# FULL-INFORMATION BEST CHOICE PROBLEMS WITH RECALL OF OBSERVATIONS, UNCERTAINTY OF SELECTION AND A RANDOM NUMBER OF OBSERVATIONS 

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Received April 3, 2000; revised September 29, 2000


#### Abstract

A full-information best choice problem is considered. A sequence of $N$ iid random variables ( $r v$ 's) with a known continuous distribution function ( $d f$ ) is observed. The number of observations $N$ is a positive $r v$ independent of observations. The objective is to maximize the probability of selecting the best (largest) observation when one choice can be made. At each stage a solicitation of the present observation as well as of any previous ones is allowed. If the $(k-t)$ th observation with the value $x$ is solicited at the $k$ th stage, the probability of successful solicitation may depend on $t$ and $x$. General properties of optimal strategies are shown and some natural cases are examined in detail. Optimal strategies and their probability of success (selecting the best) are derived.


## 1. Introduction

The following full-information best choice problem was studied by Gilbert and Mosteller [5]. A known number, $N$, of iid $r v$ 's $X_{1}, X_{2}, \ldots, X_{N}$ from a known continuous df $F$ are observed sequentially. After $X_{n}$ is observed it must be either accepted (then the observation process is terminated) or rejected (then the observation is continued) considering the possibility of obtaining a better offer against the risk of losing the current offer. The objective is to maximize the probability of selecting the largest observation assuming that one choice can be made and neither recall nor uncertainty of selection is allowed.

For a finite number of observations the so-called monotone case was obtained. The optimal strategy is to accept (if possible) the first $X_{n}$ which is largest one so far and exceeds $x_{n}$, where the sequence of optimal decision levels $\left(x_{n}\right)$ is non-increasing.

The full-information best choice problem when the number of observations $N$ is random was considered by Porosiński [10]. In this model the observer incurs an additional risk. Since $N$ is unknown, if he rejects any observation, in case it was the last one he receives nothing. A class of $d f$ 's of $N$ for which the monotone case occurs was characterized and the solution for this case was given.

Petruccelli [8] studied problems with recall of observations and uncertainty of selection depending on the observation when $N$ is fixed. Optimal strategies and their probabilities of success in some important cases were derived. Similar problems were considered by Tamaki [17] and Ano [2].

The results of this paper extend those obtained by Petruccelli [8]. The full-information best choice problem with a random number of observations is discussed. In Section 2 we

[^0]state assumptions and provide basic formulae being useful in finding an optimal strategy. Section 3 presents several special cases in detail.

## 2. Model formulation

Let $X_{1}, X_{2}, X_{3}, \ldots$ be iid rv's with a continuous $d f F$, defined on the probability space $(\Omega, \mathcal{F}, P)$. Since $F$ is known and continuous, without loss of generality it can be additionally assumed that $X_{n}$ has the uniform $d f$ on the interval $[0,1], n=1,2, \ldots$.

Let $L_{k}=\max \left\{X_{1}, \ldots, X_{k}\right\}$ and call $X_{i}$ a candidate if $L_{k}=X_{i}, 1 \leq i \leq k$.
For each $k$, the state of the process after having observed $X_{k}$ is denoted by $(x, k, t), t<k$, if $X_{k-t}=L_{k}=x$ and if the $(k-t)$ th observation has not yet been solicited (i.e. $t$ is the relative position of the candidate). The state ( $x, k, \infty$ ) describes the situation when $L_{k}=x$ and the largest among the first $k$ observations has been unsuccessfully solicited.

Let $\alpha(x, t)$ denote the probability of successful solicitation in the state $(x, k, t)$. It seems reasonable to assume that $\alpha(u, \infty)=0,0 \leq u \leq 1$ (i.e. only one solicitation is allowed for each observation) and $\alpha(u, t)$ is non-increasing in $u$ for fixed $t$ and in $t$ for fixed $u$. The intuitive reasoning for the last assumption is that the probability of accepting an offer of employment by an applicant decreases as time between the interview and offer increases and that the more competitive is the applicant the lower are chances of his acceptance an offer.

The number of observations $N$ is assumed to be a positive $r v$ with a known $d f$ given by $P(N=n)=p_{n}, n=0,1, \ldots$

In the state $(x, k, t)$ the observer has two options: he may solicit the current best observation or to wait for $(k+1)$ th one. We will denote the probability of selecting the largest (winning) by $\delta_{b}(x, k, t)$ if the observer solicits the candidate (and continues in an optimal manner if the candidate rejects an offer) and by $\delta_{f}(x, k, t)$ if the observer does not choose the current best and waiting for the next observation (assuming optimal behaviour for the future). If ( $x, k, t$ ) was the last offer the observer is forced to solicit it in state $(x, k+1, t+1)$.

In $(x, k, t)$, an optimal strategy solicits the candidate iff $\delta_{b}(x, k, t)$ exceeds $\delta_{f}(x, k, t)$.
Let $\delta(x, k, t)=\max \left\{\delta_{b}(x, k, t), \delta_{f}(x, k, t)\right\}$ and let $\delta(x, k, \infty)$ stand for the probability of winning in state $(x, k, \infty)$ when the observer has to wait for the next candidate.

The following recursive formulae are easily derived:

$$
\begin{align*}
\delta_{f}(x, k, t)= & \frac{\pi_{k+1}}{\pi_{k}}\left(\int_{x}^{1} \delta(y, k+1,0) d y+x \delta(x, k+1, t+1)\right)  \tag{1}\\
& +\frac{p_{k}}{\pi_{k}} \alpha(x, t+1)  \tag{2}\\
\delta_{b}(x, k, t) & =\alpha(x, t) \sum_{n=k}^{\infty} \frac{p_{n}}{\pi_{k}} x^{n-k}+(1-\alpha(x, t)) \delta(x, k, \infty)  \tag{3}\\
\delta(x, k, \infty) & =\frac{\pi_{k+1}}{\pi_{k}}\left(\int_{x}^{1} \delta(y, k+1,0) d y+x \delta(x, k+1, \infty)\right)
\end{align*}
$$

where $\pi_{k}=P(N \geq k)=\sum_{n=k}^{\infty} p_{n}$. They imply

$$
\begin{align*}
\pi_{k} \delta_{f}(x, k, t)= & \pi_{k} \delta(x, k, \infty)+x \pi_{k+1}(\delta(x, k+1, t+1)  \tag{4}\\
& -\delta(x, k+1, \infty)) \\
& +p_{k} \alpha(x, t+1)  \tag{5}\\
\pi_{k} \delta(x, k, \infty)= & \sum_{j=0}^{\infty} x^{j} \int_{x}^{1} \pi_{j+k+1} \delta(y, j+k+1,0) d y  \tag{6}\\
\pi_{k} \delta_{b}(x, k, t)= & x \pi_{k+1} \delta_{b}(x, k+1, t)+\alpha(x, t) p_{k} \\
& +(1-\alpha(x, t)) \int_{x}^{1} \pi_{k+1} \delta(y, k+1,0) d y
\end{align*}
$$

and

$$
\begin{align*}
\pi_{k}\left(\delta_{f}(x, k, t)-\delta_{b}(x, k, t)\right)= & x \pi_{k+1}\left(\delta(x, k+1, t+1)-\delta_{b}(x, k+1, t)\right)  \tag{7}\\
& +\alpha(x, t) \int_{x}^{1} \pi_{k+1} \delta(y, k+1,0) d y \\
& +p_{k}(\alpha(x, t+1)-\alpha(x, t))
\end{align*}
$$

## 3. Special cases

3.1. Case 1. $\alpha(x, t)=\alpha(x)$.

In this case the probability of a successful solicitation is independent of time $t$ and, intuitively, an optimal strategy should induce the observer to observe the next observation since nothing is lost in doing so.

Theorem 1. For the Case 3.1:
An optimal strategy is to wait until no observation remain and to solicit the best candidate. The probability of success $P$ (win) using the optimal strategy is

$$
\begin{equation*}
P(w i n)=\int_{0}^{1} \sum_{n=1}^{\infty} p_{n} \alpha(x) n x^{n-1} d x \tag{8}
\end{equation*}
$$

Proof. From (2)

$$
\begin{equation*}
\pi_{k} \delta_{b}(x, k, t)=(1-\alpha(x)) \pi_{k} \delta(x, k, \infty)+\alpha(x) \sum_{j=0}^{\infty} p_{j+k} x^{j} \tag{9}
\end{equation*}
$$

and $\delta_{b}(x, k, t)$ is independent of $t$. Thus putting $\delta_{b}(x, k+1, t+1)$ instead of $\delta_{b}(x, k+1, t)$ in the right-hand side of (7), we have that $\delta_{f}(x, k, t) \geq \delta_{b}(x, k, t)$ for every $k, t$.

This and (4) give

$$
\begin{aligned}
\pi_{k} \delta_{f}(x, k, t)- & x \pi_{k+1} \delta(x, k+1, t+1) \\
& =\pi_{k} \delta_{f}(x, k, t)-x \pi_{k+1} \delta_{f}(x, k+1, t+1) \\
& =\pi_{k} \delta(x, k, \infty)-x \pi_{k+1} \delta(x, k+1, \infty)+p_{k} \alpha(x)
\end{aligned}
$$

and, as a consequence,

$$
\begin{equation*}
\pi_{k} \delta_{f}(x, k, t)=\pi_{k} \delta(x, k, \infty)+\alpha(x) \sum_{i=0}^{\infty} p_{k+i} x^{i} \tag{10}
\end{equation*}
$$

So, $\delta_{f}(x, k, t)$ also does not depend on $t$ and satisfies the equation

$$
\pi_{k} \delta_{f}^{\prime}(x, k, t)-x \pi_{k+1} \delta_{f}^{\prime}(x, k+1, t)=p_{k} \alpha^{\prime}(x)
$$

(if derivative $\alpha^{\prime}(x)$ with respect to $x$ exists) which gives

$$
\pi_{k} \delta_{f}^{\prime}(x, k, t)=\alpha^{\prime}(x) \sum_{i=0}^{\infty} p_{i+k} x^{i}
$$

Integrating it by parts, taking into account the initial condition $\delta_{f}(1, k, t)=\alpha(1)$, we obtain

$$
\begin{equation*}
\pi_{k} \delta_{f}(x, k, t)=\alpha(x) \sum_{i=0}^{\infty} p_{i+k} x^{i}+\int_{x}^{1} \alpha(y) \sum_{i=1}^{\infty} i p_{i+k} y^{i-1} d y \tag{11}
\end{equation*}
$$

From (11), (10) and (9) we have the following formulae

$$
\begin{align*}
\pi_{k} \delta(x, k, \infty) & =\int_{x}^{1} \alpha(y) \sum_{i=1}^{\infty} i p_{i+k} y^{i-1} d y  \tag{12}\\
\pi_{k} \delta_{b}(x, k, t) & =\alpha(x) \sum_{i=0}^{\infty} p_{i+k} x^{i}+(1-\alpha(x)) \int_{x}^{1} \alpha(y) \sum_{i=1}^{\infty} i p_{i+k} y^{i-1} d y \tag{13}
\end{align*}
$$

It is easy to check that in this case the probability of success $P(w i n)$ is given by (8). $\quad \Lambda$
In a special case when $N$ has the geometric distribution with a parameter $p$ (i.e. $P(N=$ $n)=p q^{n}, p, q>0, p+q=1$ ), we have

$$
\begin{aligned}
\delta(x, k, \infty) & =\int_{x}^{1} \alpha(y) p q /(1-q y)^{2} d y \\
\delta_{b}(x, k, t) & =\alpha(x) p /(1-q x)+(1-\alpha(x)) \delta(x, k, \infty) \\
\delta_{f}(x, k, t) & =\alpha(x) p /(1-q x)+\delta(x, k, \infty)
\end{aligned}
$$

(all the functions are independent of $k$ and $t$ ) and

$$
P(w i n)=\int_{0}^{1} \alpha(y) p q /(1-q y)^{2} d y
$$

3.2. Case 2. $\alpha(x, 0)=\alpha(x), \alpha(x, t)=\beta(x), t \geq 1, \alpha(x) \geq \beta(x), x \in[0,1]$.

In Case 2 , from (2), $\delta_{b}(x, k, t)$ has the same value for all $t \geq 1$. Thus, putting $\delta_{b}(x, k+1, t+1)$ instead of $\delta_{b}(x, k+1, t)$ in (7), we have

$$
\begin{equation*}
\delta_{f}(x, k, t) \geq \delta_{b}(x, k, t) \tag{14}
\end{equation*}
$$

or, equivalently, $\delta(x, k, t)=\delta_{f}(x, k, t)$ for $t=1, \ldots, k$. This implies

$$
\begin{gather*}
\delta_{b}(x, k, 0)=\alpha(x) \sum_{n=k}^{\infty} \frac{p_{n}}{\pi_{k}} x^{n-k}+(1-\alpha(x)) \delta(x, k, \infty)  \tag{15}\\
\delta_{f}(x, k, 0)=\beta(x) \sum_{n=k}^{\infty} \frac{p_{n}}{\pi_{k}} x^{n-k}+\delta(x, k, \infty) \tag{16}
\end{gather*}
$$

and $\delta_{f}(x, k, 0) \geq \delta_{b}(x, k, 0) \quad$ iff

$$
\begin{equation*}
\pi_{k} \delta(x, k, \infty) \geq\left(1-\frac{\beta(x)}{\alpha(x)}\right) \sum_{n=k}^{\infty} p_{n} x^{n-k} \tag{17}
\end{equation*}
$$

Suppose $\beta(x) / \alpha(x)$ is non-increasing in $x$. Then the right-hand side of (17) is non-decreasing in $x$ and is positive for $x=1$. The left-hand side of this formula takes the value 0 in $x=1$. It is decreasing in $x$ because

$$
\begin{aligned}
\frac{d}{d x}\left(\pi_{k} \delta(x, k, \infty)\right) & =\frac{d}{d x}\left(\sum_{j=0}^{\infty} x^{j} \int_{x}^{1} \pi_{j+k+1} \delta(y, j+k+1,0) d y\right) \\
& =\sum_{j=0}^{\infty} x^{j} \pi_{j+k+1}(\delta(x, j+k+1, \infty)-\delta(x, j+k+1,0))
\end{aligned}
$$

and from (16)

$$
\delta(x, k, \infty)-\delta(x, k, 0) \leq \delta(x, k, \infty)-\delta_{f}(x, k, 0)=-\beta(x) \sum_{n=k}^{\infty} \frac{p_{n}}{\pi_{k}} x^{n-k} \leq 0
$$

for every $k$. Thus there is a $d_{k}$ such that (17) holds iff $x \leq d_{k}$.

The problem is to find values of decision levels $d_{k}, k=1,2, \ldots$, and to show the monotonicity of $\left(d_{k}\right)$ (in order to get the monotone case). It is hard in a general case (cf. Petruccelli [8] for fixed $N$ ) but relatively easy for an important case of geometric $N$.

Let the number of observations have the geometric distribution with a parameter $p$, i.e. $p_{n}=p q^{n}, p, q>0, p+q=1$.

Theorem 2. In Case 3.2 for $N$ being geometric with the parameter $p$ :
If $\beta(x) / \alpha(x)$ is non-increasing in $x$ then there exists an optimal level d, independent of $t$, such that an optimal procedure is to solicit the first candidate $X_{k}$ which exceeds the level d. If this solicitation is unsuccesful then each candidate that appears is solicited until solicitation is successful or until no observation remain.
The optimal level $d$ is the largest $x \in(0,1]$ satisfying the inequality

$$
\begin{align*}
\exp \left(-\int_{x}^{1} \frac{q}{1-q s} \alpha(s) d s\right) \int_{x}^{1} \frac{p q \alpha(t)}{(1-q t)^{2}} \exp ( & \left.-\int_{t}^{1} \frac{q}{1-q s} \alpha(s) d s\right) d t  \tag{18}\\
& \geq \frac{p}{1-q x}\left(1-\frac{\beta(x)}{\alpha(x)}\right)
\end{align*}
$$

if the inequality (18) has a solution in $(0,1]$ and $d=0$ otherwise.
Proof. Memoryless property of a geometric distribution let us suppose that

$$
\begin{equation*}
\delta_{b}(x, k, t), \delta_{f}(x, k, t), \delta(x, k, \infty) \quad \text { do not depend on } k \tag{19}
\end{equation*}
$$

and $k$ will be omitted. The inequality (17) giving an optimal level $d_{k}$ now can be rewrite as

$$
\begin{equation*}
\delta(x, \infty) \geq \frac{p}{1-q x}\left(1-\frac{\beta(x)}{\alpha(x)}\right) \tag{20}
\end{equation*}
$$

This implies that the optimal level $d_{k}=d$ is constant and

$$
\begin{align*}
\delta(x, \infty) & =\frac{q}{1-q x} \int_{x}^{1} \delta(y, 0) d y  \tag{21}\\
\delta_{f}(x, t) & =q\left(\int_{x}^{1} \delta(y, 0) d y+x \delta(x, t+1)\right)+p \alpha(x, t+1)  \tag{22}\\
& =(1-q x) \delta(x, \infty)+q x \delta(x, t+1)+p \beta(x) \\
\delta_{b}(x, t) & =\alpha(x, t) \frac{p}{1-q x}+(1-\alpha(x, t)) \delta(x, \infty)  \tag{23}\\
& = \begin{cases}\delta(x, \infty)+\alpha(x)(p /(1-q x)-\delta(x, \infty)) & \text { for } t=0 \\
\delta(x, \infty)+\beta(x)(p /(1-q x)-\delta(x, \infty)) & \text { for } t>0\end{cases}
\end{align*}
$$

From (22) for $t \geq 0$ and (14) it follows that

$$
\delta_{f}(x, t)-q x \delta_{f}(x, t+1)=(1-q x) \delta(x, \infty)+p \beta(x)
$$

which gives

$$
\begin{equation*}
\delta_{f}(x, t)=\delta(x, \infty)+\beta(x) \frac{p}{1-q x} \tag{24}
\end{equation*}
$$

Therefore $\delta_{f}(x, t)$ does not depend on $t$ and only $\delta(x, \infty)$ should be found.
Taking into account (14) and (20) and an assumption on $\alpha(x, t)$ for this case the form of $\delta(x, \infty)$ can be calculated in two steps.

Let $x>d$. Then

$$
\delta(x, \infty)=\frac{q}{1-q x} \int_{x}^{1} \delta_{b}(y, 0) d y
$$

and using (23) for $t=0$ we obtain a derivative

$$
\begin{aligned}
\delta^{\prime}(x, \infty) & =\frac{q^{2}}{(1-q x)^{2}} \int_{x}^{1} \delta_{b}(y, 0) d y-\frac{q}{1-q x} \delta_{b}(x, 0) \\
& =\frac{q}{1-q x} \alpha(x) \delta(x, \infty)-\frac{p q \alpha(x)}{(1-q x)^{2}}
\end{aligned}
$$

This linear differential equation with an initial condition $\delta(1, \infty)=0$ has a solution of the form

$$
\delta(x, \infty)=\exp \left(-\int_{x}^{1} \frac{q}{1-q s} \alpha(s) d s\right) \int_{x}^{1} \frac{p q \alpha(t)}{(1-q t)^{2}} \exp \left(-\int_{t}^{1} \frac{q}{1-q s} \alpha(s) d s\right) d t
$$

Consider now $x \leq d$. Then

$$
\delta(x, \infty)=\frac{q}{1-q x}\left(\int_{x}^{d} \delta_{f}(y, 0) d y+\int_{d}^{1} \delta_{b}(y, 0) d y\right)
$$

and using (24) we obtain a derivative

$$
\delta^{\prime}(x, \infty)=-\frac{p q \beta(x)}{(1-q x)^{2}}
$$

For an initial condition $\delta(d, \infty)=\delta(d+, \infty)$ a solution has the form

$$
\delta(x, \infty)=\delta(d+, \infty)-\int_{x}^{d} \frac{p q \beta(t)}{(1-q t)^{2}} d t
$$

It is easy to check that obtained $\delta(x, \infty), \delta_{f}(x, t), \delta_{b}(x, t)$ fulfil (1)-(3) which confirmes the supposition (19).

If $\alpha(x)$ is assumed to be constant $\alpha(x)=\alpha$ then a solution has an especially nice form. Since for $x \geq d$

$$
\delta(x, \infty)= \begin{cases}\frac{\alpha}{1-\alpha}\left(\left(\frac{p}{1-q x}\right)^{\alpha}-\frac{p}{1-q x}\right) & \text { for } \alpha<1 \\ -\frac{p}{1-q x} \ln \left(\frac{p}{1-q x}\right) & \text { for } \alpha=1\end{cases}
$$

the optimal level $d$ is the largest $x \in(0,1]$ for which the inequality

$$
\frac{\alpha}{1-\alpha}\left(\left(\frac{p}{1-q x}\right)^{\alpha-1}-1\right) \geq 1-\frac{\beta(x)}{\alpha}
$$

for $\alpha<1$ or

$$
-\ln \left(\frac{p}{1-q x}\right) \geq 1-\beta(x)
$$

for $\alpha=1$ is satisfied, if this inequality has a solution in $(0,1]$, and $d=0$ otherwise.
The probability $P($ win $)$ of choosing the largest observation using the optimal strategy can be calculated in the following way.
Consider the case of constant $\alpha$. Let $\nu(n, k, x)$ be the number of $X_{k+1}, \ldots, X_{n}$, given $N=n$, which exceed $x . \nu(n, k, x)$ has a binomial distribution with parameters $n-k$ and $1-x$. Let $g(n, \nu, x)$ for $\nu \geq 1$ be the probability, given $N=n, \nu$ and $X_{k}=L_{k}=x$, of choosing
the largest observation using the strategy of immediately soliciting each candidate among $X_{k+1}, \ldots, X_{n}$ that appears. Petruccelli [8] has shown that

$$
g(n, \nu, x)=\alpha \prod_{j=2}^{\nu}\left(1-\frac{\alpha}{j}\right)
$$

Let $\tau$ denote the optimal strategy with a constant level $d$. That is $\tau=k$ iff the optimal strategy chooses $X_{k}$. Since

$$
\begin{aligned}
P(\text { win } \mid N=n, \tau=k)= & P\left(\left(\left\{X_{k}=L_{n}\right\} \cap\left\{L_{k-1} \leq d<X_{k}\right\}\right)\right. \\
& =d^{k-1} \int_{d}^{1} \alpha x^{n-k} d x \\
& +d^{k-1} \sum_{\nu=1}^{n-k}\binom{n-k}{\nu} \int_{d}^{1}(1-\alpha)(1-x)^{\nu} x^{n-k-\nu} g(n, \nu, x) d x, \\
P(\text { win })= & \sum_{n=1}^{\infty} p q^{n} \sum_{k=1}^{n} P(\text { win } \mid N=n, \tau=k)=\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} p q^{n} P(\text { win } \mid N=n, \tau=k) \\
= & \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} p q^{k} d^{k-1}\left\{\alpha \int_{d}^{1}(q x)^{m} d x\right. \\
& \left.+\sum_{\nu=1}^{m} q^{m}\binom{m}{\nu}(1-\alpha) \int_{d}^{1}(1-x)^{\nu} x^{m-v} d x \alpha \prod_{j=2}^{\nu}\left(1-\frac{\alpha}{j}\right)\right\} \\
= & \frac{\alpha p q}{1-q d} \sum_{\nu=1}^{\infty} \prod_{j=1}^{\nu}\left(1-\frac{\alpha}{j}\right) \sum_{s=0}^{\infty}\binom{\nu+s}{\nu} \int_{d}^{1}(1-x)^{\nu} q^{\nu}(q x)^{s} d x \\
= & \alpha \frac{p}{1-q d}\left\{q \sum_{\nu=1}^{\infty} \int_{d}^{1} \frac{(q-q x)^{\nu}}{(1-q x)^{\nu+1}} d x \prod_{j=1}^{\nu}\left(1-\frac{\alpha}{j}\right)-\ln \left(\frac{p}{1-q d}\right)\right\} .
\end{aligned}
$$

If $d \in(0,1]$, the probability $P($ win $)$ attains its local maximum at optimal $d$ (the derivative of $P($ win $)$ with respect to $p$ is 0 at $d$ ) which gives

$$
\begin{equation*}
P(w i n)=\alpha \frac{p}{1-q d}\left(1+\sum_{\nu=1}^{\infty}\left(\frac{q-q d}{1-q d}\right)^{\nu} \prod_{j=1}^{\nu}\left(1-\frac{\alpha}{j}\right)\right) \tag{25}
\end{equation*}
$$

3.3. Case 3. $\alpha(x, t)=\alpha(x)(\beta(x))^{t}, t \geq 0,0<\alpha(x), \beta(x) \leq 1$.

In order to get an evident form of an optimal solution consider the problem with a number of observations $N$ having the geometric distribution with a parameter $p$, i.e. $p_{n}=p q^{n}, p, q>$ $0, p+q=1$.

Theorem 3. In Case 3.3 for geometric $N$ :
There exists an optimal level $d$, independent of $t$, being the largest $x \in(0,1]$ satisfying the
inequality

$$
\begin{align*}
\exp \left(-\int_{x}^{1} \frac{q}{1-q s} \alpha(s) d s\right) \int_{x}^{1} \frac{p q \alpha(t)}{(1-q t)^{2}} \exp ( & \left.-\int_{t}^{1} \frac{q}{1-q s} \alpha(s) d s\right) d t  \tag{26}\\
& \geq \frac{p}{1-q x}-\frac{p \beta(x)}{1-q x \beta(x)}
\end{align*}
$$

if this inequality has a solution in $(0,1]$ and $d=0$ otherwise.
An optimal procedure is to solicit the first candidate $X_{k}$ which exceeds the level d. If this solicitation is unsuccesful then each candidate that appears is solicited until solicitation is obtained or until no observation remain.

Proof. Geometric $d f$ for $N$ let us assume, as in Case 3.2, that

$$
\begin{equation*}
\delta_{b}(x, k, t), \delta_{f}(x, k, t), \delta(x, k, \infty) \text { do not depend on } k \tag{27}
\end{equation*}
$$

and $k$ will be omitted. This implies that (1) - (3) are equivalent to

$$
\begin{align*}
\delta(x, \infty) & =\frac{q}{1-q x} \int_{x}^{1} \delta(y, 0) d y  \tag{28}\\
\delta_{b}(x, t) & =\delta(x, \infty)+\alpha(x)(\beta(x))^{t}\left(\frac{p}{1-q x}-\delta(x, \infty)\right)  \tag{29}\\
\delta_{f}(x, t) & =(1-q x) \delta(x, \infty)+q x \delta(x, t+1)+p \alpha(x)(\beta(x))^{t+1} \tag{30}
\end{align*}
$$

Now we will show that

$$
\begin{equation*}
\delta_{b}(x, t+1) \geq \delta_{f}(x, t+1) \Longrightarrow \delta_{b}(x, t) \geq \delta_{f}(x, t) \tag{31}
\end{equation*}
$$

for every $t \geq 0$.
Taking into account reccurence relations

$$
\begin{aligned}
\delta_{b}(x, t+1)-\delta(x, \infty) & =\beta(x)\left(\delta_{b}(x, t)-\delta(x, \infty)\right) \\
\delta_{f}(x, t)-\delta(x, \infty) & =q x(\delta(x, t+1)-\delta(x, \infty))+\alpha(x)(\beta(x))^{t+1}
\end{aligned}
$$

and an assumption in (31) the proposition can be proved as follows

$$
\begin{aligned}
\beta(x) & \left(\delta_{b}(x, t)-\delta_{f}(x, t)\right) \\
& =\beta(x)\left(\left(\delta_{b}(x, t)-\delta(x, \infty)\right)-\left(\delta_{f}(x, t)-\delta(x, \infty)\right)\right) \\
& =\left(\delta_{b}(x, t+1)-\delta(x, \infty)\right)-q x \beta\left((\delta(x, t+1)-\delta(x, \infty))-p \alpha(x)(\beta(x))^{t+2}\right. \\
& =\delta_{b}(x, t+1)-\delta(x, \infty)-q x\left(\left(\delta_{b}(x, t+2)-\delta(x, \infty)\right)-p \alpha(x)(\beta(x))^{t+2}\right. \\
& \geq \delta_{b}(x, t+1)-\delta(x, \infty)-\left(q x\left((\delta(x, t+2)-\delta(x, \infty))+p \alpha(x)(\beta(x))^{t+2}\right)\right. \\
& =\delta_{b}(x, t+1)-\delta(x, \infty)-\left(\delta_{f}(x, t+1)-\delta(x, \infty)\right) \\
& =\delta_{b}(x, t+1)-\delta_{f}(x, t+1) \geq 0 .
\end{aligned}
$$

So, for every $t$, if $\left(\delta_{b}(x, t) \geq \delta_{f}(x, t)\right)$ then $\left(\delta_{b}(x, 0) \geq \delta_{f}(x, 0)\right)$ and the observer should solicit a current candidate at the state $(x, k, 0)$ only.
Now we will find a set of $x$ such that $\delta_{f}(x, 0)>\delta_{b}(x, 0)$. Proposition (31) gives (30) for such $x$ as

$$
\begin{equation*}
\delta_{f}(x, t)=(1-q x) \delta(x, \infty)+q x \delta_{f}(x, t+1)+p \alpha(x)(\beta(x))^{t+1} \tag{32}
\end{equation*}
$$

and, as a consequence, a formula

$$
\delta_{f}(x, t)=\delta(x, \infty)+(\beta(x))^{t} \frac{p \alpha(x) \beta(x)}{1-q x \beta(x)}
$$

This and (29) for $t=0$ implies that $\delta_{f}(x, 0) \geq \delta_{b}(x, 0)$ iff

$$
\begin{equation*}
\delta(x, \infty) \geq \frac{p}{1-q x}-\frac{p \beta(x)}{1-q x \beta(x)} \tag{33}
\end{equation*}
$$

The function $\delta(x, \infty)$ is decreasing in $x$ and $\delta(1, \infty)=0$ (cf. Case 3.2). Let $R(x)$ state for the right-hand side of (33) as a function of $x$. Since $R(1)=(1-\beta(1)) /(1-q \beta(1)) \geq 0$ and the derivative $R^{\prime}(x)=p\left(q(1-\beta(x))^{2}-\beta^{\prime}(x)(1-q x)^{2}\right) /(1-q x \beta(x))^{2}$ is non-negative if $\beta(x)$ is assumed to be a non-increasing function, there is a $d$ such that the condition (33) holds iff $x \leq d$, where the optimal level $d$ is independent of $t$ (and $k$ ).

Now we are to find the decision level $d$.
Let $x>d$. Then $\delta_{b}(x, 0)>\delta_{f}(x, 0)$,

$$
\delta(x, \infty)=\frac{q}{1-q x} \int_{x}^{1} \delta_{b}(y, 0) d y
$$

and using (29) for $t=0$ we obtain a derivative

$$
\delta^{\prime}(x, \infty)=\frac{q}{1-q x} \alpha(x) \delta(x, \infty)-\frac{p q \alpha(x)}{(1-q x)^{2}}
$$

This linear differential equation with an initial condition $\delta(1, \infty)=0$ has a solution of the form

$$
\delta(x, \infty)=\exp \left(-\int_{x}^{1} \frac{q}{1-q s} \alpha(s) d s\right) \int_{x}^{1} \frac{p q \alpha(t)}{(1-q t)^{2}} \exp \left(-\int_{t}^{1} \frac{q}{1-q s} \alpha(s) d s\right) d t
$$

This gives (26).
Consider now $x \leq d$. Then $\delta_{b}(x, 0)>\delta_{f}(x, 0)$ for $x>d, \delta_{b}(x, t) \leq \delta_{f}(x, t)$ for $x \leq d$ and

$$
\delta(x, \infty)=\frac{q}{1-q x}\left(\int_{x}^{d} \delta_{f}(y, 0) d y+\int_{d}^{1} \delta_{b}(y, 0) d y\right)
$$

and using (32) we obtain a derivative

$$
\delta^{\prime}(x, \infty)=-\frac{p q \alpha(x) \beta(x)}{(1-q x)(1-q x \beta(x)}
$$

For an initial condition $\delta(d, \infty)=\delta(d+, \infty)$ a solution has the form

$$
\delta(x, \infty)=\delta(d+, \infty)-\int_{x}^{d} \frac{p q \alpha(t) \beta(t)}{(1-q t)(1-q t \beta(t)} d t
$$

It is easy to check that obtained $\delta(x, \infty), \delta_{f}(x, t), \delta_{b}(x, t)$ fulfil (1) - (3) which confirmes the supposition (27).

If $\alpha(x)$ is assumed to be constant $\alpha(x)=\alpha$ then the solution has an especially nice form. Since for $x \geq d$

$$
\delta(x, \infty)= \begin{cases}\frac{\alpha}{1-\alpha}\left(\left(\frac{p}{1-q x}\right)^{\alpha}-\frac{p}{1-q x}\right) & \text { for } \alpha<1 \\ -\frac{p}{1-q x} \ln \left(\frac{p}{1-q x}\right) & \text { for } \alpha=1\end{cases}
$$

where the optimal level $d$ is the largest $x$ in $(0,1]$ satisfying the inequality

$$
\frac{\alpha}{1-\alpha}\left(\left(\frac{p}{1-q x}\right)^{\alpha-1}-1\right) \geq \frac{1-\beta(x)}{1-q x \beta(x)}
$$

for $\alpha<1$ or

$$
-\ln \left(\frac{p}{1-q x}\right) \geq \frac{1-\beta(x)}{1-q x \beta(x)}
$$

for $\alpha=1$ if this inequality has a solution in $(0,1]$, and $d=0$ otherwise.
The probability $P($ win $)$ of choosing the largest observation using the optimal strategy with the level $d$ for constant $\alpha$ can be calculated in the same way as in Case 3.2 and has the value given by (25) if $d>0$.

## 4. Remarks

1. The condition that $\beta(x) / \alpha(x)$ is non-increasing in Theorem 2 seems to be natural. Petruccelli [8] writes: "the relative loss in the probability of succesful solicitation by not soliciting immediately is greater the larger the observation is. Thus, for example, the better applicant for the job, the less is the relative probability of his acceptance of an offer later compared to his probability of accepting an immediate offer. This is reasonable if one believes that better applicants are chosen faster by the market than lesser applicants".
2. For Cases 3.2 and 3.3 when $N$ is geometric the optimal strategy occurs to be of barrier type. It is, of course, also optimal in class of single level strategies. This is a natural class in full-information best choice problems with imperfect observations. It means that exact values of observations are not known - the observer is informed only whether a current observation exceeds or not a decision level he specified. Such problems were considered by Enns [4], Sakaguchi [13], Porosiński [12].
3. As far as the author knows, the full-information best choice problems with discrete time have been solved (besides problems mentioned in Introduction) only for choosing the largest with two choices allowed when $N$ is fixed (Tamaki [16]) and for choosing one of two bests for geometric number of observations (Porosiński and Szajowski [11]).
4. Problems with backward solicitation and uncertain selection were first posed for noinformation best choice problem (secretary problem). See for instance papers by Yang [18], Smith [14], Karni and Schwartz [6], Petruccelli [7] and [9], Szajowski [15], Ano [1] and Ano, Tamaki and Hu [3] and the references given there.

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[^0]:    1991 Mathematics Subject Classification. 62L15; Secondary 60G40.
    Key words and phrases. best choice problem, optimal stopping, full information, secretary problem, Poisson process, random horizon.

    Research supported by the grant KBN 341541.
    This paper is in final form and no version of it will be submitted for publication elsewhere.

