n}: n \in \quad\right\}$ be a countable point finite open covering of a normal space $X$. It is well known that there exists a precise closed shrinking-covering $\mathcal{K}=\left\{K_{n}: n \in \quad\right\}$ of $\mathcal{U}$. Define open sets $G(k, n) \quad(k, n \in \quad)$ such that

$$
K_{k} \subseteq G(k, n) \subseteq \overline{G(k, n)} \subseteq G(k, n+1) \subseteq U_{k} \quad(k \in \quad)
$$

Let us now define

$$
W_{1,1}=G(1,2) \quad, \quad W_{k, n}=G(k, n+1)-\prod_{1 \leq i \leq n-1} G(i, n-1) \quad 2 \leq n, k \leq n
$$

Then $\mathcal{W}=\left\{W_{k, n}: 1 \leq k \leq n, n \in \quad\right\}$ is the required open refinement of $\mathcal{U}$. Since we have

$$
\begin{gather*}
\overline{\bigcup_{1 \leq i \leq n} G(i, n)} \subseteq \bigcup \mathcal{W}  \tag{1}\\
W_{k, n} \cap W_{i, m}=\emptyset \text { if } n+3 \leq m \tag{2}
\end{gather*}
$$

The statement (1) can be proved easily by induction, since $\overline{G(1,1)} \subseteq G(1,2)=W_{1,1}$ and if (1) holds for $n \in$ then for any $1 \leq k \leq n+1$ we have following:

$$
\begin{gathered}
\overline{G(k, n+1)} \subseteq G(k, n+2)=\left(G(k, n+2)-\overline{\bigcup_{1 \leq i \leq n} G(i, n)}\right) \cup\left(G(k, n+2) \cap \overline{\bigcup_{1 \leq i \leq n} G(i, n)}\right) \\
=W_{k, n+1} \cup\left(G(k, n+2) \cap \overline{\bigcup_{1 \leq i \leq n} G(i, n)}\right) \subseteq \bigcup \mathcal{W} .
\end{gathered}
$$

The second assertion holds easily, since $n+3 \leq m$ gives

$$
W_{k, n} \subseteq G(k, n+1) \subseteq \overline{G(k, n+2)} \subseteq \overline{G(k, m-1)} \subseteq \overline{\bigcup_{1 \leq i \leq m-1} G(i, m-1)}
$$

for any $k \leq n$. Thus $\operatorname{ord}\left(W_{k, n}, \mathcal{W}\right)<\aleph_{0}$ for any $n \in \quad, k \leq n$. Since $\mathcal{K}$ is covering, (1) says that $\mathcal{W}$ is an open covering; and the last observation says on the other hand that $\mathcal{W}$ is star-finite.

Proposition 2.10 A normal $\omega$-starcompact space $X$ is $\omega_{0}$-compact.

Proof. Suppose there exists a countable infinite closed discrete set $K=\left\{x_{n}: n \in \omega\right\}$ in a normal space $X$. Since $X$ is $\aleph_{0}$ - collectionwise normal there exists a discrete open family $\mathcal{G}=\left\{G_{n}\right\}_{n \in \omega}$ so that $x_{n} \in G_{n}$ for each $n \in \omega$. Let $\mathcal{W}$ be an open star-finite (and thus point finite) refinement of $\mathcal{G} \cup\{X-K\}$. Then we evidently have
$X \neq s t^{n}(A, \mathcal{W})$ for any $n \in \omega$ and $A \in[X]^{<\omega}$.
Pareek has proved in $[\mathrm{P}]$ that starlindelöf collectionwise Hausdorff spaces are $\omega_{1}$-compact, i.e. in such spaces closed-discrete subsets are countable. We will prove on the other hand that $\omega$-starlindelöf $\aleph_{1}$ - collectionwise normal spaces are $\omega_{1}$-compact. We need first the following result with a long proof which is based on a slightly modified technique of J.C.Smith, see [Sm]. Actually this technique was used first by E.Micheal in his well known paper [M].

One should remember here before the following proof that i) a family $\mathcal{A}$ is discrete in a topological space iff $\{\bar{A}: A \in \mathcal{A}\}$ is so, ii) a closed subset $K$ in a normal space is contained in an open set $G$ iff there exists a cozero set $U$ such that $K \subseteq U \subseteq \bar{U} \subseteq G$ and iii) the union set of a locally finite family of cozero sets in any space is again a cozero set.

Proposition 2.11 In any $\aleph_{1}$-collectionwise normal space $X$, an open covering $\mathcal{G}=\bigcup_{n=1}^{\infty} \mathcal{G}_{n}$ satisfying the following conditions, has a locally finite open refinement:
i) Each $\mathcal{G}_{n}=\left\{G_{n \alpha}: \alpha<\kappa_{n}<\omega_{1}\right\}$ is open (not necessarily covering) family,
ii) The countable open covering $\mathcal{G}^{*}=\left\{G_{n}^{*}\right\}_{n=1}^{\infty}$ where $G_{n}^{*}=\bigcup \mathcal{G}_{n}=\bigcup_{\alpha<\kappa_{n}} G_{n \alpha}$, is point finite, i.e. $1 \leq \operatorname{ord}\left(x, \mathcal{G}^{*}\right)<\omega_{0} \quad$ for each $x \in X$,
iii) For each $x \in X$ there exists an $n_{x} \in \quad$ such that $1 \leq \operatorname{ord}\left(x, \mathcal{G}_{n_{x}}\right)<\omega_{0}$.

Proof. The hypothesis guarantees the existence of the positive integers $k_{x}$ and $m_{x}$ for each $x \in X$ such that

$$
\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=k_{x} \text { and } \operatorname{ord}\left(x, \mathcal{G}_{n_{x}}\right)=m_{x}
$$

Let us introduce the sets

$$
\begin{aligned}
K(k, n, m) & =\left\{x \in X: \operatorname{ord}\left(x, \mathcal{G}^{*}\right) \leq k, \operatorname{ord}\left(x, \mathcal{G}_{n}\right) \leq m\right\} \\
G_{n}(\Lambda) & =\bigcap\left\{G_{n \alpha}: \alpha \in \Lambda\right\} \quad, \quad \Lambda \in\left[\kappa_{n}\right]^{<\omega}
\end{aligned}
$$

These are respectively closed and open sets evidently. It is easy to observe that $X=$ $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K(k, n, m)$. We construct by induction a collection $\bigcup_{k=1}^{\infty} \mathcal{U}(k, n, m)=$ $\mathcal{U}(n, m)$ of cozero sets for each positive integer $n$ and $m$ satisfying the following conditions:

1) $\mathcal{U}(k, n, m) \prec \mathcal{G}$ for each $k$,
2) $\bigcup \mathcal{U}(k, n, m)$ is a cozero set for each $k$,
3) $X=\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(k, n, m))$.

We construct first the family $\mathcal{U}(k, 1,1)$ for each positive integer $k$. Let

$$
\mathcal{A}(k, 1,1)=\left\{A(k, 1, \Lambda, 1): \Lambda \in\left[\kappa_{k}\right]^{1}\right\}
$$

where

$$
A(k, 1, \Lambda, 1)=K(1, k, 1) \cap G_{k}(\Lambda) \quad\left(\forall \Lambda \in\left[\kappa_{k}\right]^{1}\right)
$$

Notice that if $x \in X-K(1, k, 1)$ then it is easy to define an open nbhd of $x$ disjoint with all $A(k, 1, \Lambda, 1)$ sets. If $x \in K(1, k, 1)$ then $x$ belongs exactly one member say $G_{k \alpha}$ of $\mathcal{G}_{k}$ and this member is disjoint with all $A(k, 1, \Lambda, 1)$ sets where $\Lambda=\{\beta\}$ and $\beta \neq \alpha$. Thus the family $\mathcal{A}(k, 1,1)$ is discrete in $X$ and therefore there exists an open discrete family $\mathcal{U}(k, 1,1)=\left\{U(k, 1, \Lambda, 1): \Lambda \in\left[\kappa_{k}\right]^{1}\right\}$ for each $k$ such that $\overline{A(k, 1, \Lambda, 1)} \subseteq U(k, 1, \Lambda, 1)$ and $\left.\left.1^{\prime}\right) \mathcal{U}(k, 1,1) \prec \mathcal{G}, 2^{\prime}\right) \bigcup \mathcal{U}(k, 1,1)$ is a cozero set for each $k$ and $\left.3^{\prime}\right)$ if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=$ 1 and $\operatorname{ord}\left(x, \mathcal{G}_{k}\right)=1$ for some $k$ then $x \in \bigcup \mathcal{U}(k, 1,1)$. Let us define now $\mathcal{U}(1,1)=$ $\bigcup_{k=1}^{\infty} \mathcal{U}(k, 1,1)$. Suppose now that all the families $\mathcal{U}(1,1), \ldots, \mathcal{U}(1, m)$ have been defined so that if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=1$ and $\operatorname{ord}\left(x, \mathcal{G}_{k}\right) \leq i \leq m$ for some positive integer $k$ then $x \in$ $\bigcup_{i=1}^{m}(\bigcup \mathcal{U}(1, i))$ holds. Define now for each positive integer $k$

$$
A(k, 1, \Lambda, m+1)=\left[K(1, k, m+1)-\bigcup_{i=1}^{m}(\bigcup \mathcal{U}(1, i))\right] \cap G_{k}(\Lambda)
$$

for each $\Lambda \in\left[\kappa_{k}\right]^{m+1}$. Then $\mathcal{A}(k, 1, m+1)=\left\{A(k, 1, \Lambda, m+1): \Lambda \in\left[\kappa_{k}\right]^{m+1}\right\}$ is a discrete family in $X$. In fact take any point $x \in X$; if $x \in X-K(1, k, m+1) \cup \bigcup_{i=1}^{m}(\bigcup \mathcal{U}(1, i))$ then $x$ evidently have an open nbhd disjoint with all members of this family; if $x$ belongs to the closed set written in bracket in above then $x$ necessarily belong exactly $m+1$ number of members of $\mathcal{G}_{k}$ and their intersection set intersects at most one member of $\mathcal{A}(k, 1, m+1)$. Since $X$ is $\aleph_{1}$-cwn there exists an open discrete collection $\mathcal{U}(k, 1, m+1)=$ $\left\{U(k, 1, \Lambda, m+1): \Lambda \in\left[\kappa_{k}\right]^{m+1}\right\}$ for each $k$ such that $\overline{A(k, 1, \Lambda, m+1)} \subseteq U(k, 1, \Lambda, m+1)$ and $\left.\left.1^{\prime \prime}\right) \mathcal{U}(k, 1, m+1) \prec \mathcal{G}, 2^{\prime \prime}\right) \cup \mathcal{U}(k, 1, m+1)$ is a cozero set for each $k$ and $\left.3^{\prime \prime}\right)$ if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=1, x \notin \bigcup_{i=1}^{m}(\bigcup \mathcal{U}(1, i))$ and $\operatorname{ord}\left(x, \mathcal{G}_{k}\right) \leq m+1$ for some positive integer $k$ then $x \in \bigcup \mathcal{U}(k, 1, m+1)$. Let us define $\mathcal{U}(1, m+1)=\bigcup_{k=1}^{\infty} \mathcal{U}(k, 1, m+1)$. Thus all $\mathcal{U}(1, n)$ families have been defined now by induction on $n$. At the next step we define the open families $\mathcal{U}(n, 1), \mathcal{U}(n, 2), \ldots$ for each $n>1$ satisfying the conditions 1$)$ and 2$)$ in above together with the fact if $1<\operatorname{ord}\left(x, \mathcal{G}^{*}\right)$ and $0<\operatorname{ord}\left(x, \mathcal{G}_{k}\right)<\omega_{0}$ for some positive integer $k$ then $x$ belongs to some member of one of these families. For this purpose suppose now that all $\mathcal{U}(i, m)$ families for each $i<n$ and for all positive $m$ have already been defined such that if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right) \leq n-1$ and $0<\operatorname{ord}\left(x, \mathcal{G}_{k}\right)<\omega_{0}$ for some $k$ then $x \in \bigcup_{i<n} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(i, m))$ holds. Define now

$$
A(k, n, \Lambda, 1)=\left[K(n, k, 1)-\bigcup_{i=1}^{n-1} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(i, m))\right] \cap G_{k}(\Lambda)
$$

for each $\Lambda \in\left[\kappa_{k}\right]^{1}$ and then let $\mathcal{A}(k, n, 1)$ be the family of all these sets. This family is discrete in $X$. In fact take any $x \in X$. If we have

$$
x \in(X-K(n, k, 1)) \cup \bigcup_{i=1}^{n-1} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(i, m))
$$

then $x$ have evidently an open nbhd disjoint with all members of $\mathcal{A}(k, n, 1)$. Suppose now that $x$ does not belong this union. Then we necessarily have $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=n$. If $x \notin \bigcup \mathcal{G}_{k}$ then $x \in \bigcap_{j=1}^{n}\left(\bigcup \mathcal{G}_{i_{j}}\right)$ where each index $i_{j}$ is different from $k$ and this last intersection set does not intersect any member of $\mathcal{A}(k, n, 1)$. If finally $x \in \bigcup \mathcal{G}_{k}$ then we have $\operatorname{ord}\left(x, \mathcal{G}_{k}\right)=1$ since $x \in K(n, k, 1)$ and so a suitable open nbhd of $x$ intersecting at most one member of $\mathcal{A}(k, n, 1)$ can easily be defined. Define now the open-discrete family $\mathcal{U}(k, n, 1)=\left\{U(k, n, \Lambda, 1): \Lambda \in\left[\kappa_{k}\right]^{1}\right\}$ for each positive integer $k$ such that $\overline{A(k, n, \Lambda, 1)} \subseteq U(k, n, \Lambda, 1)$ and $\left.\left.1^{\prime \prime \prime}\right) \mathcal{U}(k, n, 1) \prec \mathcal{G}, 2^{\prime \prime \prime}\right) \bigcup \mathcal{U}(k, n, 1)$ is a cozero set for each $k$ and $\left.3^{\prime \prime \prime}\right)$ if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right) \leq n, x \notin \bigcup_{i=1}^{n-1} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(i, m))$ and $\operatorname{ord}\left(x, \mathcal{G}_{k}\right)=1$ for some $k$ then $x \in \bigcup \mathcal{U}(k, n, 1)$. Define now $\mathcal{U}(n, 1)=\bigcup_{k=1}^{\infty} \mathcal{U}(k, n, 1)$ and suppose that each $\mathcal{U}(n, 1), \ldots, \mathcal{U}(n, m)$ have already been defined such that if $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)<n$ or $\operatorname{ord}\left(x, \mathcal{G}^{*}\right)=n$ and $\operatorname{ord}\left(x, \mathcal{G}_{k}\right) \leq m$ for some $k$ then $x$ belongs to some member of the family $\bigcup_{j \leq m} \mathcal{U}(n, j) \cup \bigcup_{i<n} \bigcup_{m=1}^{\infty} \mathcal{U}(i, m)$. Then one can prove similarly that the family $\mathcal{A}(k, n, m+1)$ of all the sets

$$
\left[K(n, k, m+1)-\left(\bigcup_{j \leq m}(\bigcup \mathcal{U}(n, j)) \cup \bigcup_{i<n} \bigcup_{m=1}^{\infty}(\bigcup \mathcal{U}(i, m))\right)\right] \cap G_{k}(\Lambda)
$$

where $\Lambda \in\left[\kappa_{k}\right]^{m+1}$ is discrete in $X$ and the corresponding open-discrete family $\mathcal{U}(k, n, m+1)$ can be defined similarly. Thus $\mathcal{U}(n, m+1)=\bigcup_{k=1}^{\infty} \mathcal{U}(k, n, m+1)$ is de-
fined and the induction process is established. Since

$$
\mathcal{U}=\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{U}(k, n, m)
$$

is an open refinement of $\mathcal{G}$ such that $\bigcup \mathcal{U}(k, n, m)$ is a cozero set for each triple $(k, n, m)$ then, $\mathcal{G}$ has a locally finite open refinement by the following result.

Proposition 2.12 If $\mathcal{U}=\bigcup_{n=1}^{\infty} \mathcal{U}_{n}$ is a $\sigma$-locally finite open cover of any space $X$ such that $\bigcup \mathcal{U}_{n}$ is cozero set for each $n$ then $\mathcal{U}$ has a locally finite open refinement.

Proof. By a well known result of K. Morita, every countable cozero-set cover of any space has a locally finite open refinement (see Theorem 11.1 in [AS] for instance). Now let $\mathcal{W}$ be an open and locally finite open refinement of countable cozero-set cover $\left\{U_{n}\right\}_{n=1}^{\infty}$ where $U_{n}=\bigcup \mathcal{U}_{n}$. Define $n(W)$ for each $W \in \mathcal{W}$ as the minimal positive integer $n$ such that $W \subseteq U_{n}$ and write $W_{n}=\bigcup\{W \in \mathcal{W}: n(W)=n\}$ for each $n$. Then $\mathcal{W}^{*}=\bigcup_{n=1}^{\infty}\left\{W_{n} \cap U: U \in \mathcal{U}_{n}\right\}$ is the required open refinement of $\mathcal{U}$.

Proposition 2.13 An $\aleph_{1}$-collectionwise normal $\omega$-starlindelöf space $X$ is $\omega_{1}$-compact.

Proof. Let $K=\left\{x_{\alpha}: \alpha<\omega_{1}\right\}$ be a closed-discrete subset of $X$. Let $\left\{G_{\alpha}: \alpha<\omega_{1}\right\}$ be an open discrete family satisfying $x_{\alpha} \in G_{\alpha}$ for each $\alpha<\omega_{1}$. Express $\mathcal{G}=\left\{G_{\alpha}\right\}_{\alpha<\omega_{1}} \cup\{X-K\}$ as $\bigcup_{n=0}^{\infty} \mathcal{G}_{n}$ as in the above proposition where $\mathcal{G}_{n} \cap \mathcal{G}_{m}=\emptyset \quad(n \neq m)$ and $\mathcal{G}_{0}=\{X-K\}$. Then $\mathcal{G}$ has a sequence of normal refinements $\left\{\mathcal{U}_{n}\right\}_{n=1}^{\infty}$ by the above theorem, i.e. $\mathcal{U}_{1}$ is a star refinement of $\mathcal{G}$ and for each $U \in \mathcal{U}_{n+1}$ there exists an $U^{*} \in \mathcal{U}_{n}$ such that $\operatorname{st}\left(U, \mathcal{U}_{n+1}\right) \subseteq U^{*}$. Then $X$ is not $\omega$-starlindelöf since it is not $n$-starlindelöf for any positive integer $n$, because for any countable subset $C$ of $X$ we have $X \neq s t^{n}\left(C, \mathcal{U}_{n}\right)$, for

$$
s t^{n}\left(C, \mathcal{U}_{n}\right)=\bigcup\left\{s t^{n}\left(x, \mathcal{U}_{n}\right): x \in C\right\}
$$

is contained by the union set of a suitable subfamily of $\mathcal{G}$. This proves the assertion.

Lemma 2.14 If there is an infinite locally finite (resp. uncountable locally countable) open family $\mathcal{U}$, then there is a locally finite (resp. locally countable) cellular family $\mathcal{U}^{*}$ such that $\bigcup \mathcal{U}^{*} \subseteq \bigcup \mathcal{U}$ and $\left|\mathcal{U}^{*}\right|=|\mathcal{U}|$.

Proof. We only prove the first statement. Let $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha \in \Lambda}$ be a locally finite infinite family of open sets. We may even suppose that $\Lambda$ is well ordered. Let $\omega_{0} \leq \kappa=|\Lambda|$. Define $\alpha_{0}=\min \Lambda$ and determine a finite $\Lambda_{0}(\subseteq \Lambda)$ and a non-empty open subset $W_{\alpha_{0}} \subseteq U_{\alpha_{0}}$ such that

$$
W_{\alpha_{0}} \cap \bigcup\left\{U_{\gamma}: \gamma \in \Lambda-\Lambda_{0}\right\}=\emptyset
$$

Take now a fixed $\mu<\kappa$ and furthermore suppose that for each $\beta<\mu$, a finite subset $\Lambda_{\beta}$ of $\Lambda-\bigcup_{\gamma<\beta} \Lambda_{\gamma}$ and a non-empty open subset $W_{\alpha_{\beta}}$ of $U_{\alpha_{\beta}}$ satisfying

$$
W_{\alpha_{\beta}} \cap \bigcup\left\{U_{\alpha}: \alpha \in \Lambda-\bigcup_{\gamma \leq \beta} \Lambda_{\gamma}\right\}=\emptyset
$$

have already been defined. Since $\left|\Lambda-\bigcup_{\beta<\mu} \Lambda_{\beta}\right|=\kappa$ and since the family $\left\{U_{\alpha}: \alpha \in \Lambda-\bigcup_{\beta<\mu} \Lambda_{\beta}\right\}$ is locally finite, there exist a finite subset $\Lambda_{\mu}$ of $\Lambda-\bigcup_{\beta<\mu} \Lambda_{\beta}$ and a non-empty open subset $W_{\alpha_{\mu}} \subseteq U_{\alpha_{\mu}}$ where $\alpha_{\mu}=\min \left(\Lambda-\bigcup_{\beta<\mu} \Lambda_{\beta}\right)$ such that

$$
W_{\alpha_{\mu}} \cap \bigcup\left\{U_{\alpha}: \alpha \in \Lambda-\left(\Lambda_{\mu} \cup \bigcup_{\beta<\mu} \Lambda_{\beta}\right)\right\}=\emptyset
$$

Notice that the locally finite family $\left\{W_{\alpha_{\gamma}}: \gamma<\kappa\right\}$ is also a cellular family. In fact whenever $\gamma \neq \beta$ and say for instance $\gamma<\beta$ then we have

$$
W_{\alpha_{\gamma}} \cap W_{\alpha_{\beta}} \subseteq W_{\alpha_{\gamma}} \cap U_{\alpha_{\beta}} \subseteq W_{\alpha_{\gamma}} \cap \bigcup\left\{U_{\alpha}: \alpha \in \Lambda-\bigcup_{\eta \leq \gamma} \Lambda_{\eta}\right\}=\emptyset
$$

since $\alpha_{\beta} \in \Lambda-\bigcup_{\gamma<\beta} \Lambda_{\gamma} \subseteq \Lambda-\bigcup_{\eta \leq \gamma} \Lambda_{\eta}$ holds.

Now a result which is similar to Proposition 2.6 comes

Proposition 2.15 A quasi-regular almost starlindelöf space is pseudo $\omega_{1}$-compact.

Proof. Let $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha<\omega_{1}}$ be an uncountable locally finite family of open sets in a quasi regular space X . Without losing the generality we may suppose by the above Lemma that $\mathcal{U}$ is an uncountable celular family. Choose a non-empty open set $W_{\alpha}$ for each $\alpha<\omega_{1}$ such that $\overline{W_{\alpha}} \subseteq U_{\alpha}$ and define $\mathcal{W}=\left\{W_{\alpha}: \alpha<\omega_{1}\right\}$. We evidently have $\overline{\bigcup \mathcal{W}} \subseteq \bigcup \mathcal{U}$ and $|\mathcal{W}|=\omega_{1}$. One easily observes that $\operatorname{st}\left(C, \mathcal{W}^{*}\right)$ is not dense for any $C \in[X] \leq \omega$ where $\mathcal{W}^{*}=\mathcal{U} \cup\{X-\overline{\bigcup \mathcal{W}}\}$ since for any countable subset $\Lambda$ of $\omega_{1}$, the closed set

$$
K(\Lambda)=\overline{\bigcup_{\alpha \in \Lambda} U_{\alpha}} \cup \overline{X-\overline{\bigcup \mathcal{W}}}
$$

does not intersect any member of the subfamily $\left\{W_{\alpha}: \alpha \in \omega_{1}-\Lambda\right\}$ and so

$$
\operatorname{st}\left(K(\Lambda), \mathcal{W}^{*}\right) \cap \bigcup\left\{W_{\alpha}: \alpha \in \omega_{1}-\Lambda\right\}=\emptyset
$$

Thus $X$ is not almost starlindelöf.

Corollary 2.16 (Pareek) An almost starlindelöf normal $T_{1}$ space is pseudo $\omega_{1}$-compact.

## 3 Results on certain generalizations of Lindelöf spaces

Now we obtain some results on certain generalizations of Lindelöf spaces which were defined in the first section.

Proposition 3.1 Perfectly normal pseudo $\omega_{1}$-Lindelöf spaces are almost starlindelöf.

Proof. If a perfectly normal, pseudo $\omega_{1}$-Lindelöf space $X$ is not almost starlindelöf then there exists an open cover $\mathcal{G}$ and an uncountable closed-discrete subset $A=\left\{x_{\alpha}: \alpha \in \omega_{1}\right\}$ such that $\operatorname{st}\left(x_{\alpha}, \mathcal{G}\right) \cap A=\left\{x_{\alpha}\right\}$. Furthermore each $x_{\alpha}$ would have an open neighborhood $W_{\alpha}$ satisfying

$$
W_{\alpha} \subseteq \operatorname{st}\left(x_{\alpha}, \mathcal{G}\right)-\overline{s t\left(A_{\alpha}, \mathcal{G}\right)} \quad\left(\alpha<\omega_{1}\right)
$$

where $A_{\alpha}=\left\{x_{\beta}: \beta<\alpha\right\}$. Notice that $\partial(\bigcup \mathcal{W})$ is non empty, since otherwise the cellular family $\mathcal{W}=\left\{W_{\alpha}\right\}_{\alpha<\omega_{1}}$ would be uncountable and discrete. Now for any open subset $U$ of $X$ satisfying

$$
\partial(\bigcup \mathcal{W}) \subseteq U \subseteq \bar{U} \subseteq X
$$

the family $\left\{W_{\alpha}-\bar{U}: \alpha<\omega_{1}\right\}$ is locally countable (even discrete) and cellular since $X=$ $U \cup \bigcup \mathcal{W} \cup(X-\overline{\bigcup \mathcal{W}})$ holds. Thus there exists an $\beta=\beta(U)<\omega_{1}$ for this open $U$ such that $W_{\alpha}-\bar{U}=\emptyset$ for each $\beta \leq \alpha<\omega_{1}$. Now write

$$
\partial(\bigcup \mathcal{W})=\bigcap_{n<\omega_{0}} U_{n}=\bigcap_{n<\omega_{0}} \overline{U_{n}} \quad, \quad \overline{U_{n+1}} \subseteq U_{n}=\operatorname{int} U_{n}\left(n<\omega_{0}\right)
$$

Then we easily get

$$
W_{\alpha} \subseteq \bigcap_{n<\omega_{0}} \overline{U_{n}}=\partial(\bigcup \mathcal{W})
$$

and thus $W_{\alpha}=\emptyset$ for each $\alpha>\beta_{0}=\sup \beta_{n}$ where each $\beta_{n}=\beta\left(U_{n}\right)$ is defined as before. This contradictory result proves the assertion.

Proposition 3.2 Open discrete families in a quasi-regular $\omega$-starlindelöf space are countable.

Proof. Suppose that there exists an uncountable discrete family $\mathcal{G}$ of open sets in a quasi-regular space $X$. We may suppose without loosing the generality that $\mathcal{G}=\left\{G_{\beta}\right\}_{\beta<\omega_{1}}$. Let $\lim \left(\omega_{1}\right)$ be the set of all limit ordinals in $\left[0, \omega_{1}\right)$ and let us write $\Lambda=\lim \left(\omega_{1}\right) \cup\{0\}$. It is well known that for each $\beta<\omega_{1}$ there exists a unique ordinal number $\alpha \in \Lambda$ and a unique $n \in \omega_{0}$ such that $\beta=\alpha+n$. We will write $G_{\alpha+n}$ instead of $G_{\beta}$ whenever $\beta=\alpha+n$. Since $X$ is quasi-regular, there exists a monotonically non-increasing finite sequence of open sets $U_{\alpha+n, k}$ such that

$$
\emptyset \neq U_{\alpha+n, 0} \subseteq \bar{U}_{\alpha+n, 0} \subseteq \ldots \subseteq U_{\alpha+n, n} \subseteq \bar{U}_{\alpha+n, n} \subseteq G_{\alpha+n}
$$

Define first the point $x_{\alpha, n} \in U_{\alpha+n, 0}$ for each $\alpha \in \Lambda$ and $n \in \omega_{0}$. Let also now write

$$
\begin{gathered}
W=X-\overline{\bigcup_{\alpha \in \Lambda} \bigcup_{n \in \omega_{0}} U_{\alpha+n, n}}, \\
W_{\alpha+n, k}=U_{\alpha+n, k+1}-\bar{U}_{\alpha+n, k-1} \quad(k=1,2, \ldots, n-1), \\
W_{\alpha+n, n}=G_{\alpha+n}-\bar{U}_{\alpha+n, n-1}, \quad W_{\alpha+n, 0}=U_{\alpha+n, 1}
\end{gathered}
$$

Then we have, just as in the proof of Proposition $2.8 W_{\alpha+n, k} \cap W_{\beta+m, i} \neq \emptyset$ for $\alpha \neq \beta, \alpha, \beta \in$ $\Lambda$ and $W_{\alpha+n, k} \cap W_{\alpha+n, i} \neq \emptyset$ necessarily gives $|k-i| \leq 1$. Furthermore $\mathcal{W}=\{W\} \cup\left\{W_{\alpha+n, k}\right.$ : $\left.\alpha \in \Lambda, n \in \omega_{0}, k \leq n\right\}$ is an open cover for $X$ since $X=W \cup \bigcup\left\{G_{\alpha+n}: \alpha \in \Lambda, n \in \omega_{0}\right\}$ and $G_{\alpha+n}=\bigcup_{k \leq n} W_{\alpha+n, k}$. We furthermore have $s t^{n}\left(x_{\alpha, n}, \mathcal{W}\right) \subseteq \bigcup_{k \leq n} W_{\alpha+n, k}=G_{\alpha+n}$. For any countable subset $C$ of $X$ and for any $n \in \omega_{0}$ we evidently have a limit ordinal $\alpha$ such that $C \cap G_{\alpha+n}=\emptyset$ since members of $\mathcal{G}$ are pairwise disjoint. Thus we easily get, $x_{\alpha, n} \in X-s t^{n}(C, \mathcal{W})$ just like in the proof of Proposition 2.8. Thus the proposition follows.

Proposition 3.3 An $\aleph_{1}$-collectionwise normal $T_{1}$ space is $\omega$-starlindelöf iff it is almost starlindelöf.

Proof. Every almost starlindelöf space is evidently $\omega$-starlindelöf. Since there exists an uncountable open discrete family in any $\aleph_{1}$-collectionwise normal non almost starlindelöf $T_{1}$ space as will be shown in Proposition 3.5, the necessity follows easily from the preceding result.

Proposition 3.4 Almost Lindelöf spaces are pseudo $\omega_{1}$ - Lindelöf and pseudo $\omega_{1}$-Lindelöf spaces are almost $\omega_{1}-$ Lindelöf.

Proof. Define the open set $G(J)=X-\widetilde{\bigcup_{\alpha \in \Lambda-J} G_{\alpha}}$ for each $J \in[\Lambda] \leq \omega$ where $\mathcal{G}=\left\{G_{\alpha}\right\}_{\alpha \in \Lambda}$ is any uncountable locally countable family of open sets in almost Lindelöf space $X$. One can easily notice that $G(J) \subseteq G\left(J^{\prime}\right)$ whenever $J \subseteq J^{\prime}$. Since $\{G(J): J \in[\Lambda] \leq \omega\}$ is an open covering, by the aid of countable number of $J_{n}$ 's, one can write $X=\overline{G\left(\bigcup_{n<\omega_{0}} J_{n}\right)}$ and thus we obtain

$$
G_{\alpha}=\emptyset \quad\left(\forall \alpha \in \Lambda-\bigcup_{n<\omega_{0}} J_{n}\right)
$$

which establishes the first assertion. Now suppose secontly that, $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha<\omega_{1}}$ is an open covering of the pseudo $\omega_{1}$-Lindelöf space $X$. Since the family $\mathcal{W}=\left\{W_{\alpha}=U_{\alpha}-\overline{\bigcup_{\beta<\alpha} U_{\beta}}\right.$ : $\left.\alpha<\omega_{1}\right\}$ is locally countable, all but a countable number of members of it are necessarily empty. This proves the second assertion.

The following is a revised form of Theorem 3.10 of [P]:

Proposition 3.5 An $\aleph_{1}$-collectionwise normal almost $\omega_{1}$-Lindelöf $T_{1}$ space is almost starlindelöf.

Proof. Suppose that an open cover $\mathcal{U}$ of an almost $\omega_{1}$-Lindelöf, $\aleph_{1}$-collectionwise normal $T_{1}$ space $X$ satisfies $X \neq \overline{s t(C, \mathcal{U})}$ for each countable subset $C$ of $X$. Then by transfinite induction we can easily define the subsets $A_{\alpha}=\left\{x_{\beta}: \beta<\alpha\right\}$ and points $x_{\alpha} \in X$ such that $x_{\alpha} \in X-\overline{\operatorname{st}\left(A_{\alpha}, \mathcal{U}\right)}$ for each $\alpha<\omega_{1}$. Then the open sets $U_{\alpha}^{*}=\operatorname{st}\left(x_{\alpha}, \mathcal{U}\right)-\overline{\operatorname{st}\left(A_{\alpha}, \mathcal{U}\right)}$ are pairwise disjoint since $U_{\alpha}^{*} \cap U_{\beta}^{*} \subseteq \operatorname{st}\left(x_{\alpha}, \mathcal{U}\right)-\overline{\operatorname{st}\left(A_{\beta}, \mathcal{U}\right)}$ whenever $\alpha<\beta$ and besides $x_{\alpha} \in U_{\alpha}^{*}$ holds for $\alpha<\omega_{1}$. Furthermore the supposition $x_{\alpha}, x_{\beta} \in U \in \mathcal{U}$ for $\alpha<\beta$ easily yields the contradiction $x_{\beta} \in \operatorname{st}\left(x_{\alpha}, \mathcal{U}\right)-\overline{\operatorname{st}\left(A_{\beta}, \mathcal{U}\right)}=\emptyset$. Thus $A=\left\{x_{\alpha}: \alpha<\omega_{1}\right\}$ is closed-discrete and since $X$ is an $\aleph_{1}$-collectionwise normal, one can define an open discrete family $\mathcal{W}=\left\{W_{\alpha}: \alpha<\omega_{1}\right\}$ such that $x_{\alpha} \in W_{\alpha} \subseteq \overline{W_{\alpha}} \subseteq U_{\alpha}^{*}$ for each $\alpha<\omega_{1}$. Then the open cover $\{X-\overline{\bigcup \mathcal{W}}\} \cup\left\{U_{\alpha}^{*}: \alpha<\omega_{1}\right\}$ of $X$ would have no countable subcollection whose union set is dense in $X$. Thus the statement is established.

Various strong chain conditions have been defined and examined in [EGW]. One of these is CPRCC i.e. the closure preserving refinement chain condition. A topological space $X$ satisfies CPRCC (or briefly $X$ is called as a CPRCC space) iff any cellular family $\mathcal{G}=\left\{G_{\alpha}: \alpha \in \Lambda\right\}$ (not necessarily covering) has a non-degenerate closure preserving open refinement $\mathcal{U}=\left\{U_{\alpha}: \alpha \in \Lambda\right\}$ in which non degenerate means $U_{\alpha} \neq \emptyset$ whenever $G_{\alpha}$ is non-empty and $U_{\alpha} \subseteq G_{\alpha}$ for each $\alpha \in \Lambda$. It is proved in [EGW] among several results that i) there exists a Tikhonov CPRCC space which is not discrete, ii) there also exists a CPRCC $T_{2}$ space $X$ possessing a cellular family having no non-degenerate locally finite open refinement. We give now a characterization of being almost $\omega_{1}$-Lindelöf in quasi-regular CPRCC spaces:

Proposition 3.6 A quasi-regular CPRCC space $X$ is almost starlindelöf iff $X$ is almost $\omega_{1}$-Lindelöf.

Proof. If an open covering $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha<\omega_{1}}$ of $X$ have no countable subfamily with a dense union set then we may suppose without loosing the generality that open set $G_{\alpha}=$ $U_{\alpha}-\overline{\bigcup\left\{U_{\beta}: \beta<\alpha\right\}}$ is non-empty for each $\alpha<\omega_{1}$. Then $\mathcal{G}=\left\{G_{\alpha}\right\}_{\alpha<\omega_{1}}$ would be a cellular family and since $X$ is quasi-regular CPRCC space there exists a non-degenerate closure preserving open family $\mathcal{W}=\left\{W_{\alpha}: \alpha<\omega_{1}\right\}$ such that $\overline{W_{\alpha}} \subseteq G_{\alpha}$ for each $\alpha<\omega_{1}$ and thus $\overline{\bigcup\left\{W_{\alpha}: \alpha<\omega_{1}\right\}}=\bigcup\left\{\overline{W_{\alpha}}: \alpha<\omega_{1}\right\} \subseteq \bigcup \mathcal{G}$. Then the open cover $\mathcal{G}^{*}=\mathcal{G} \cup\{X-\overline{\bigcup \mathcal{W}}\}$ would evidently be a point finite open covering satisfying

$$
W_{\beta} \cap\left(\overline{\bigcup_{\alpha \in \Lambda} G_{\alpha}} \cup \overline{X-\overline{\bigcup \mathcal{W}}}\right)=\emptyset
$$

for any countable $\Lambda \subseteq \omega_{1}$ and for any $\beta>\sup \Lambda$. Thus $\operatorname{st}\left(C, \mathcal{G}^{*}\right)$ is not dense for any countable $C \subseteq X$ and therefore necessity follows easily. Suppose now conversely that $X$ is an almost $\omega_{1}$-Lindelöf quasi-regular CPRCC space and suppose additionally that there exists an open cover $\mathcal{U}$ of $X$ such that $X \neq \overline{s t(C, \mathcal{U})}$ for any countable subset $C$ of $X$. Then as in above proof, there exists an open family $\mathcal{V}=\left\{V_{\alpha}: \alpha<\omega_{1}\right\}$ such that

$$
V_{\alpha} \subseteq U_{\alpha}^{*}=\operatorname{st}\left(x_{\alpha}, \mathcal{U}\right)-\overline{\operatorname{st}\left(A_{\alpha}, \mathcal{U}\right)} \quad\left(\alpha<\omega_{1}\right)
$$

and furthermore $\overline{\bigcup\left\{V_{\alpha}: \alpha \in \Lambda\right\}} \subseteq \bigcup\left\{U_{\alpha}^{*}: \alpha \in \Lambda\right\}$ for each subset $\Lambda$ of $\omega_{1}$. Thus the open cover $\{X-\overline{\bigcup \mathcal{V}}\} \cup\left\{U_{\alpha}^{*}: \alpha<\omega_{1}\right\}$ would have no countable subcollection whose union
set is dense in $X$. Thus sufficiency follows.

We close this section with the following characterizations:

Proposition 3.7 i) A space is compact iff it is metacompact and starcompact iff it is strongly paracompact and $\omega$-starcompact. ii) A space is Lindelöf iff it is strongly paralindelöf and $\omega$ starlindelöf.

Proof. i) The first assertion is actually known (see Thm 2.4.5. of [vDRRT]). We give here another proof. Let $\mathcal{G}$ be any open covering of the metacompact and starcompact space $X$. Then a pointwise Worell refinement $\mathcal{U}$ of $\mathcal{G}$ exists (see [B], Thm 3.5.) i.e., for each $x \in X$ a finite subfamily $\mathcal{G}_{x}$ of $\mathcal{G}$ exists for which $\operatorname{st}(x, \mathcal{U}) \subseteq \bigcup \mathcal{G}_{x}$ holds. By the starcompactness a finite subset $A$ of $X$ satisfiying $X=\operatorname{st}(A, \mathcal{U})=\bigcup_{x \in A}\left(\bigcup \mathcal{G}_{x}\right)$ exists, meaning that $\mathcal{G}^{*}=\left\{G: \exists x \in A, G \in \mathcal{G}_{x}\right\}$ is the required finite subcovering of $\mathcal{G}$. The second characterization is an easy consequence of the definitions since any open covering $\mathcal{G}$ of a strongly paracompact space may be taken as satisfying the condition $\operatorname{ord}(G, \mathcal{G})<\aleph_{0}$ for any $G \in \mathcal{G}$. ii) Left to the reader.

## $\underline{4}$ Some related questions

We finally close the paper by posing some non straightforward related questions.

Question 4.1: Does there exist a non countably compact Hausdorff space satisfying "every point finite countable open covering has a finite subcovering" condition and having an infinite point-finite open cover?

Question 4.2: Does there exist a regular almost countably compact Hausdorff space which is not almost starcompact?

Question 4.3: Does there exist a non quasi-regular, almost starcompact Hausdorff space which is not almost countably compact?

Question 4.4: Almost starlindelöf is equivalent to almost $\omega_{1}$-Lindelöf in collectionwise normal Hausdorff spaces. What about if "collectionwise" removed?

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