GENERAL EXPANSION MAPPINGS ON TOPOLOGICAL SPACES

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Abstract. In this paper, first, we shall prove some fixed point theorems on general expansion mappings in arbitrary topological spaces. Secondly, we formulate a new way in fixed point theory under Expansion Monotone Principle. Also, we describe a class of conditions sufficient for the existence of fixed points.

1. Introduction and definitions. In recent years a great number of papers have presented extensions of the well-known Banach-Picard contraction principle. The purpose of the present paper is to consider general expansion mappings by introducing "monotonicity" conditions on topological spaces. In this sense, we describe a class of conditions sufficient for the existence of fixed points.

In this paper we formulate some new monotone principles of fixed point and, for former monotone principles, see Tasković [2] and [3].

In [5] Wang, Gao, Li and Iseki proved the following statement for a class of expansive mappings. Naimely, if (X, d) is a complete metric space, if a mapping $T : X \to X$ is onto and if there exists $q > 1$ such that

$$d\Big(T(x), T(y)\Big) \geq q d(x, y) \tag{Is}$$

for each $x, y \in X$, then T has a unique fixed point in the metric space X.

In this paper, we extend Wang, Gao, Li and Iseki's theorem and we think that our conditions may be adapted for other classes of mappings to obtain some extensions of new fixed point results.

Let X be a topological space, $T : X \to X$ and $A : X \times X \to \mathbb{R}_+^\circ := [0, +\infty)$. We shall introduce the concept of CS-convergence in a space X; i.e., a topological space X satisfies the condition of **CS-convergence** if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X and $A(x_n, Tx_n) \to 0$ $(n \to \infty)$ implies that $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Also, we shall introduce the concept of invariant property for space X; i. e., a topological space X satisfies the condition of **invariant property** if there is a nonempty subset A of X such that $T(A) = A$. Obviously, if $T : X \to X$ is an onto mapping, then X is with the invariant property for $A = X$. Also, if $T : X \to X$ continuous on a compact space X, then X has the invariant property.

2. General expansion mappings. In this section, we begin with the following statement which is fundamental.

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Lemma 1. *Let the mapping* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ := (0, +\infty)$ *have the following properties*

$$(\varphi) \qquad \varphi(t) > t \quad \textit{and} \quad \liminf_{z \to t-0} \varphi(z) > t$$

for every $t \in \mathbb{R}_+$. *If the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of real nonnegative numbers satisfies the inequality*

$$x_n \geq \varphi(x_{n+1}) \qquad \textit{for all} \quad n \in \mathbb{N},$$

then it converges to zero. The velocity of this convergence is not necessarily geometric.

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is a nonincreasing bounded sequence in $\mathbb{R}_+$, there is a $t \geq 0$ such that $x_n \to t$ $(n \to \infty)$. We claim that $t = 0$. If $t > 0$, then

$$t = \liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} \varphi(x_{n+1}) \geq \liminf_{z\to t-0} \varphi(z) > t,$$

which is a contradiction. Consequently $t = 0$ and so $x_n \to 0$ $(n \to \infty)$. The proof is complete.

We are now in a position to formulate the following general statements.

Theorem 1. (General expansion). *Let* T *be a mapping of a topological space* X *into itself, where* X *with the invariant property and with the condition of CS-convergence. If there is a mapping* $\varphi : \mathbb{R}^\circ_+ \to \mathbb{R}^\circ_+$ *such that the condition* (φ) *holds and*

$$\text{(A)} \qquad A(Tx, Ty) \geq \varphi\Big(A(x, y)\Big)$$

for all $x, y \in X$, *where* $A : X \times X \to \mathbb{R}^\circ_+$, $x \mapsto A(x, Tx)$ *is lower semicontinuous and* $A(x, y) = 0$ *implies* $x = y$, *then* T *has a unique fixed point in* X.

As immediate consequences of the preceding statement we obtain results in [5] of Wang, Gao, Li and Iseki's and in [1] of Daffer and Kaneko's.

Corollary 1. (Wang, Gao, Li and Iseki). *Let* (X, d) *be a complete metric space. If* T *is a mapping of* X *onto itself and if there exists* $q > 1$ *such that*

$$\text{(Is)} \qquad d\Big(T(x), T(y)\Big) \geq q d(x, y)$$

for all $x, y \in X$, *then* T *has a unique fixed point in* X.

Proof. Since $T : X \to X$ is onto, we have that X satisfies the condition of invariant property $T(A) = A$ for $A = X$. Let $A(x, y) = d(x, y)$ and $\varphi(t) = qt$ for $q > 1$ and $t \in \mathbb{R}^\circ_+$. It is easy to see that A and φ satisfy all the required hypotheses in Theorem 1. By hypothesis, X is complete and, therefore, X satisfies the condition of CS-convergence. Hence, it follows from Theorem 1 that T has a unique fixed point in X.

Corollary 2. (Daffer and Kaneko). *Let* T *be a continuous compact mapping of a metric space* (X, d) *into itself satisfying the expansive condition* (Is). *Then* T *has a unique fixed point in* X.

Proof. Since $T : X \to X$ is a continuous and compact mapping, we have that $T(A) = A$ for $A := \cap_{n=1}^{\infty} T^n(Y)$ and for some compact subset Y of X. Thus we have that X satisfies the condition of invariant property. Further, the proof is a totally analogous with the preceding proof of Corollary 1.

Proof of Theorem 1. Since X is with invariant property, there exists a nonempty subset A of X such that $T(A) = A$. Then the set $A_x := T^{-1}(\{x\}) \subset A$ is a nonempty subset of A for every $x \in A$. If $g|A$ is a function of choice, then there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X defined by $a_{n+1} = g(A_{a_n})$ for $n \in \mathbb{N}$, where $a_1 \in A$ is an arbitrary point. Thus we obtain that is $a_n = T(a_{n+1})$ for all $n \in \mathbb{N}$, in X. From (A) we have

$$A(a_n, a_{n+1}) = A\Big(T(a_{n+1}), T(a_{n+2})\Big) \geq \varphi\Big(A(a_{n+1}, a_{n+2})\Big)$$

for all $n \in \mathbb{N}$. Applying Lemma 1 to the sequence $\{A(a_n, a_{n+1})\}_{n\in\mathbb{N}}$, we obtain $A(a_n, a_{n+1}) \to 0$ $(n \to \infty)$. This implies (from CS-convergence) that the sequence $\{a_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence $\{a_{n(k)}\}_{k\in\mathbb{N}}$ with limit $\xi \in X$. Since $x \mapsto A(x, Tx)$ is lower semicontinuous at ξ, we obtain

$$A(\xi, T\xi) \leq \liminf A(a_{n(k)}, a_{n(k)-1}) = \liminf A(a_n, a_{n-1}) = 0,$$

i.e., $T\xi = \xi$. We complete the proof by showing that T can have at most one fixed point. In fact, if $\xi \neq \eta$ were two fixed points, then $A(\xi, \eta) = A(T\xi, T\eta) \geq \varphi(A(\xi, \eta)) > A(\xi, \eta)$, which is a contradiction. The proof is complete.

3. Some localizations. Let X be a topological space, let $T : X \to X$ and let $B : X \to \mathbb{R}_+^\circ$ be a lower semicontinuous function on X.

In this section, we shall introduce the concept of LCS-convergence in a space X; i.e., a topological space X satisfies the condition of **LCS-convergence** if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X and $B(x_n) \to 0$ $(n \to \infty)$ implies that $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Theorem 2. (Localization of general expansion). *Let T be a mapping of a topological space X into itself, where X with invariant property and with the condition of LCS-convergence. If there is a mapping $\varphi : \mathbb{R}_+^\circ \to \mathbb{R}_+^\circ$ such that the conditon (φ) holds and*

$$\text{(LA)} \qquad B(Tx) \geq \varphi(Bx) \qquad \textit{for every} \quad x \in X,$$

where $B : X \to \mathbb{R}_+^\circ$ is lower semicontinuous and $B(x) = 0$ implies $Tx = x$, then T has a fixed point in X.

Proof. Totally analoguous as in the proof of Theorem 1, we obtain that there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X defined by $a_n = T(a_{n+1})$ for $n \in \mathbb{N}$. Thus, from (LA), we have

$$B(a_n) = B\Big(T(a_{n+1})\Big) \geq \varphi\Big(B(a_{n+1})\Big)$$

for all $n \in \mathbb{N}$. Applying Lemma 1 to the sequence $\{B(a_n)\}_{n\in\mathbb{N}}$, we obtain $B(a_n) \to \to 0 (n \to \infty)$. This implies (from LCS-convergence) that the sequence $\{a_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence $\{a_{n(k)}\}_{k\in\mathbb{N}}$ with limit $\xi \in X$. Since $B : X \to \mathbb{R}_+^\circ$ is lower semicontinuous, we have

$$B(\xi) \leq \liminf B(a_{n(k)}) = \liminf B(a_n) = 0,$$

which implies that $B(\xi) = 0$, i.e., $T\xi = \xi$. The proof is complete.

Corollary 3. (Wang, Gao, Li and Iseki [5]). *Let (X,d) be a complete metric space. If T is a continuous mapping of X onto itself and if there is a real number $a>1$ such that*

$$d\Big(T(x),T^2(x)\Big)\geq ad(x,Tx)$$

for each $x\in X$, then T has a fixed point in X.

Proof. Let $B(x)=d(x,T(x))$ and $\varphi(t)=at$ $(a>1, t\in\mathbb{R}_+^\circ)$. Then $B(x)$ is a lower semicontinuous function (T is a continuous mapping). Since X satisfies the condition of invariant property ($T:X\to X$ is onto) and the condition of LCS-convergence (X is a complete metric space), applying Theorem 2 gives $T\xi=\xi$ for some $\xi\in X$.

Corollary 4. (Wang, Gao, Li and Iseki [5]). *Let (X,d) be a complete metric space. If T is a continuous mapping of X onto itself and if there is a real number $a>1$ such that*

$$\text{(W)}\qquad d\Big(T(x),T(y)\Big)\geq a\min\Big\{d(x,y),d(x,T(x)),d(y,T(y))\Big\}$$

for all $x,y\in X$, then T has a fixed point in X.

Proof. Let x be an arbitrary point in X. Then, for $y=Tx$, from the preceding inequality (W), we have

$$d\Big(T(x),T^2(x)\Big)\geq a\min\Big\{d(x,T(x)),d(T(x),T^2(x))\Big\}=ad\Big(x,Tx\Big),$$

which means that (LA) holds. Hence, for $B(x)=d(x,T(x))$, $\varphi(t)=at$ $(a>1, t\in\mathbb{R}_+^\circ)$ and, since X satisfies the condition of LCS-convergence (X is complete) and the condition of invariant property ($T:X\to X$ is onto), applying Theorem 2, we obtain $T\xi=\xi$ for some $\xi\in X$.

4. Further expansion mappings. In this section, we extend the preceding results and we describe a class of conditions sufficient for the existence of fixed and other points.

In connection with this, we shall introduce the concept of upper BCS-convergence in a space X for a bounded above function $B:X\to\mathbb{R}$; i.e., a topological space X satisfies the condition of **upper BCS-convergence** if $\{a_n(x)\}_{n\in\mathbb{N}}$ is a sequence in X with arbitrary $x\in X$ and $B(a_n(x))\to b$ $(n\to\infty)$ implies that $\{a_n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{a_{n(k)}(x)\}_{k\in\mathbb{N}}$ which converges to $\xi\in X$, where

$$\text{(Us)}\qquad B(\xi)\geq\sup_{x\in X}\limsup_{k\to\infty}B(a_{n(k)}(x)).$$

We are now in a position to formulate our main statement (Expansion Monotone Principle) in the following form.

Theorem 3. *Let T be a mapping of a topological space X into itself, where X satisfies the condition of upper BCS-convergence. If*

$$\text{(B)}\qquad B(Tx)\geq B(x)\quad\text{for every}\quad x\in X,$$

then there exists a point $\xi\in X$ such that

$$\text{(M)}\qquad B(T\xi)=B(\xi)=\alpha:=\sup_{x\in X}\lim_{n\to\infty}B(b_n(x))$$

for some sequence $\{b_n(x)\}_{n\in\mathbb{N}}$ in X which converges to ξ. If $B(Tx)=B(x)=\alpha$ implies $Tx=x$, then T has a fixed point in X.

Proof. Let x be an arbitrary point in X. Then, from (B), we have the following inequalities

$$\cdots \geq B(T^{n+1}x) \geq B(T^n x) \geq \cdots \geq B(Tx) \geq B(x) \tag{1}$$

for every $n \in \mathbb{N} \cup \{0\}$ and for every $x \in X$. Thus, since B is bounded above, for the sequence $\{B(T^n x)\}_{n\in\mathbb{N}}$, we obtain $B(T^n x) \to b$ $(n \to \infty)$ with arbitrary $x \in X$. This implies (from upper BCS-convergence) that its sequence $\{T^n x\}_{n\in\mathbb{N}}$ contains a convergent subsequence $\{T^{n(k)}x\}_{k\in\mathbb{N}}$ with limit $\xi \in X$. Since X satisfies the condition of upper BCS-convergence, from (1), we have

$$\alpha := \sup_{x\in X} \lim_{n\to\infty} B(T^n x) \geq \lim_{n\to\infty} B(T^n \xi) \geq \cdots \geq B(T^n \xi) \geq$$
$$\cdots \geq B(T\xi) \geq B(\xi) \geq \sup_{x\in X} \limsup_{k\to\infty} B(T^{n(k)}x) = \alpha,$$

i.e., $B(T\xi) = B(\xi) = \alpha$. This means that (M) holds, where the existing sequence $\{b_n(x)\}_{n\in\mathbb{N}}$, is the preceding subsequence of iterates $\{T^{n(k)}x\}_{k\in\mathbb{N}}$. If $B(T\xi) = B(\xi) = \alpha$ implies that $T\xi = \xi$, then $\xi \in X$ is a fixed point of T. The proof is complete.

We now show that the following our statement is a special case of the above Theorem 3.

Corollary 5. (Tasković [4]). *Let T be a self-map on a complete metric space (X, d). Suppose that there exists an upper semicontinuous bounded above function $G : X \to \mathbb{R}$ such that*

$$d\Big(x, T(x)\Big) \leq G(Tx) - G(x) \tag{T}$$

for every $x \in X$. Then T has a fixed point in X.

Proof. Let $B(x) = G(x)$, which is a bounded above and an upper semicontinuous function on X, and thus with the property (Us). Thus $B(Tx) \geq B(x)$ for every $x \in X$; i.e., (B) in Theorem 3. Since X satisfies the condition of upper BCS-convergence (X is a complete metric space and, for $x_n := T^n(x)$, from (T), we have

$$\sum_{i=0}^{n} d(x_i, x_{i+1}) \leq G(x_{n+1}) - G(x),$$

where G is a bounded above functional and an upper semicontinuous functional; i.e., $B(T^n x) \to b$ $(n \to \infty)$ implies that $\{T^n x\}_{n\in\mathbb{N}}$ converges to some $\xi \in X$ and the property (Us) holds), applying Theorem 3 we obtain $B(T\xi) = B(\xi) = \alpha$. Thus, from (T), we obtain

$$d\Big(\xi, T(\xi)\Big) \leq B(T\xi) - B(\xi) = \alpha - \alpha;$$

i.e., $\xi = T\xi$ for some $\xi \in X$. The proof is complete.

In connection with the preceding, from the proof of Theorem 3, we obtain, as a directly extension of Theorem 3, the following general result.

Theorem 4. *Let T be a mapping of a topological space X into itself, where X satisfies the condition of upper BCS-convergence. If*

$$B(Tx) \geq B(x) \qquad \textit{for every} \quad x \in X, \tag{B}$$

then there exists a point $\xi \in X$ such that

$$B(T^k\xi) = \cdots = B(T\xi) = B(\xi) = \alpha := \sup_{x\in X} \lim_{n\to\infty} B(b_n(x)) \tag{Mk}$$

for arbitrary fixed positive integer $k \geq 1$ and for some sequence $\{b_n(x)\}_{n\in\mathbb{N}}$ in X which converges to ξ. If (Mk) implies $\xi = T\xi$, then T has a fixed point in X.

As an immediate consequence of this result we obtain our the following statement which is an extension of Corollary 5.

Theorem 5. (Tasković [4]). *Let T be a self-map on a complete metric space (X, d). Suppose that there exist an upper semicontinuous bounded above function $G : X \to \mathbb{R}$ and an arbitrary fixed integer $k \geq 0$ such that*

$$d(x, Tx) \leq G(Tx) - G(x) + \cdots + G\Big(T^{2k+1}x\Big) - G\Big(T^{2k}x\Big) \tag{Tk}$$

and $G(T^{2i}x) \leq G(T^{2i+1}x)$ for $i = 0, 1, \ldots, k$ and for every $x \in X$. Then T has a fixed point in X.

Proof. Since $B(x) = G(x)$ is upper semicontinuous, thus is B with the property (Us). Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let n and m $(n < m)$ be any positive integers. From the property (Tk), we have

$$\sum_{i=0}^{n} d\Big(T^i x, T^{i+1}x\Big) \leq G\Big(T^{n+1}x\Big) - G(x),$$

and thus, since G is a bounded above functional, we obtain the following fact:

$$d\Big(T^n x, T^m x\Big) \leq \sum_{i=n}^{m-1} d\Big(T^i x, T^{i+1}x\Big) \to 0 \quad (n, m \to \infty).$$

Hence $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and, by completeness, there is $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Therefore, X satisfies the condition of upper BCS-convergence for $B(x) = G(x)$, where the property (Us) holds.

Also, from the property (Tk), we have $B(Tx) \geq B(x)$ for every $x \in X$, i.e., (B) in Theorem 4. Applying Theorem 4, we obtain a form of (Mk), i.e, $B(T^{2k+1}\xi) = B(T^{2k}\xi) = \cdots = B(T\xi) = B(\xi) = \alpha$, for some $\xi \in X$. Thus, from the property (Tk), we have

$$d\Big(\xi, T(\xi)\Big) \leq G(T\xi) - G(\xi) + \cdots + G\Big(T^{2k+1}\xi\Big) - G\Big(T^{2k}\xi\Big) = (2k+1)(\alpha - \alpha) = 0,$$

i.e., $\xi = T\xi$ for some $\xi \in X$. The proof is complete.

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