# STRONGLY FLUID AND WEAKLY UNSOLID VARIETIES 

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#### Abstract

A variety $V$ is called solid if every identity in $V$ is satisfied as hyperidentity, i. e. if for every substitution of terms of $V$ of appropriate arity for the operation symbols in $s \approx t$, the resulting identity holds in $V$. In the most cases it is very difficult to check whether an identity is satisfied as hyperidentity. The reason is that there are too much substitutions of terms for operation symbols. In this paper we will present methods to reduce the number of substitutions which are to check. This will be done by using some equivalence relations on the set of all substitutions of terms for operation symbols.

Particular properties of the corresponding equivalence classes lead to the concepts of strongly fluid and weakly unsolid varieties. The results will be applied to varieties of bands, of overcommutative semigroups and to some varieties of non-commutative groupoids.


1. Preliminaries. By $\left\{f_{i} \mid i \in I\right\}$ we denote an indexed set of operation symbols of type $\tau$ where $f_{i}$ is $n_{i}$-ary. Let $W_{\tau}(X)$ be the set of all terms built up by variables from $X$ and operation symbols from $\left\{f_{i} \mid i \in I\right\}$.

To precisizes what we understand under substitution of a term for an operation symbol we introduce the concept of a hypersubstitution of type $\tau$ as a map which associates to each fundamental operation symbol $f_{i}$ of type $\tau$ a term $\sigma\left(f_{i}\right)$ of type $\tau$ of the same arity. Any such map can be inductively extended to a map $\hat{\sigma}$ defined on the set of all terms of type $\tau$ as follows:
(i) $\hat{\sigma}[x]:=x$ for every variable $x \in X$,
(ii) $\hat{\sigma}\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right]:=\sigma\left(f_{i}\right)\left(\hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right)$ for any terms $t_{1}, \ldots, t_{n_{i}}$ and any operation symbol $f_{i}$.

On the set $\operatorname{Hyp}(\tau)$ of all hypersubstitutions of type $\tau$ we define a binary associative operation by $\sigma_{1} \circ_{h} \sigma_{2}:=\hat{\sigma}_{1} \circ \sigma_{2}$ where $\circ$ is the usual composition of functions and obtain a monoid $\underline{H y p(\tau)}:=\left(H y p(\tau) ; \circ_{h}, \sigma_{i d}\right)$ with $\sigma_{i d}\left(f_{i}\right)=f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$.

An identity $s \approx t$ satisfied in the variety $V$ of type $\tau$ is a hyperidentity if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $V$ for all $\sigma \in H y p(\tau)$.

A weaker concept is that of an $M$-hyperidentity where we request this property only for all hypersubstitutions from a submonoid $\underline{M} \subseteq \underline{H y p(\tau)}$. A variety $V$ is called solid if every of its identities is a hyperidentity. The weaker concept defined by $M$-hyperidentities is called $M$-solidity. The importance of these concepts consists in the fact that all $M$-solid varieties of type $\tau$ form a complete sublattice $\mathcal{S}_{M}(\tau)$ of the lattice of all varieties of type $\tau$ with

[^0]$$
\underline{M_{1}} \subseteq \underline{M_{2}} \Rightarrow \mathcal{S}_{M_{2}}(\tau) \subseteq \mathcal{S}_{M_{1}}(\tau)
$$

Further we will use the following denotations:
IdV $\quad-\quad$ the set of all identities satisfied in the variety $V$, we write $V \models s \approx t$ and $\underline{A} \mid=s \approx t$ if $s \approx t$ is an identity in the variety $V$ or in the algebra $\underline{A}$, repectively,
$\operatorname{Mod} \Sigma \quad-\quad$ the class of all algebras of type $\tau$ satisfying all equations from $\Sigma$ as identities,
$H I d V-\quad$ the set of all hyperidentities satisfied in the variety $V$, i. e. $H I d V:=\{s \approx t \mid \forall \sigma \in H y p(\tau)(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in I d V)\}$,
$P(V) \quad-\quad$ the set of all hypersubstitutions of $\operatorname{Hyp}(\tau)$ which preserve all identities of the variety $V$, i. e.

$$
P(V):=\{\sigma \mid \forall s \approx t \in \operatorname{Id} V(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in I d V)\}
$$

Hypersubstitutions from $P(V)$ are called $V$-proper. The set $P(V)$ forms a submonoid of $H y p(\tau)$ and $V$ is solid iff $P(V)=H y p(\tau)$. A hypersubstitution $\sigma$ is called $V$-inner if $\bar{\sigma}\left(f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)\right) \approx f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$ is an identity in $V$. The set $P_{0}(V)$ of all $V$-inner hypersubstitutions forms a submonoid of $P(V)$ ([5]).
2. Equivalence relations on sets of hypersubstitutions. To reduce the complexity of checking J. Płonka introduced the following binary relation on $H y p(\tau)$ with respect to a variety $V$ of type $\tau$ ([5]).

$$
\sigma_{1} \sim_{V} \sigma_{2}: \Leftrightarrow \sigma_{1}\left(f_{i}\right) \approx \sigma_{2}\left(f_{i}\right) \in I d V \text { for every } i \in I
$$

Clearly, $\sim_{V}$ is an equivalence relation on $\operatorname{Hyp}(\tau)$ and satisfies the following property: if $\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t] \in I d V$ and $\sigma_{1} \sim_{V} \sigma_{2}$ then $\hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t] \in I d V$. Because of this property it is enough to consider one representative from each equivalence class with respect to $\sim_{V}$ if we want to test whether $s \approx t$ is a hyperidentity in the variety $V$. Therefore using a choice function $\Phi$ we have to select a set $N_{\Phi}^{H y p(\tau)}(V) \subseteq H y p(\tau)$ of representatives for the quotient set $H y p(\tau) / \sim_{V}$. Further, because of $\sigma_{i d}\left(f_{i}\right)=f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)$ the set $P_{0}(V)$ is the equivalence class of $\sigma_{i d}$ and $P(V)$ is a union of equivalence classes with respect to $\sim_{V}$.

For a hypersubstitution $\sigma \in H y p(\tau)$ and an algebra $\underline{A}=\left(A ;\left(f_{i}^{A}\right)_{i \in I}\right)$ of type $\tau$ we define the derived algebra $\sigma[\underline{A}]$ by $\sigma[\underline{A}]:=\left(\underline{A} ;\left(\sigma\left(f_{i}\right) \underline{A}\right)_{i \in I}\right)$.

Here $\sigma\left(f_{i}\right)^{A}$ is the term operation induced by the term $\sigma\left(f_{i}\right)$ on the algebra $\underline{A}$.
The following property plays an important role in the theory of hyperidentities:

$$
\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t] \Leftrightarrow \sigma[\underline{A}] \models s \approx t \quad \text { (conjugate property) }
$$

Further one has
Proposition 2.1. ([5]) Let $V$ be a variety of type $\tau$ and let $\sigma_{1}, \sigma_{2}$ be hypersubstitutions of type $\tau$. Then

$$
\sigma_{1} \sim_{V} \sigma_{2} \Leftrightarrow \sigma_{1}[\underline{A}]=\sigma_{2}[\underline{A}]
$$

for all $\underline{A} \in V$.
As counterpart of solid varieties in [3] the concepts of unsolid and completely unsolid varieties were defined.

Definition 2.2. A variety $V$ of type $\tau$ is unsolid if $P(V)=P_{0}(V)$ and completely unsolid if $P(V)=P_{0}(V)=\left\{\sigma_{i d}\right\}$.

In [6] D. Schweigert defined fluid varieties to express the opposite of solidity.
Definition 2.3. A variety $V$ of type $\tau$ is called fluid if for every algebra $\underline{A} \in V$ and for every hypersubstitution $\sigma \in H y p(\tau)$ there holds

$$
\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}] \cong \underline{A} .
$$

It is easy to prove that every subvariety of a fluid variety is fluid and that all subvarieties of a fluid variety $V$ of type $\tau$ form a sublattice of the lattice of all varieties of type $\tau$. In [3] was proved:

Proposition 2.4. Let $V$ be a variety of type $\tau$ and let $H y p(\tau)$ be the set of all hypersubstitutions of type $\tau$. Then
(i) If $V$ is fluid then for every hypersubstitution $\sigma \in \operatorname{Hyp}(\tau)$ there holds: $\sigma \in P(V) \Rightarrow \forall \underline{A} \in V(\sigma[\underline{A}] \cong \underline{A})$.
(ii) If $V$ is a minimal variety, (i.e. an atom in the lattice of all varieties of type $\tau$ ) and if $V$ is unsolid then $V$ is fluid.

We define a new relation on $\operatorname{Hyp}(\tau)$ generalizing the property of the relation $\sim_{V}$ of Proposition 2.1.

Definition 2.5. Let $V$ be a variety of type $\tau$ and let $\sigma_{1}, \sigma_{2}$ be hypersubstitutions of type $\tau$. Then we define

$$
\sigma_{1} \sim_{V}^{I} \sigma_{2}: \Leftrightarrow \forall \underline{A} \in V\left(\sigma_{1}[\underline{A}] \cong \sigma_{2}[\underline{A}]\right)
$$

Clearly, $\sim_{V}^{I}$ is also an equivalence relation on $\operatorname{Hyp}(\tau)$ and contains the relation $\sim_{V}$. Further we have:

## Proposition 2.6.

(i) The relation $\sim_{V}^{I}$ is a right congruence on $\operatorname{Hyp}(\tau)$.
(ii) If $V$ is solid then $\sim_{V}^{I}$ is a congruence on $\operatorname{Hyp}(\tau)$.

Proof. (i) Let $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ and $\sigma \in \operatorname{Hyp}(\tau)$. Then $\sigma_{1}[\underline{A}] \cong \sigma_{2}[\underline{A}]$ for all $\underline{A} \in V$. If two algebra are isomorphic then the derived algebras are also isomorphic (see [6], Lemma 3.1). Therefore, $\sigma\left[\sigma_{1}[\underline{A}]\right] \cong \sigma\left[\sigma_{2}[\underline{A}]\right]$. Now we prove that $\left(\sigma_{1} \circ_{h} \sigma\right)[\underline{A}]=\sigma\left[\sigma_{1}[\underline{A}]\right]$. We use the fact that if $t$ is a term and if $t^{\sigma[\underline{A}]}$ is the term operation induced by $t$ on the algebra $\sigma[\underline{A}]$ then $t^{\sigma[\underline{A}]}=\hat{\sigma}[t] \underline{A}$. (see [4]). Then $f_{i}^{\sigma\left[\sigma_{1}[\underline{A}]\right]}=\hat{\sigma}\left[f_{i}\right]^{\sigma_{1}[\underline{A}]}=\hat{\sigma}_{1}\left[\sigma\left(f_{i}\right)\right] \underline{A}=f_{i}^{\left(\sigma_{1} \circ_{h} \sigma\right)[\underline{A}]}$ for all $i \in I$. As a consequence we have $\left(\sigma_{1} \circ_{h} \sigma\right)[\underline{A}]=\sigma\left[\sigma_{1}[\underline{A}]\right] \cong \sigma\left[\sigma_{2}[\underline{A}]\right]=\left(\sigma_{2} \circ_{h} \sigma\right)[\underline{A}]$ and $\sigma_{1} \circ_{h} \sigma \sim_{V}^{I} \sigma_{2} \circ_{h} \sigma$.
(ii) If $V$ is solid then for every $\underline{A} \in V$ and every $\sigma \in \operatorname{Hyp}(\tau)$, one obtains $\sigma[\underline{A}] \in V$. But then from $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ one has $\sigma_{1}[\sigma[\underline{A}]] \cong \sigma_{2}[\sigma[\underline{A}]]$, i.e. $\left(\sigma \circ_{h} \sigma_{1}\right)[\underline{A}] \cong\left(\sigma \circ_{h} \sigma_{2}\right)[\underline{A}]$ and $\sigma \circ_{h} \sigma_{1} \sim_{V}^{I} \sigma \circ_{h} \sigma_{2}$. This shows that $\sim_{V}^{I}$ is also a left congruence and thus a congruence.

The following theorem shows that the relation $\sim_{V}^{I}$ makes it easier to check hyperidentities:

Theorem 2.7. Let $V$ be a variety of type $\tau$. Then the following hold:
(i) For all $\sigma_{1}, \sigma_{2} \in H y p(\tau)$, if $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ then $\sigma_{1}$ is $V$-proper iff $\sigma_{2}$ is $V$-proper.
(ii) For all $s, t \in W_{\tau}(X)$ and all $\sigma_{1}, \sigma_{2} \in \operatorname{Hyp}(\tau)$, if $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ then the equation $\hat{\sigma}_{1}[s] \approx$ $\hat{\sigma}_{1}[t]$ is an identity in $V$ iff $\hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t]$ is an identity in $V$.

Proof. (i) Let $\sigma_{1}$ be a $V$-proper hypersubstitution. Then for all identities $s \approx t$ in $I d V$ the equation $\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t]$ is an identity in $V$, i.e. for all $\underline{A} \in V$ we have $\underline{A}=\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t]$. Using the conjugate property we have $\sigma_{1}[\underline{A}]=s \approx t$. Then $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ gives $\sigma_{1}[\underline{A}] \cong \sigma_{2}[\underline{A}]$ for all $\underline{A} \in V$. Since $I d \sigma_{1}[\underline{A}]=I d \sigma_{2}[\underline{A}]$ we have $\sigma_{2}[\underline{A}]=s \approx t$ and using again the conjugate property this gives $\underline{A} \models \hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t]$ for all $\underline{A} \in V$ and this means $\hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t]$ is an identity in $V$ and $\sigma_{2}$ is also $V$-proper. The converse direction follows in the same way. (ii) If $\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t]$ is an identity in $V$ then for all $\underline{A} \in V$ we get $\underline{A}=\hat{\sigma}_{1}[s] \approx \hat{\sigma}_{1}[t]$ and then $\sigma_{1}[\underline{A}] \mid=s \approx t$. Since $\sigma_{1} \sim_{V}^{I} \sigma_{2}$ we have $\sigma_{1}[\underline{A}] \cong \sigma_{2}[\underline{A}]$ for all $\underline{A} \in V$ and then $\sigma_{2}[\underline{A}] \mid=s \approx t$ and thus $\underline{A} \mid=\hat{\sigma}_{2}[s] \approx \hat{\sigma}_{2}[t]$. The converse follows in the same way.

Theorem 2.7 (i) means that the submonoid $P(V)$ of $\operatorname{Hyp}(\tau)$ is a union of equivalence classes with respect to the relation $\sim_{V}^{I}$.

If we restrict our attention to a submonoid $M$ of $\operatorname{Hyp}(\tau)$ and to the restricted relation $\sim_{V}^{I}$ we get:
Lemma 2.8. Let $M \subseteq H y p(\tau)$ and let $V$ be a variety of type $\tau$. Then the monoid $P(V) \cap M$ is a union of equivalence classes of the restricted relation $\sim_{V}^{I} \mid M$.

Proof. Let $\sigma$ be a hypersubstitution in $P(V) \cap M$, and let $\varrho \in M$ satisfy $\sigma \sim_{V}^{I} \mid M \varrho$. We want to show that $\varrho$ is also in $P(V)$. Let $s \approx t$ be any identity of $V$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in I d V$, i.e. for all $\underline{A} \in V(\underline{A} \mid=\hat{\sigma}[s] \approx \hat{\sigma}[t])$, but then $\sigma[\underline{A}] \mid=s \approx t$ and $\sigma[\underline{A}] \cong \varrho[\underline{A}]$ shows $\varrho[\underline{A}]|=s \approx t, \underline{A}|=\hat{\varrho}[s] \approx \hat{\varrho}[t]$ for all $\underline{A} \in V$ and $\varrho \in P(V) \cap M$.

Let $M$ be a monoid of hypersubstitutions. Consider the quotient set $M / \sim_{V}^{I} \mid M$. Let $\Phi$ be a choice function which chooses from $M$ one hypersubstitution from each equivalence class of $\sim_{V}^{I} \mid M$ and let ${ }^{I} N_{\Phi}^{M}(V)$ be the set of hypersubstitutions so chosen. From $\left[\sigma_{i d}\right]_{\sim_{V}^{I} \mid M}$ the function $\Phi$ chooses $\sigma_{i d}$. The set ${ }^{I} N_{\Phi}^{M}(V)$ is called set of $V$-normal form hypersubstitutions with respect to $\Phi$ and $\sim_{V}^{I} \mid M$.

By $\sigma_{1} \circ_{N I} \sigma_{2}:=\Phi\left(\sigma_{1} \circ_{h} \sigma_{2}\right)$ we define a binary operation on ${ }^{I} N_{\Phi}^{M}(V)$ and obtain a groupoid with identity element.

A variety $V$ is called ${ }^{I} N_{\Phi}^{M}(V)$-solid if every identity of $V$ is preserved by every hypersubstitution from ${ }^{I} N_{\Phi}^{M}(V)$. Then we have

Lemma 2.9. Let $M$ be a monoid of hypersubstitutions of type $\tau$ and let $V$ be a variety of type $\tau$. For any choice function $\Phi$ the variety $V$ is $M$-solid iff $V$ is ${ }^{I} N_{\Phi}^{M}(V)$-solid.

Proof. It is clear that if $V$ is $M$-solid then it is certainly also ${ }^{I} N_{\Phi}^{M}(V)$-solid. Conversely, suppose that $V$ is ${ }^{I} N_{\Phi}^{M}(V)$-solid. This means that all the members of the set ${ }^{I} N_{\Phi}^{M}(V)$ are also members of $P(V) \bigcap M$. Since by Lemma 2.8 $P(V) \bigcap M$ is a union of $\sim_{V}^{I} \mid M$-classes, a hypersubstitution that is equivalent to an element of ${ }^{I} N_{\phi}^{M}(V)$ is also in $P(V) \bigcap M$. But by construction any element of $M$ is equivalent to an element of ${ }^{I} N_{\phi}^{M}(V)$. Thus $M \subseteq P(V)$, and $V$ is $M$-solid.

In analogy to [4], Theorem 4.4 .9 we obtain also:
Theorem 2.10. Let $V$ be a variety of type $\tau$, and let $M$ be a monoid of hypersubstitutions of type $\tau$. Let $\Phi$ be a choice function which chooses one hypersubstitution from each class with respect to $\sim_{V}^{I} \mid M$. If the set ${ }^{I} N_{\Phi}^{M}(V)$ is finite, and if $V$ has a finite equational basis $\Sigma$, then the hypermodel class $H_{M} \operatorname{Mod} \Sigma$ is also finitely based.

Proof. It follows from the general theory of hypersubstitutions (see [4] ) that the set

$$
\Gamma:=\{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid s \approx t \in \Sigma, \sigma \in M\}
$$

is a basis for $H_{M} \operatorname{Mod} \Sigma$. Now let

$$
\Theta:=\left\{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid s \approx t \in \Sigma, \sigma \in{ }^{I} N_{\Phi}^{M}(V)\right\}
$$

Using the five derivation rules for identities $\Theta$ yields $\Gamma$, so that $\Theta$ is also a basis for $H_{M} \operatorname{Mod} \Sigma$. But under the hypotheses that both $\Sigma$ and ${ }^{I} N_{\phi}^{M}(V)$ are finite, we see that $\Theta$ is also finite.
3. Application to fluid varieties. Using the definition of the relation $\sim_{V}^{I}$ Definition 2.3 can be reformulated to: A variety $V$ of type $\tau$ is fluid if for every algebra $\underline{A} \in V$ and for every hypersubstitution $\sigma \in H y p(\tau)$ there holds:

$$
\sigma[\underline{A}] \in V \Rightarrow \sigma \sim_{V}^{I} \sigma_{i d}
$$

Inner hypersubstitutions form the equivalence class of $\sigma_{i d}$ with respect to the relation $\sim_{V}$. In parallel we define

Definition 3.1. A hypersubstitution $\sigma \in H y p(\tau)$ is called weakly inner hypersubstitution with respect to $\sim_{V}^{I}$ if $\sigma \sim_{V}^{I} \sigma_{i d}$. By $P_{f l}(V)$ we denote the set of all weakly inner hypersubstitutions with respect to $V$.

The set $P_{f l}(V)$ has the following properties.
Proposition 3.2. $P_{f l}(V)$ forms a submonoid of $P(V)$ which contains $P_{0}(V)$ as submonoid:

$$
P_{0}(V) \subseteq P_{f l}(V) \subseteq P(V)
$$

Proof. Since $\sim_{V} \subseteq \sim_{V}^{I}$ and since $P_{0}(V)=\left[\sigma_{i d}\right]_{\sim_{V}}$ we have

$$
P_{0}(V)=\left[\sigma_{i d}\right]_{\sim_{V}} \subseteq\left[\sigma_{i d}\right]_{\sim_{V}^{I}}=P_{f l}(V)
$$

Assume that $\sigma \in P_{f l}(V)$. Then for all $\underline{A} \in V$ we get $\sigma[\underline{A}] \cong \underline{A}$. If $s \approx t \in I d V$, i.e. if $\underline{A}=s \approx t$ for all $\underline{A} \in V$, then using the isomorphism, $\sigma[\underline{A}] \models s \approx t$ and by the conjugate property we get $\underline{A} \mid=\hat{\sigma}[s] \approx \hat{\sigma}[t]$, i.e. $\sigma$ is a $V$-proper hypersubstitution and then $P_{f l}(V) \subseteq$ $P(V)$. Since the product of two inner hypersubstitutions is an inner hypersubstitution the set $P_{0}(V)$ forms a submonoid of $P_{f l}(V)$. We have to show that the product of two hypersubstitutions from $P_{f l}(V)$ belongs to $P_{f l}(V)$. Assume that $\sigma_{1}, \sigma_{2} \in P_{f l}(V)$. Then $\sigma_{1}[\underline{A}] \cong \underline{A}$ and $\sigma_{2}[\underline{A}] \cong \underline{A}$ for all $\underline{A} \in V$. But then also $\left(\sigma_{1} \circ_{h} \sigma_{2}\right)[\underline{A}]=\sigma_{2}\left[\sigma_{1}[\underline{A}]\right] \cong \sigma_{2}[\underline{A}] \cong$ $\underline{A}$ since every isomorphism between two algebras is an isomorphism between the derived algebras (see [6]). But then $\sigma_{1} \circ_{h} \sigma_{2} \sim_{V}^{I} \sigma_{i d}$ and $\sigma_{1} \circ_{h} \sigma_{2} \in P_{f l}(V)$.

Then from Proposition 2.4 we obtain:
Corollary 3.3. If $V$ is a fluid variety then $P(V)=P_{f l}(V)$.
Proof. In any case there holds $P_{f l}(V) \subseteq P(V)$. By Proposition 2.4 for a fluid variety we have: if $\sigma \in P(V)$ then for all $\underline{A} \in V \quad(\sigma[\underline{A}] \cong \underline{A})$, but this means, if $\sigma \in P(V)$ then $\sigma \in P_{f l}(V)$ and so $P(V) \subseteq P_{f l}(V)$.

The set ${ }^{I} N_{\Phi}^{M}(V)$ can be used to test whether $V$ is fluid or not.
Lemma 3.4. The variety $V$ is fluid iff for all $\underline{A} \in V$, for all $\sigma \in N_{\Phi}^{H y p(\tau)}(V)$ with respect to some choice function $\Phi$ the following implication is satisfied:

$$
\begin{equation*}
\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}] \cong \underline{A} . \tag{*}
\end{equation*}
$$

Proof. If $V$ is fluid then this implication is satisfied for all $\sigma \in \operatorname{Hyp}(\tau)$, especially for all hypersubstitutions coming from ${ }^{I} N_{\Phi}^{H y p(\tau)}(V)$. If conversely for all $\sigma \in{ }^{I} N_{\Phi}^{\text {Hyp ( })}(V)$ the implication $\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}] \cong \underline{A}$ is satisfied and if $\sigma^{\prime} \in H y p(\tau)$ then we can find a hypersubstitution $\sigma \in{ }^{I} N_{\Phi}^{H y p(\tau)}(V)$ with $\sigma^{\prime} \sim_{V}^{I} \sigma$. But this means $\sigma^{\prime}[\underline{A}] \cong \sigma[\underline{A}]$ and $\sigma^{\prime}[\underline{A}] \in V$ implies $\sigma^{\prime}[\underline{A}] \cong \sigma[\underline{A}] \cong \underline{A}$ and $\left(^{*}\right)$ is satisfied for all hypersubstitutions $\sigma^{\prime}$ from Hyp( $\tau$ ).
4. Generalization of fluid and unsolid varieties. We can sharpen the definition of a fluid variety if we request the equality $\sigma[\underline{A}]=\underline{A}$ on the right hand side of the implication in Definition 2.3.

Definition 4.1. A variety $V$ of type $\tau$ is strongly fluid if for every algebra $\underline{A} \in V$ and for every hypersubstitution $\sigma \in H y p(\tau)$ there holds

$$
\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}]=\underline{A} .
$$

Clearly, every strongly fluid variety is also fluid. Further we have:
Proposition 4.2. If a variety $V$ of type $\tau$ is strongly fluid then it is unsolid
Proof. If $V$ is strongly fluid then for all $\sigma \in \operatorname{Hyp}(\tau)$ we have

$$
\sigma \in P(V) \Rightarrow \sigma[\underline{A}]=\underline{A} .
$$

The right hand side of this implication means $\sigma \in P_{0}(V)$ and thus $P(V) \subseteq P_{0}(V)$. Together with $P_{0}(V) \subseteq P(V)$ we have $P(V)=P_{0}(V)$. This means, $V$ is unsolid.

Proposition 4.2 and Corollary 3.3 mean that for strongly fluid varieties all three monoids $P_{0}(V), P_{f l}(V)$, and $P(V)$ are equal.

Further we define
Definition 4.3. A variety $V$ of type $\tau$ is weakly unsolid if $P(V)=P_{f l}(V)$.
Clearly, if $V$ is unsolid, then by Proposition 3.2 it is also weakly unsolid. By 3.3, every fluid variety $V$ is weakly unsolid. In the following case fluidity implies also unsolidity.
Proposition 4.4. Let $V$ be a fluid variety of type $\tau$. If $\sim \sim_{V}^{I} \subseteq \sim_{V}$ then $V$ is unsolid.
Proof. We have to show that $P(V) \subseteq P_{0}(V)$. Assume that $\sigma \in P(V)$. Then for any $\underline{A} \in V$ we have $\sigma[\underline{[ }] \in V$ and thus for all $\underline{A} \in V, \sigma[\underline{A}] \cong \underline{A}=\sigma_{i d}[\underline{A}]$ since $V$ is fluid. Because of $\sim_{V}^{I} \subseteq \sim_{V}$ we have also $\sigma \sim_{V} \sigma_{i d}$ and $\sigma$ belongs to $P_{0}(V)$.

Example: In [3] the authors showed that the variety $C O M$ of all commutative semigroups is unsolid, but not fluid. Then $C O M$ is also not strongly fluid. The variety $C O M$ is also weakly unsolid.
5. Applications to varieties of groupoids. At first we compare the relations $\sim_{V}^{I}$ and $\sim_{V}$ for the variety of all bands, i.e. idempotent semigroups, and for the class of all semigroups containing the variety of all commutative semigroups (overcommutative semigroups).
Theorem 5.1. Let $V$ be a nontrivial variety of bands. Then $\sim I_{V}^{I} \subseteq \sim_{V}$.
Proof. There are exactly the following six binary terms over the variety $B$ of all bands: $x, y, x y, y x, x y x, y x y$. That means, the quotient set $\operatorname{Hyp}(\tau) / \sim_{V}$ where $V$ is a variety of bands is a subset of the set $\left\{\left[\sigma_{x}\right]_{\sim_{V}},\left[\sigma_{y}\right]_{\sim_{V}},\left[\sigma_{x y}\right]_{\sim_{V}},\left[\sigma_{y x}\right]_{\sim_{V}},\left[\sigma_{x y x}\right]_{\sim_{V}},\left[\sigma_{y x y}\right]_{\sim_{V}}\right\}$. We check
all possible pairs of representatives. Assume that $\sigma_{x} \sim_{V}^{I} \sigma_{y}$. Let $F_{2}(V)$ be the two-generated free algebra of $V$. Then there exists an isomorphism

$$
\varphi: \sigma_{x}\left[\underline{F_{2}(V)}\right] \rightarrow \sigma_{y}\left[\underline{F_{2}(V)}\right]
$$

and we have

$$
\begin{aligned}
\varphi\left([x]_{I d V}\right) & =\varphi\left(\sigma_{x}(f) \underline{F_{2}(V)}\left([x]_{I d V},[y]_{I d V}\right)\right) \\
& =\sigma_{y}(f) \underline{F_{2}(V)}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\varphi\left([y]_{I d V}\right) .
\end{aligned}
$$

Here $f$ denotes the binary operation symbol. Since $\varphi$ is one-to-one we get $[x]_{I d V}=[y]_{I d V}$, a contradiction.

Assume that $x y \approx x \notin I d V$ and that $\sigma_{x y} \sim_{V}^{I} \sigma_{x}$. Then there exists an isomorphism

$$
\varphi: \sigma_{x y}\left[\underline{F_{2}(V)}\right] \rightarrow \sigma_{x}\left[\underline{F_{2}(V)}\right]
$$

and we have

$$
\begin{aligned}
\varphi\left([x]_{I d V}\right) & =\sigma_{x}(f) \frac{F_{2}(V)}{}\left(\varphi\left([x]_{I d V}\right), \varphi\left([x]_{I d V}\right)\right) \\
& =\sigma_{x}(f) \frac{F_{2}(V)}{}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\varphi\left(\sigma_{x y}(f) \underline{F_{2}(V)}\left([x]_{I d V},[y]_{I d V}\right)\right) \\
& =\varphi\left([x y]_{I d V}\right) .
\end{aligned}
$$

Since $\varphi$ is one-to-one we have $[x]_{I d V}=[x y]_{I d V}$. This is a contradiction to our assumption that $x \approx x y \notin I d V$.

In the same manner one can show that for each $u \in\{x y, y x, x y x, y x y\}$ both, $\sigma_{x} \sim_{V}^{I} \sigma_{u}$ and $\sigma_{y} \sim_{V}^{I} \sigma_{u}$ are impossible if $x \approx u \notin I d V$ and $y \approx u \notin I d V$, respectively.

Assume that $x y \approx x y x \notin I d V$ and that $\sigma_{x y} \sim_{V}^{I} \sigma_{x y x}$. Then there is an isomorphism

$$
\varphi: \sigma_{x y}\left[\underline{F_{2}(V)}\right] \rightarrow \sigma_{x y x}\left[\underline{F_{2}(V)}\right]
$$

and we have

$$
\begin{aligned}
\varphi\left([x y]_{I d V}\right) & =\varphi\left(\sigma_{x y}(f) \underline{F_{2}(V)}\left([x]_{I d V},[y]_{I d V}\right)\right) \\
& =\sigma_{x y x}(f) \underline{F_{2}(V)}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right)
\end{aligned}
$$

Because of $\varphi\left([y x]_{I d V}\right)=\varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right)$ and using the idempotent law one can easily verify that

$$
\begin{aligned}
\varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right) & =\varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right) \varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \\
& =\varphi\left([x]_{I d V}\right) \varphi\left([y x]_{I d V}\right) \varphi\left([x]_{I d V}\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\varphi\left([x y]_{I d V}\right) & =\varphi\left([x]_{I d V}\right) \varphi\left([y x]_{I d V}\right) \varphi\left([x]_{I d V}\right) \\
& =\sigma_{x y x}(f) \frac{F_{2}(V)}{}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y x]_{I d V}\right)\right) \\
& =\varphi\left(\sigma_{y x}(f) \frac{F_{2}(V)}{}\left([x]_{I d V},[y x]_{I d V}\right)\right) \\
& =\varphi\left([x y x]_{I d V}\right)
\end{aligned}
$$

This gives $[x]_{I d V}=[x y x]_{I d V}$, a contradiction.
In a similar way one proves that if $x y \approx y x y \notin I d V$ and $y x \approx x y x \notin I d V$ and $y x \approx$ $y x y \notin I d V$, then $\sigma_{x y} \sim_{V}^{I} \sigma_{y x y}$ in the first, $\sigma_{y x} \sim_{V}^{I} \sigma_{x y x}$ in the second, and $\sigma_{y x} \sim_{V}^{I} \sigma_{y x y}$ in the third case, respectively, are impossible.

Now we show that $\sigma_{x y} \sim_{V}^{I} \sigma_{y x}$ is impossible if $x y \approx y x \notin I d V$. We consider the two cases $x y \approx y x y \in I d V$ or $x y \approx y x y \notin I d V$. In the first case we have $y x \approx y x y \notin I d V$ and by the previous fact $\sigma_{y x} \sim_{V}^{I} \sigma_{y x y}$ is impossible. But then $\sigma_{y x} \sim_{V}^{I} \sigma_{x y}$ is impossible since otherwise from $\sigma_{x y} \sim I V \sigma_{y x y}$ and $\sigma_{x y} \sim \sim_{V}^{I} \sigma_{y x}$ it would follow $\sigma_{y x} \sim \sim_{V}^{I} \sigma_{y x y}$.

Assume that $x y \approx y x y \notin I d V$ and that $\sigma_{x y} \sim_{V}^{I} \sigma_{y x}$. Let $\underline{S}$ be the semigroup defined by the following finite presentation:

$$
<\{x, y, z\} \mid\{a=b \mid a, b \in W(\{x, y, z\}), a \approx b \in I d V\} \cup\{x y=x, x z=x\}>
$$

Clearly, $\{x, y, z\} \subseteq S$ and $\underline{S} \in V$. Therefore, there is an isomorphism

$$
\varphi: \sigma_{x y}[\underline{S}] \rightarrow \sigma_{y x}[\underline{S}]
$$

and we have

$$
\begin{equation*}
\varphi\left(\sigma_{x y}(f) \underline{S}(x, x)\right)=\varphi\left(\sigma_{x y}(f) \underline{S}(x, u)\right)=\sigma_{y x}(f) \underline{S}(\varphi(x), \varphi(u)) \text { for all } u \in S \tag{*}
\end{equation*}
$$

Since $x y \approx y x y \notin I d V$ for each $u \in S$ there is an element $w_{u} \in\{x, y, z\}$ such that both, $\varphi(u)$ and $\varphi(u) \varphi(x)$ start with $w_{u}$. Since $\{x, y, z\} \subseteq\{\varphi(u) \mid u \in S\}$ we have $\mid\{\varphi(u) \varphi(x) \mid u \in$ $S\} \mid \geq 3$. This contradicts (*).

The last case is that $x y x \approx y x y \notin I d V$, but $\sigma_{x y x} \sim_{V}^{I} \sigma_{y x y}$. If in this case $x y x \approx x y \in$ $I d V$ then $y x y \approx x y \notin I d V$ and from one of the previous cases we get that $\sigma_{x y} \sim_{V}^{I} \sigma_{y x y}$ is impossible. If $\sigma_{x y x} \sim_{V}^{I} \sigma_{y x y}$ then from $\sigma_{x y x} \sim I \quad \sigma_{x y}$ we would get a contradiction. Thus $\sigma_{x y x} \sim{ }_{V}^{I} \sigma_{y x y}$ is impossible.

Consider now the case that $x y x \approx x y \notin I d V$. Then we have

$$
\begin{aligned}
\varphi\left([x y x]_{I d V}\right) & =\varphi\left(\sigma_{x y x}(f) \frac{F_{2}(V)}{}\left([x]_{I d V},[y]_{I d V}\right)\right) \\
& =\sigma_{y x y}(f) \frac{F_{2}(V)}{}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi\left([x y]_{I d V}\right) & \left.=\varphi\left(\sigma_{x y x}(f)\right)^{F_{2}(V)}\left([x y]_{I d V},[y]_{I d V}\right)\right) \\
& =\sigma_{y x y}(f) \underline{F_{2}(V)}\left(\varphi\left([x y]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\varphi\left([y]_{I d V}\right) \varphi\left(\left[[x y]_{I d V}\right) \varphi\left([y]_{I d V}\right)\right. \\
& =\varphi\left([y]_{I d V}\right) \varphi\left([x]_{I d V}\right) \varphi\left([y]_{I d V}\right) .
\end{aligned}
$$

Altogether we have $\varphi\left([x y x]_{I d V}\right)=\varphi\left([x y]_{I d V}\right)$ and then $x y x \approx x y \in I d V$, a contradiction. This shows that also in this case $\sigma_{x y x} \sim_{V}^{I} \sigma_{y x y}$ is impossible.

Let now $u, v \in W(\{x, y\})$ with $u \approx v \notin I d V$. Then there are $u^{\prime}, v^{\prime} \in\{x, y, x y, y x, x y x$, $y x y\}$ such that $u \approx u^{\prime}, v \approx v^{\prime} \in I d V$ and $u^{\prime} \approx v^{\prime} \notin I d V$. In the previous part of the proof we showed that $\sigma_{u^{\prime}} \sim_{V}^{I} \sigma_{v^{\prime}}$ is impossible. For all $\underline{A} \in V$ we have $\sigma_{u^{\prime}}[\underline{A}]=\sigma_{u}[\underline{A}] \quad\left(\operatorname{sigma}_{v^{\prime}}[\underline{A}]=\right.$ $\left.\sigma_{v}[\underline{A}]\right)$. Therefore $\sigma_{u} \sim_{v}^{I} \sigma_{V}$ is impossible and we have $\sim_{V}^{I} \subseteq \sim_{V}$.

For varieties of overcommutative semigroups the relations $\sim_{V}^{I}$ and $\sim_{V}$ are also equal, i.e.

Theorem 5.2. Let $V$ be a variety of semigroups which contains the variety of commutative semigroups. Then $\sim_{V}^{I} \subseteq \sim_{V}$.
Proof. Assume that $\sigma_{u} \sim_{V}^{I} \sigma_{v}$ for two binary terms $u$ and $v$. Let $\ell(u)$ be the number of variables occurring in $u$. We can assume that $\ell(u) \leq \ell(v)$. Put $n:=\ell(v)$. The variety $W:=\operatorname{Mod}\left\{\operatorname{IdV} \cup\left\{x_{1} \ldots x_{n+1} \approx y_{1} \ldots y_{n+1}\right\}\right.$ is a subvariety of $V$. We consider the free algebra with respect to $W$ generated by $\{x, y, z\}: \underline{F_{3}(W)}$. By $u(x, z)$ we denote the term obtained from the term $u$ replacing $x$ by $z$. In the same way the terms $u(y, z), u(y, x), u(z, y), u(y, z), v(y, x), v(z, x), v(z, y)$ and $v(x, z)$ are defined. Without loss of generality we assume that $v$ contains the variable $y$. Consider $S:=F_{3}(W) \backslash$ $\left\{[v]_{I d W},[v(x, z)]_{I d W}\right\}$. On the set S we define an operation $f \underline{\underline{S}}$ by $f \underline{\underline{S}}\left([a]_{I d V},[b]_{I d V}\right)=$
 $\left\{[v]_{I d W},[v(x, z)]_{I d W}\right\}$. Then it is easy to see that $\underline{S} \in W \subseteq V$. Because of $\sigma_{u} \sim_{V}^{I} \sigma_{v}$ there is an isomorphism $\varphi: \sigma_{u}[\underline{S}] \rightarrow \sigma_{v}[\underline{S}]$. Assume that $\varphi\left([x]_{I d W}^{n+1}\right) \in\left\{[x]_{I d W},[y]_{I d W},[z]_{I d W}\right\}$. Then we have $\varphi\left([x]_{I d W}^{n+1}\right)=\varphi\left(\sigma_{u}(f) \underline{S}\left([x]_{I W W}^{n+1},[x]_{I d W}^{n+1}\right)\right)=\sigma_{v}(f) \underline{S}\left(\varphi\left([x]_{I d W}^{n+1}\right), \varphi\left([x]_{I d W}^{n}\right)\right)=$ $\left(\varphi\left([x]_{I d W}^{n+1}\right)\right)^{n}$, i.e. $[w]_{I d W}=[w]_{I d W}^{n}$ for some $w \in\{x, y, z\}$. But then $w \approx w^{n} \in I d W$, a contradiction. Therefore, $\varphi\left([x]_{I d W}^{n+1}\right) \notin\left\{[x]_{I d W},[y]_{I d W},[z]_{I d W}\right\}$ and we have $\varphi\left([x]_{I d W}^{n+1}\right)=$ $\varphi\left(\sigma_{u}(f) \underline{S}\left([x]_{I d W}^{n+1},[x]_{I d W}^{n+1}\right)\right)=\sigma_{v}(f)^{\underline{S}}\left(\varphi\left([x]_{I d W}^{n+1}\right), \varphi\left([x]_{I d W}^{n+1}\right)\right)=[x]_{I d W}^{n+1}$, i.e. $\varphi\left([x]_{I d W}^{n+1}\right)=$ $[x]_{I W W}^{n+1}$.

For the elements $\varphi\left([x]_{I d W}\right), \varphi\left([y]_{I d W}\right)$, and $\varphi\left([z]_{I d W}\right)$ we have to consider the following seven cases:
(i) $\varphi\left([x]_{I d W}\right)=[x]_{I d W}, \varphi\left([y]_{I d W}\right)=[y]_{I d W}, \varphi\left([z]_{I d W}\right)=[z]_{I d W}$,
(ii) $\varphi\left([x]_{I d W}\right)=[x]_{I d W}, \varphi\left([y]_{I d W}\right)=[z]_{I d W}, \varphi\left([z]_{I d W}\right)=[y]_{I d W}$,
(iii) $\varphi\left([x]_{I d W}\right)=[y]_{I d W}, \varphi\left([y]_{I d W}\right)=[x]_{I d W}, \varphi\left([z]_{I d W}\right)=[z]_{I d W}$,
(iv) $\varphi\left([x]_{I d W}\right)=[y]_{I d W}, \varphi\left([y]_{I d W}\right)=[z]_{I d W}, \varphi\left([z]_{I d W}\right)=[x]_{I d W}$,
(v) $\varphi\left([x]_{I d W}\right)=[z]_{I d W}, \varphi\left([y]_{I d W}\right)=[y]_{I d W}, \varphi\left([z]_{I d W}\right)=[x]_{I d W}$,
(vi) $\varphi\left([x]_{I d W}\right)=[z]_{I d W}, \varphi\left([y]_{I d W}\right)=[x]_{I d W}, \varphi\left([z]_{I d W}\right)=[y]_{I d W}$,
(vii) $\left\{\varphi\left([x]_{I d W}\right), \varphi\left([y]_{I d W}\right), \varphi\left([z]_{I d W}\right)\right\} \nsubseteq\left\{[x]_{I d W},[y]_{I d W},[z]_{I d W}\right\}$.

Case (i): We have

$$
\begin{aligned}
\left.\varphi\left(\sigma_{u}(f)\right)^{\underline{S}}\left([x]_{I d V},[y]_{I d V}\right)\right) & =\sigma_{v}(f) \underline{\underline{S}}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right) \\
& =\sigma_{v}(f) \underline{\underline{S}}\left([x]_{I d V},[y]_{I d V}\right) \\
& =[x]_{I d W}^{n+1} .
\end{aligned}
$$

Because of $\varphi\left([x]_{I d W}^{n+1}\right)=[x]_{I d W}^{n+1}$ we have

$$
\left.\varphi\left(\sigma_{u}(f)\right)^{\underline{S}}\left([x]_{I d W},[y]_{I d W}\right)\right)=\varphi\left([x]_{I d W}^{n+1}\right),
$$

i.e.

$$
\sigma_{u}(f)^{\underline{S}}\left([x]_{I d W},[y]_{I d W}\right)=[x]_{I d W}^{n+1}
$$

This implies $u \approx x^{n+1} \in I d W$ or $u \approx v \in I d W$ or $u \approx v(x, z) \in I d W$. Clearly, $u \approx x^{n+1} \in$ $I d W$ as well as $u \approx v(x, z) \in I d W$ are impossible. Consequently, $u \approx v \in I d W$.
Case (ii): In the same way as in the first case we obtain $u \approx v \in I d W$.
Case (iii): We have

$$
\begin{aligned}
\varphi\left(\sigma_{u}(f)^{\underline{S}}\left([x]_{I d W},[z]_{I d W}\right)\right) & =\sigma_{v}(f) \underline{S}\left(\varphi\left([y]_{I d W}\right), \varphi\left([z]_{I d W}\right)\right) \\
& =\sigma_{v}(f)^{\underline{S}}\left([x]_{I d W},[z]_{I d W}\right)=[x]_{I d W}^{n+1} .
\end{aligned}
$$

Because of $\varphi\left([x]_{I d W}^{n+1}\right)=[x]_{I d W}^{n+1}$ we have then $\varphi\left(\sigma_{u}(f) \underline{S}^{n}\left([y]_{I d W},[z]_{I d W}\right)\right)=\varphi\left([x]_{I d W}^{n+1}\right)$. This implies $u(y, z) \approx x^{n+1} \in I d W$ or $u(y, z) \approx v \in I d W$ or $u(y, z) \approx v(x, z) \in I d W$. Clearly,
$u(y, z) \approx x^{n+1} \notin I d W$. The case $u(y, z) \approx v \in I d W$ is only possible if both terms $u$ and $v$ have the same length and are costructed only by the variable $y$. But $u(y, z)$ is a term in the variable $y$ if $u$ is a term in the variable $x$, i.e. $[u]_{I d W}=[x]_{I d W}^{n}$ and $[v]_{I d W}=$ $[y]_{I d W}^{n}$. So we have the following situation: $\varphi\left([x]_{I d W}^{n+1}\right)=\varphi\left(\sigma_{u}(f) \underline{S}\left([x]_{I d W}^{n+1},[y]_{I d W}\right)\right)=$ $\sigma_{v}(f)^{\underline{S}}\left(\varphi\left([x]_{I d W}^{n+1}\right), \varphi\left([y]_{I d W}\right)\right)=\left(\varphi\left([y]_{I d W}\right)\right)^{n}=[x]_{I d W}^{n}$. But there holds $[x]_{I d W}^{n} \neq[x]_{I d W}^{n+1}$ (because of $y^{n} \approx x^{n} \notin I d W$ and $\left.z^{n} \approx x^{n} \notin I d W\right)$, which contradicts $\varphi\left([x]_{I d W}^{n+1}\right)=[x]_{I d W}^{n+1}$. Consequently, $u(y, z) \approx v \in I d W$ is impossible. From $u(y, z) \approx v(x, z) \in I d W$ one obtains $u \approx v \in I d W$ if one substitutes $y$ by $x$ and $z$ by $y$.

In the cases (iv), (v)and (vi) we get $u \approx v \in I d W$ in a similar way as in case (iii).
Case (vii): There is an element $w \in\{x, y, z\}$ such that $\varphi\left([w]_{I d W}\right) \notin\left\{[x]_{I d W},[y]_{I d W},[z]_{I d W}\right\}$.
a) If $w \in\{x, y\}$ then there holds

$$
\begin{aligned}
\varphi\left(\sigma_{u}(f)^{\underline{s}}\left([x]_{I d W},[y]_{I d W}\right)\right) & =\sigma_{v}(f)^{\underline{S}}\left(\varphi\left([x]_{I d W}\right), \varphi\left([y]_{I d W}\right)\right) \\
& =\sigma_{v}(f) \underline{-}\left([x]_{I d W},[y]_{I d W}\right) \\
& =[x]_{I d W}^{n+1}
\end{aligned}
$$

and we get $u \approx v \in I d W$ in the same way as in case (i).
b) If $w=z$ then we have

$$
\begin{aligned}
\varphi\left(\sigma_{u}(f)^{\underline{S}}\left([x]_{I d W},[z]_{I d W}\right)\right) & =\sigma_{v}(f)^{\underline{S}}\left(\varphi\left([x]_{I d W}\right), \varphi\left([z]_{I d W}\right)\right) \\
& =\sigma_{v}(f) \underline{S}\left([x]_{I d W},[z]_{I d W}\right) \\
& =[x]_{I d W}^{n+1}
\end{aligned}
$$

and we get $u \approx v \in I d W$ in the same manner as in case (i). Consequently $u \approx v \in I d W$.
Altogether this shows that $u \approx v \in I d W$. Since both terms, $u$ and $v$ have a length less than $n+1$, the identity $u \approx v$ is also valid in $V$, i.e. $u \approx v \in I d V$. This means $\sigma_{u} \sim_{V} \sigma_{v}$.

Now we consider all varieties $V$ of groupoids where the two-generated free algebra $\underline{F_{2}(V)}$ consists of at most three elements.

We will use the following denotations for varieties of groupoids:

$$
\begin{array}{ll}
I & - \text { trivial variety, } \\
L Z= & \operatorname{Mod}\{x y \approx x\}, \\
R Z= & \operatorname{Mod}\{x y \approx y\} \\
S L= & \operatorname{Mod}\left\{x(y z) \approx(x y) z, x^{2} \approx x, x y \approx y x\right\} \\
& (I, L Z, R Z, \text { and } S L \text { are varieties of semigroups }) \\
V_{1,2}= & \operatorname{Mod}\left\{x y \approx y x, x^{2} \approx x,(x y) y \approx x y\right\}-\text { the variety of near semilattices, } \\
V_{2}= & \operatorname{Mod}\{x y \approx y x, x \approx x,(x y) y \approx x\}-\text { the variety of Steiner quasigroups, } \\
V_{3}= & \operatorname{Mod}\left\{x y \approx y x,(x y) y \approx x y, x^{2} \approx x y\right\}
\end{array}
$$

By Hyp we denote the monoid of hypersubstitutions of typ (2) and by $N_{\Phi}^{H y p}(V)$ the groupoid of normal form hypersubstitutions with respect to the relation $\sim_{V}$.

In [2] we showed

## Theorem 5.3.

(i) If $V$ is a variety of groupoids and if $\Phi$ is an arbitrary choice function $\Phi: H y p / \sim_{V} \rightarrow$ Hyp then $\left|N_{\Phi}^{H y p}(V)\right| \leq 3$ iff $V \in\left\{I, L Z, R Z, S L, V_{1,2}, V_{2}, V_{3}\right\}$.
(ii)

$$
\begin{aligned}
N_{\Phi}^{H y p}(L Z) & =\left\{\sigma_{x y}, \sigma_{u}\right\} \text { with } \sigma_{u} \sim_{L Z} \sigma_{y} \\
N_{\Phi}^{H y p}(R Z) & =\left\{\sigma_{x y}, \sigma_{v}\right\} \text { with } \sigma_{v} \sim_{R Z} \sigma_{x} \\
N_{\Phi}^{H y p}(S L) & =\left\{\sigma_{x y}, \sigma_{u}, \sigma_{v}\right\} \text { with } \sigma_{u} \sim_{S L} \sigma_{x}, \sigma_{v} \sim_{S L} \sigma_{y} \\
N_{\Phi}^{H y p}\left(V_{1,2}\right) & =\left\{\sigma_{x y}, \sigma_{u}, \sigma_{v}\right\} \text { with } \sigma_{u} \sim_{V_{1,2}} \sigma_{x}, \sigma_{v} \sim_{V_{1,2}} \sigma_{y} \\
N_{\Phi}^{H y p}\left(V_{2}\right) & =\left\{\sigma_{x y}, \sigma_{u}, \sigma_{v}\right\} \text { with } \sigma_{u} \sim_{V_{2}} \sigma_{x}, \sigma_{v} \sim_{V_{2}} \sigma_{y} \\
N_{\Phi}^{H y p}\left(V_{3}\right) & =\left\{\sigma_{x y}, \sigma_{u}, \sigma_{v}\right\} \text { with } \sigma_{u} \sim_{V_{3}} \sigma_{x}, \sigma_{v} \sim_{V_{3}} \sigma_{y}
\end{aligned}
$$

(iii) The following equalities are satisfied for proper and inner hypersubstitutions:

$$
P(L Z)=P_{0}(L Z), P(R Z)=P_{0}(R Z), P(S L)=P_{0}(S L), P\left(V_{1,2}\right)=P_{0}\left(V_{1,2}\right), P\left(V_{2}\right)=
$$

$$
P_{0}\left(V_{2}\right), P\left(V_{3}\right)=P_{0}\left(V_{3}\right)
$$

It follows:
Corollary 5.4. The varieties $L Z, R Z, S L, V_{1,2}, V_{2}, V_{3}$ are unsolid.
For $L Z, R Z, S L$ in [3] was already proved that these varieties are fluid.
Theorem 5.5. Each of the varieties $I, L Z, R Z, S L, V_{1,2}, V_{2}, V_{3}$ is fluid.
Proof. We have only to consider the varieties $V_{1,2}, V_{2}$, and $V_{3}$. By the previous theorem the hypersubstitutions $\sigma_{x}, \sigma_{y}, \sigma_{x y}$ have to be checked. If $\underline{A} \in V \in\left\{V_{1,2}, V_{2}, V_{3}\right\}$ then $\underline{A}$ is commutative, but $\sigma_{x}[\underline{A}]=\left(A ; e_{1}^{2, \underline{A}}\right), \sigma_{y}[\underline{A}]=\left(A ; e_{2}^{2, \underline{A}}\right)$ where $e_{1}^{2, \underline{A}}, e_{2}^{2, \underline{A}}$ are the binary projections on $A$ on the first component and on the second component, respectively, are not commutative. Thus $\sigma_{x}[\underline{A}], \sigma_{y}[\underline{A}] \notin V$. Moreover, $\sigma_{x y}[\underline{A}]=\underline{A} \in V$ for all $\underline{A} \in V$ and $\sigma_{x y}[\underline{A}] \cong \underline{A}$.

By Proposition 4.4 the unsolidity of all these varieties would also follow from Theorem 5.5 if we could prove that $\sim_{V}^{I} \subseteq \sim_{V}$.

Theorem 5.6. If $V$ is a nontrivial variety of groupoids such that the two-generated free algebra $\underline{F_{2}(V)}$ consists of at most three elements then $\sim_{V}^{I} \subseteq \sim_{V}$.

Proof. $\left|F_{2}(V)\right| \leq 3$ holds iff $\left|N_{\Phi}^{H y p}(V)\right| \leq 3$ for any choice function $\Phi$. By Theorem 5.3 we have $F_{2}(V)=\left\{[x y]_{I d V},[x]_{I d V}\right\}$ if $V \in\{L Z, R Z\}$ and $F_{2}(V)=\left\{[x y]_{I d V},[x]_{I d V},[y]_{I d V}\right\}$ if $V \in\left\{S L, V_{1,2}, V_{2}, V_{3}\right\}$. Clearly, if $V$ is nontrivial then $\sigma_{x} \sim_{V}^{I} \sigma_{y}$ is impossible since otherwise for every algebra $\underline{A} \in V$ which is not trivial there existed an isomorphism $\varphi: \sigma_{x}[\underline{A}] \rightarrow \sigma_{y}[\underline{A}]$ with $\varphi\left([x]_{I d V}\right)=\varphi\left(\sigma_{x}(f) \underline{A}\left([x]_{I d V},[y]_{I d V}\right)\right)=\sigma_{y}(f) \underline{A}\left(\varphi\left([x]_{I d V}\right), \varphi\left([y]_{I d V}\right)\right)=\varphi\left([y]_{I d V}\right)$ and from $\varphi\left([x]_{I d V}\right)=\varphi\left([y]_{I d V}\right)$ it follows $[x]_{I d V}=[y]_{I d V}$, i.e. $x \approx y \in I d V$.

The varieties $L Z, R Z, S L$ are varieties of bands. Therefore, by 5.1 we have to consider $V_{1,2}, V_{2}$, and $V_{3}$. Assume that $\sigma_{x y} \sim_{V}^{I} \sigma_{y}$ for $V \in\left\{V_{1,2}, V_{2}, V_{3}\right\}$. Then we have for $V_{1,2}$ :

$$
\begin{aligned}
\varphi\left([x y]_{I d V_{1,2}}\right) & =\varphi\left(\sigma_{x y}(f) \underline{F_{2}\left(V_{1,2}\right)}\left([x y]_{I d V_{1,2}},[y]_{I d V_{1,2}}\right)\right) \\
& =\sigma_{y}(f) \underline{F_{2}\left(V_{1,2}\right.}\left(\varphi\left([x y]_{I d V_{1,2}}\right), \varphi\left([y]_{I d V_{1,2}}\right)\right) \\
& =\varphi\left([y]_{I d V_{1,2}}\right)
\end{aligned}
$$

and thus $[x y]_{I d V_{1,2}}=[y]_{I d V_{1,2}}$ and $x y \approx y \in I d V_{1,2}$ which is a contradiction, for $V_{2}$ :

$$
\begin{aligned}
\varphi\left([x y]_{I d V_{2}}\right) & =\varphi\left(\sigma_{x y}(f) \frac{F_{2}\left(V_{2}\right)}{}\left([x]_{I d V_{2}},[y]_{I d V_{2}}\right)\right) \\
& =\sigma_{y}(f) \underline{F_{2}\left(V_{2}\right)}\left(\varphi\left([x]_{I d V_{2}}\right), \varphi\left([y]_{I d V_{2}}\right)\right) \\
& =\varphi\left([y]_{I d V_{2}}\right)
\end{aligned}
$$

and then $[x y]_{I d V_{2}}=[y]_{I d V_{2}}$ and $x y \approx y \in I d V_{2}$, a contradiction, for $V_{3}$ :

$$
\begin{aligned}
\varphi\left([x y]_{I d V_{3}}\right) & =\varphi\left(\sigma_{x y}(f) \underline{F_{2}\left(V_{3}\right)}\left([x]_{I d V_{3}},[x]_{I d V_{3}}\right)\right. \\
& =\sigma_{x}(f) \underline{F_{2}\left(V_{3}\right)}\left(\varphi\left([x]_{I d V_{3}}\right), \varphi\left([x]_{I d V_{3}}\right)\right) \\
& =\varphi\left([x]_{I d V_{3}}\right)
\end{aligned}
$$

and then $[x y]_{I d V_{3}}=[x]_{I d V_{3}}$ and $x y \approx x \in I d V_{3}$, a contradiction.
All three varieties are commutative, i.e. $\sigma_{x y} \sim_{V} \sigma_{y x}$ and then $\sigma_{x y} \sim_{V}^{I} \sigma_{y x}, V \in$ $\left\{V_{1,2}, V_{2}, V_{3}\right\}$. Assume that $\sigma_{x y} \sim_{V}^{I} \sigma_{x}$, then by $\sigma_{x y} \sim_{V}^{I} \sigma_{y x}$ and transitivity we have also $\sigma_{y x} \sim_{V}^{I} \sigma_{x}$, but this contradicts the fact that $\sigma_{x y} \sim_{V}^{I} \sigma_{y}$ is impossible as we showed before.

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