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REMOVABILITY OF EXCEPTIONAL SETS ON THE BOUNDARY FOR SOLUTIONS TO SOME NONLINEAR EQUATIONS*

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ABSTRACT. We consider the boundary value problem for nonlinear elliptic equations on the supposition that the solution assumes boundary values on the boundary given except on a certain subset. We study the removability of boundary singularities, and give a sufficient condition for that. Moreover, the equivalence notion of removable boundary singularities via probabilistic approach is also discussed.

1. Introduction

Let D be a bounded domain in R^d with C^2-boundary ∂D. K denotes a closed subset of ∂D. The uniformly elliptic operator L is defined by

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

where the coefficients $A = (a_{ij})$, $b = (b_i)$ are all bounded continuous functions on D. More precisely, the Hölder continuity with exponent λ is assumed, namely, $a_{ij}, b_i \in C^{0,\lambda}(D)$ for every i, j. We assume, in addition, that $a_{ij} = a_{ji}$ and

$$(A.1)\;\; a_{ij} \in C^2(D), \quad b_i \in C^1(D); \quad (A.2)\;\; \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(x) \le \sum_{i=1}^{d} \frac{\partial}{\partial x_i} b_i(x).$$

Our main concern is the problem on the removable singularity for solutions to nonlinear differential equations. We consider the boundary value problem for nonlinear elliptic equations:

$$(1) \qquad Lu = u^{\alpha} \quad \text{in} \quad D \quad (\alpha > 1), \quad \text{with} \quad u|_{\partial D \setminus K} = f.$$

Here the set K is an exceptional set on the boundary. We would like to know when the restriction $\partial D \setminus K$ of the solution u is replaced by the whole boundary ∂D. Then if that is possible, K is called the removable boundary singularity (RBS). It is a not only interesting but also important problem to think about what kind of characterization for removability of the singularity K is possible. Another interesting problem is on the explosive solution at the boundary. Consider the following problem:

$$(2) \qquad Lu = u^{\alpha} \quad \text{in} \quad D \quad \text{with} \quad u|_{\partial D} = \infty.$$

The second expression in the above means that $\lim_{D \ni y \to x} u(y) = \infty$ for $\forall x \in \partial D$. We are also interested in describing the probabilistic characterization of the solution with explosion at the boundary. This is another stimulating problem (cf. [LG93]). These two problems are mutually related, however, we shall treat the former problem only and leave the latter

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one for our next paper. The motivation of our study consists in those investigations on the uniqueness and exceptional sets on the boundary of the Dirichlet problem, inspired by Gaǰdenko's remarkable work [G81], and our guiding idea is due to Sheu's results [S94]. For a function space F, pbF indicates the subspace of F whose elements are all positive bounded functions.

2. Hausdorff Measure and Removability

The Hausdorff measure of A ($\subset R^d$) with parameter s is given as follows. For $\varepsilon > 0$, $\Delta(\varepsilon)$ is a countable open covering $(N(\varepsilon), \{B(x_i, r_i)\}_i)$ of A such that $A \subset \bigcup_{i=1}^{N(\varepsilon)} B(x_i,r_i)$ where $B(x_i, r_i)$ is an open ball with center x_i and radius r_i, $0 < r_i \leq \varepsilon$. Then the Hausdorff measure $\Lambda^s(A)$ of A is defined by

$$\Lambda^s(A) = \lim_{\varepsilon \downarrow 0} \left(\inf_{\Delta} \sum_{i=1}^{N(\varepsilon)} r_i^s \right).$$

The Hausdorff dimension $\dim_H(A)$ of A is the supremum of $s \in R_+$ such that $\Lambda^s(A) > 0$. The interpretation of the problem (1) as classical problem means that the nonnegative solution u lying in $C^2(D)$ satisfies

$$Lu = u^\alpha \quad \text{in} \quad D, \qquad \lim_{D \ni x \to y} u(x) = f(y), \quad \forall y \in \partial D \setminus K, \tag{3}$$

for $f \in pC(\partial D)$. Let dx be the Lebesgue measure on R^d, and n denotes the unit exterior normal vector to the boundary ∂D. $S(dy)$ is the surface measure on ∂D. We set $\mu(dx) = p(x)dx$, where $p(x)$ is the distance function from x to the boundary ∂D. The first assertion is a result on nonremovable singularity.

Theorem 1. *For some positive number $\gamma(\alpha)$, $\alpha > 1$ satisfying that γ is monotone decreasing in α and $\gamma \nearrow \infty$ as $\alpha \searrow 1$, there exists a family of solutions $\{u \equiv u_\alpha \geq 0;\, \alpha > 1\}$ of the boundary value problem* (3) *such that $d > \gamma(\alpha)$ and $\Lambda^s(K) > 0$ for some $s \in (d - \gamma(\alpha),$ $d-1]$, $(\alpha > 1)$.*

We defer for the moment giving the proof of the theorem. Instead, we shall introduce below several auxiliary results. The next two propositions are well-known, e.g. see [BM92].

Proposition 2. *For every $f \in pbC(\partial D)$, there exists the unique solution u of* (3) *with $\partial D \setminus K$ replaced by ∂D.*

Proposition 3. *Let u be the solution stated in Proposition 2. u is the unique solution of the integral equation of the form*

$$u(x) + \int_D g_L(x,y)u^\alpha(y)dy = \int_{\partial D} k_L(x,y)f(y)S(dy), \quad \text{for } \forall x \in D,$$

where g_L (resp. k_L) is the Green (resp. Poisson) kernel for the operator L in D.

Lemma 4. *Let h be an L-harmonic function in D satisfying $h \in L^\alpha(\mu(dx))$. Then there exists a nonnegative solution in D for the integral equation*

$$u(x) + \int_D g_L(x,y)u^\alpha(y)dy = h(x), \qquad \forall x \in D. \tag{4}$$

Proof. Let $g_L^{(n)}(x,y)$ be a sequence of the Green functions for L in the domain D_n, where D_n denotes a bounded regular domain such that $D_n \subset D_{n+1}$, and $D_n \nearrow D$ as $n \to \infty$. By virtue of Proposition 2 and 3, for each n there exists the unique nonnegative solution u_n in D_n satisfying

$$u_n(x) + \int_{D_n} g_L^{(n)}(x,y)u_n^\alpha(y)dy = h(x), \quad \forall x \in D_n. \tag{5}$$

Note that $\Delta u_k = u_k^\alpha$ in D_n holds for some k as far as $k \geq n$. Hence it follows immediately from the Maximum principle (cf. Theorem 0.5, p.113 in [Dy91]) that the limit $u(x) = \lim_{n\to\infty} u_n(x)$ exists for all $x \in D$. Notice that (5) is, in fact, valid even with D_n replaced by D for all $x \in D$ and all n, only if we extend $u_n(x)$, $g_L^{(n)}(x,y)$ trivially for x (or y) outside D_n. Take $\varepsilon > 0$ small enough such that $\bar{U}_\varepsilon(x) \subset D$ for fixed $x \in D$ and $g_L(x,y) \leq c(x) \cdot p(y)$ for all $y \in D \setminus U_\varepsilon(x)$, where $U_\varepsilon(x)$ denotes an ε-neighborhood of the point x. From our major assumption, we have

$$\int_D g_L(x,y)h^\alpha(y)dy \leq \int_{U_\varepsilon(x)} g_L(x,y)h^\alpha(y)dy + \int_{D\setminus U_\varepsilon(x)} c(x)p(x)h^\alpha(y)dy < \infty.$$

Since $\lim_{n\to\infty} g_L^{(n)}(x,y)u_n^\alpha(y) = g_L(x,y)u^\alpha(y) \leq g_L(x,y)h^\alpha(y)$ for all $y \in D$, the assertion yields from the Lebesgue convergence theorem applied to (5). q.e.d.

Lemma 5. *Let h be the same L-harmonic function as stated in Lemma 4. Then the maximal solution of $Lu = u^\alpha$ in D dominated by h satisfies* (4).

Proof. The assertion is easily obtained by virtue of the Maximum principle, hence omitted; see e.g. [DK96b]. q.e.d.

Proof of Theorem 1. First of all, assume that $\Lambda^s(K) > 0$. Take a measure $\pi \in M_F(K)$ such that $\pi(B) \leq r^s$, for any ball B in R^d with radius r. For the Poisson kernel $k_L(x,y)$ for the elliptic operator L ([DK96a]), the function

$$\hat{K}(x) := \int_K k_L(x,y)\pi(dy)$$

is L-harmonic in D and vanishes on $\partial D \setminus K$. We show that $\hat{K} \in L^\alpha(\mu(dx))$. By virtue of Maz'ya-Plamenevsky's argument(1985), it follows from Maz'ya's lemma(1975) that there exists a constant $C > 0$ (depending on L and D) such that $k_L(x,y) < C \cdot p(x)\ |x-y|^{-d}$ holds for all $x \in D$, $y \in \partial D$, cf. [Dy94], [DK96a]. By this estimate, it is sufficient to show that

$$l(x) := \int_{\partial D} \left(p(x)/|x-y|^d\right)\pi(dy) \in L^\alpha(\mu(dx)). \tag{6}$$

To show (6) can be attributed to finding a constant C such that

$$\int_D l(x)g(x)\mu(dx) \leq C \quad \text{for} \quad \text{any} \quad g > 0 \tag{7}$$

satisfying that $\int_D \{g(x)\}^\beta \mu(dx) = 1$ with $1/\alpha + 1/\beta = 1$. Consider the function

$$F(z) = \int_D \int_K \frac{\{g(x)\}^{\beta(1-z)}p(x)}{|x-y|^{s/\alpha+\{d-\gamma(\alpha)\}/\beta+(d-s+1)z+1}}\pi(dy)\mu(dx).$$

It is easy to verify that $|F(1+ib)| < \infty$ (technically, see the estimation method in the proof of Theorem 6.2, p.100, [C91]). Thus we attain (7). On this account, the conclusion yields from a routine work with the maximum principle and a discussion of domination of the maximal solution by some L-harmonic function. Indeed, it goes similarly as the proof of Theorem 1 (A), p.705, [S94], together with Lemma 4, Lemma 5 and the maximal solution argument (cf. [DK96b]). q.e.d.

3. Removable Exceptional Sets on The Boundary

Under the assumptions (A.1) and (A.2), the operator L has an expression of the divergence form

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}u\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(\hat{b}_i(x)u) - c(x)u$$

with $\hat{b_i} = -b_i + \sum_j \partial_j a_{ij}$, $c = -\sum_i \partial_i \hat{b_i}$, $\partial_i = \partial/\partial x_i$, $(i = 1, 2, \cdots, d)$. Then notice that $a_{ij}, \hat{b_i} \in C^1(D)$ and $c \geq 0$. The adjoint of L is given by

$$L^* u = \sum_{i,j}\partial_i(a_{ij}\partial_j u) + \sum_i \hat{b_i}\partial_i u - cu.$$

Now we shall introduce another interpretation of (1), due to the Gmira-Véron formulation [GV91]. That is, the solution is a nonnegative function $u \in C^2(D) \cup C(\bar{D} \setminus K)$ satisfying

$$\int_D \{-u \cdot L^* g + u^\alpha g\}dx + \int_{\partial D} f\frac{\partial g}{\partial n}S(dy) = 0 \tag{8}$$

for $\forall\, g \in C^{1,1}(\bar{D}) \cap W_0^{1,\infty}(D)$ with the compact support which is contained in $\bar{D} \setminus K$. The second result asserts the existence of the solution u of (1) with admissible region $\partial D \setminus K$ replaced by the whole boundary ∂D.

Theorem 6. *Let u be a solution of* (4). *If* $\dim_H(K) < d - \gamma(\alpha)$ *and*

$$u \in L^{\frac{1}{\gamma(\alpha)-1}+1}(dx) \bigcap L^\alpha(\mu(dx)),$$

then K is the removable boundary singularities (RBS).

Remarks. The above-mentioned result is an extension of Sheu's theorem (1994) (cf. Theorem 2, p.702, [S94]), where only the simple case of $L = \Delta$ was investigated. It is interesting that the above theorem suggests that the Hausdorff dimension for removability of exceptional sets on the boundary may possibly attain the optimal one in the sense of relative relation between space dimension and nonlinearity parameter.

Proof. The proof of Theorem 6 is greatly due to Chabrowski's lemma. Put $\beta = d - \gamma(\alpha)$. K is a closed set in ∂D such that $\Lambda^\beta(K) = 0$. Consequently, for $\varepsilon > 0$ there can be found a covering $\{G_n^{[\varepsilon]};\, n = 1, \cdots, N(\varepsilon)\}$ of K such that (i) $G_n^{[\varepsilon]}$ is a d-dimensional closed cube with edge of length $a_n = 2^{-k_n} < \varepsilon$, $k_n \in \mathbf{Z}^+$, and $a_1 \geq a_2 \geq \cdots \geq a_{N(\varepsilon)}$; (ii) $(G_n^{[\varepsilon]})^\circ \cap (G_m^{[\varepsilon]})^\circ = \emptyset$ if $n \neq m$; (iii) $\sum_{n=1}^{N(\varepsilon)} a_n^\beta \leq 1$. This $\{G_n^{[\varepsilon]}\}$ is called the standard covering of K corresponding to ε if

$$\sum_{n=1}^{N(\varepsilon)} {a_n}^{d-\gamma(\alpha)} \to 0$$

as $\varepsilon \searrow 0$. We need the following lemma.

Lemma 7.(Chabrowski (1991)) *Let $\{G_n^{[\varepsilon]}\}$ be the standard covering of K corresponding to some $\varepsilon > 0$. Then there exists a family of functions $\{g_n\}_n$ such that*

(a) $g_n \in pC_0^\infty(R^d)$, $\quad \mathrm{supp} g_n \subset 2G_n \quad for \quad \forall n$ (b) $0 \leq \sum_{n=1}^{N(\varepsilon)} g_n(x) \leq 1 \quad for \quad \forall x \in R^d$

(c) $\sum_n g_n(x) = 1 \quad for \quad \forall x \in \cup_{n=1}^{N(\varepsilon)}(3/2)G_n^{[\varepsilon]}$

(d) *there exists a constsnt $c = c(d) > 0$ such that for $x \in R^d$, $n = 1, \cdots, N(\varepsilon)$*

$$\left|\frac{\partial}{\partial x_i}\sum_{j=1}^{n} g_j(x)\right| \leq \frac{c}{a_n}, \quad \left|\frac{\partial^2}{\partial x_i \partial x_k}\sum_{j=1}^{n} g_j(x)\right| \leq \frac{c}{a_n^2}, \quad (i, k = 1, \cdots, d)$$

(cf. Lemma 6.1, p.91, [C91]).

For an arbitrary $\varepsilon > 0$, choose $\{g_n\}_n$ as in Lemma 7. Put $k_p(x) = \sum_{j=1}^{p} g_j(x)$, and $h_p(x) = 1 - k_p(x)$ for any $x \in R^d$, $(0 \le p \le N \equiv N(\varepsilon))$. Take $g \in C^{1,1}(\bar{D}) \cap W_0^{1,\infty}(D)$ with compact support in $\bar{D}$. Since $g \cdot h_N \in C^{1,1}(\bar{D}) \cap W_0^{1,\infty}(D)$ with $\text{supp}(g \cdot h_N)$ (which is contained in $\bar{D} \setminus K$), by (8) we obtain

$$\int_D \{-u \cdot L^*(gh_N)\}dx + \int_D u^\alpha \cdot gh_N dx = -\int_{\partial D} f \cdot \frac{\partial(gh_N)}{\partial n} S(dy). \tag{9}$$

Clearly it follows that

$$\lim_{\varepsilon \to 0} \int_D u^\alpha \cdot gh_N dx = \int_D u^\alpha \cdot g dx, \qquad \text{and} \tag{10}$$

$$\lim_{\varepsilon \to 0} \int_{\partial D} f \frac{\partial(gh_N)}{\partial n} dS = \int_{\partial D} f \frac{\partial g}{\partial n} dS. \tag{11}$$

Since $L^*(gh_N) = \sum_{i,j} \partial_i(a_{ij}\partial_j(gh_N)) + \sum_i \hat{b}_i \partial_i(gh_N) - c(gh_N)$ with $\hat{b}_i = -b_i + \sum_j \partial_j a_{ij}$ and $c = -\sum_i \partial_i \hat{b}_i$, we have

$$\begin{aligned} I_1 &:= \int_D u \cdot \sum_{i,j} \partial_i \left(a_{ij} \cdot \partial_j[gh_N]\right) dx \\ &= \int_D u \sum_{i,j} (\partial_i a_{ij})(\partial_j[gh_N])dx + \int_D u \sum_{i,j} a_{ij}(\partial_{ij}^2[gh_N])dx \equiv I_{11} + I_{12}. \end{aligned}$$

As to I_{11} it suffices to estimate the integral of the summation of those terms like $(\partial_i a_{ij})$ $(\partial_j g)h_N$, $(\partial_i a_{ij})g\cdot$ $(\partial_j h_N)$. Likewise, as to I_{12} we need to consider the sum of the terms $\partial_{ij}^2 g \cdot h_N$, $\partial_j g \cdot \partial_i h_N$, $\partial_i g \cdot \partial_j h_N$, and $g \cdot \partial_{ij}^2 h_N$. Set

$$I_2 := \int_D u \cdot \sum_i \hat{b}_i \left(\partial_i[gh_N]\right) dx = -\int_D u \sum_i b_i \cdot \partial_i[gh_N]dx + \int_D u \sum_{i,j} (\partial_j a_{ij}) \cdot \partial_i[gh_N]dx.$$

As for I_2, we have to take care of the terms $\partial_i g \cdot h_N + g \cdot \partial_i h_N$ multiplied by b_i or by $\partial_j a_{ij}$. Moreover, we put

$$I_3 := \int_D c[gh_N]dx = \int_D \sum_i \partial_i b_i \cdot [gh_N]dx - \int_D \sum_{i,j} (\partial_{ij}^2 a_{ij}) \cdot [gh_N]dx.$$

Because it is rather longsome to discuss all of the above integral terms, we shall mention below only two of them. Those calculations essentially explain almost everything involved with the others. For instance, let us consider the integral $I_{12*} = \int u \cdot \sum_{i,j} a_{ij}\partial_i g \cdot \partial_j h_N \, dx$. Since

$$\text{supp}\left(\sum_{j=1}^{N(\varepsilon)} g_j(x)\right) \subset \bigcup_{j=1}^{N(\varepsilon)} 2G_j^{[\varepsilon]}$$

from the condition (a) of Lemma 7, we have $\text{supp}(h_N(x)) \subset \cup_{j=1}^{N} 2G_j^{[\varepsilon]}$. By the assumptions on the coefficients $A = (a_{ij})$, we can find some constant $C > 0$ and I_{12*} is able to be estimated by

$$C \int_{D \cap (\cup_{j=1}^{N} 2G_j^{[\varepsilon]})} u \cdot \sum_{i=1}^{d} \left| \frac{\partial h_N}{\partial x_i} \right| dx \tag{12}$$

because $g \in C^{1,1}(\bar{D})$. For simplicity, set $\mathcal{D}(G_*, N, \varepsilon) := D\cap\ (\cup_{j=1}^N\ 2G_j^{[\varepsilon]})$, and

$$A := \int_{\mathcal{D}(G_*,N,\varepsilon)} u^{1+\frac{1}{\gamma(\alpha)-1}} dx, \quad B := \int_{\mathcal{D}(G_*,N,\varepsilon)} \left(\sum_{i=1}^d \left| \frac{\partial h_N}{\partial x_i} \right| \right)^{\gamma(\alpha)} dx.$$

An application of the Hölder inequality to (12) reads Eq.(12) $\leq C\cdot A^{1-1/\gamma(\alpha)}\cdot B^{1/\gamma(\alpha)}$. Note that $A \to 0$ as $\varepsilon \to 0$ since $u \in L^{1+1/\{\gamma(\alpha)-1\}}(dx)$ and the Lebesgue measure of $\cup_i 2G_i^{[\varepsilon]}$ vanishes as $\varepsilon \to 0$. So that, if B is bounded, then we know that I_{12*} becomes null as ε goes to zero. The boundedness of B yields from the following estimate. Put

$$U_N := 2G_N^{[\varepsilon]}, \quad \text{and} \quad U_p := 2G_p^{[\varepsilon]} - \bigcup_{i=p+1}^{N(\varepsilon)} 2G_i^{[\varepsilon]}, \quad (1 \leq p \leq N-1).$$

Notice that $h_N = h_p$ on U_p $(p = 1, 2, \cdots, N)$. On this account, we can deduce that

$$\begin{aligned} B &= \int_{D\cap(\cup_{p=1}^N U_p)} \left(\sum_{i=1}^d \left| \frac{\partial h_N}{\partial x_i} \right| \right)^{\gamma(\alpha)} dx \leq C(\gamma) \sum_{p=1}^{N(\varepsilon)} \sum_{i=1}^d \int_{D\cap U_p} \left| \frac{\partial h_N}{\partial x_i} \right|^{\gamma(\alpha)} dx \\ &\leq C'(\gamma, d) \sum_{p=1}^{N(\varepsilon)} a_p^{d-\gamma(\alpha)} \leq C'(\gamma, d), \end{aligned}$$

by employing (d) of Lemma 7 and the condition (iii) of the covering $\{G_n^{[\varepsilon]}\}$ of K. Next let us consider the integral $I_{12\diamond} = \int u\cdot\ g \sum_{i,j}\ a_{ij}\ (\partial_{ij}^2 h_N)\ dx$. Since $g \in C^{1,1}(\bar{D})$, we can estimate similarly

$$(13) \quad I_{12\diamond} \leq C\|g/p\|_\infty \int_D u \sum_{i,j} \partial_{ij}^2 h_N p(x) dx \leq C_1 \|u\|_{L^\alpha(d\mu)} \cdot \left(\int_D \left| \sum_{i,j} \partial_{ij}^2 h_N \right|^\beta \mu(dx) \right)^{1/\beta}$$

by making use of Hölder's inequality with $1/\alpha + 1/\beta = 1$. The same discussion in estmating (12) is valid, too, for (13). $\int_{D\cap(\cup_n 2G_n)}\ u^\alpha\ d\mu$ vanishes as ε tends to zero, because the covering $\{G_n^{[\varepsilon]}\}$ is standard. Thus we obtain that $I_{12\diamond} \to 0$ as $\varepsilon \to 0$. The computation goes almost similarly for the rest of other terms. Consequently we obtain

$$I_1 \to \int_D u \sum_{i,j} (\partial_i a_{ij}) \partial_j g dx + \int_D u \sum_{i,j} a_{ij} \partial_{ij}^2 g dx, \quad I_2 \to \int_D u \cdot \sum_i \hat{b}_i (\partial_i g) dx,$$

and $I_3 \to \int_D c \cdot g dx$ as $\varepsilon \to 0$. This concludes the assertion (cf. [D99a]).

4. Probabilistic Characterization

Next we shall discuss the equivalence problem to the RBS. Let $\xi = (\xi_t, \Pi_x)$ be the L-diffusion process. $\tau = \inf\{t > 0;\ \xi_t \notin D\}$ is the first exit time of the process ξ from the domain D. A boundary element $x \in \partial D$ is called a regular point if $\Pi_x(\tau = 0) = 1$ holds for the first exit time τ. When we say that the domain D is regular, we mean that D has a regular boundary. $M_F(R^d)$ denotes the totality of finite measures on R^d. $\langle \mu, f \rangle$ indicates the integral of f with respect to the measure $d\mu$. Let $X = (\Omega, \mathcal{F}, P_m,\ X_t, \mathcal{F}_t)$ be a finite measure valued branching Markov process associated with the equation $\mathcal{L} = Lu - u^\alpha = 0$ in the sense of Dynkin [Dy94]. Alternatively, for each $m \in M_F(R^d)$, there exists a probability measure P_m on $(\Omega, \mathcal{F})$ such that $X_0 = m$, P_m-a.s., and for $\varphi \in \mathrm{Dom}(L)$

$$M_t(\varphi) := \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \int_0^t \langle X_s, L\varphi \rangle ds, \quad \forall t \geq 0$$

is a continuous $(\mathcal{F}_t)$-martingale under P_m, and its quadratic variation process is given by

$$\langle M.(\varphi)\rangle_t = \int_0^t \langle X_s, \varphi^2\rangle ds, \quad \forall t \geq 0, \quad P_m - \text{a.s.}$$

The support suppX_t of a random measure X_t for each $t > 0$ is the minimal closure of closed sets $G \subset R^d$ such that $X_t(G^c) = 0$ holds. The range of X is defined by

$$\mathcal{R}(X) := \bigcup_{\varepsilon>0} \left(\overline{\bigcup_{t\geq\varepsilon} \text{supp} X_t}\right)^{closure}.$$

Note that $\mathcal{R}(X)$ is a random set. We say that a set F is $\mathcal{R}$-polar if $P_x(\mathcal{R}(X) \cap F \neq \emptyset) = 0$ holds for $\forall x \notin F$. Similarly we may define the concept of boundary polar set. We say that a set K is ∂-polar if $P_x(\mathcal{R}(\tilde{X}_D) \cap K \neq \emptyset) = 0$ holds for $\forall x \notin K$, where $\tilde{X}_D$ is a part of X in the domain D [Dy91](or see [D99b]).

Theorem 8.([D99c]) *Let D be a bounded regular domain in R^d. Then K is the RBS if and only if K is ∂-polar.*

Proof. The assertion yields directly from the result of [DK96a](see also [BP84], [D99b]). So we suppress the details of the proof. Instead, we shall rather give below only the basic idea, key point and outline. Let $1 < \alpha \leq 2$ because of the restriction on the corresponding process in the probability theory which we are relying on. From the argument in Theorem 1, the existence of singularity is allowed if $d > \gamma(\alpha)$ for $\alpha > 1$. It is well known that the sets A $(\subset R^d)$ with $\dim_H(A) > d - \gamma(\alpha)$ cannot be S-polar. Corollary in Dynkin(1991) suggests that ∂-polar K is the RBS together with Theorem 6, because the S-polarity induces the $\mathcal{R}$-polarity and then $\dim_H(K) < d - \gamma(\alpha)$. We call $\beta = d - \gamma(\alpha)$ the critical dimension for $\mathcal{R}$-polarity. We write $\text{Cap}_x^{\partial D}$ for the capacity on the boundary ∂D associated with the range $\mathcal{R}(\tilde{X}_D)$ under the measure P_x. As a matter of fact, by Choquet's capacity theory, Γ is ∂-polar iff $\text{Cap}_x^{\partial D}(\Gamma) = 0$ for all $x \in D$. While, for the Bessel capacity $\text{Cap}_{r,p}$, the class of $\mathcal{R}$-polar sets for any (L, α)-superdiffusion X is identical to the class of null sets of the capacity $\text{Cap}_{2,\{\frac{\alpha}{\alpha-1}\}}$. Based upon this result, it can be deduced that the class of ∂-polar sets is the same as the class of null sets for the Poisson capacity $\text{Cap}^L_{\alpha/(\alpha-1)}$, where

$$\text{Cap}_p^L(F) := \sup\left\{\nu(F); \int_D m(dx)\left[\int_F k_L(x,y)\nu(dy)\right]^{\frac{p}{p-1}} \leq 1\right\}$$

for a compact set F with $\nu \in M_F(K)$ and an admissible measure $m(dx)$ on D (cf. Theorem 1.2a, [DK96a]). Moreover, the above-mentioned class also coincides with the class of null sets for the Riesz capacity $\text{Cap}^{\partial}_{2/\alpha,\{\alpha/(\alpha-1)\}}$. According to the Dynkin-Kuznetsov general theory for the removability of singularity, we can show that Γ is a weak RBS if $\text{Cap}^{\partial}_{2/\alpha,\{\alpha/(\alpha-1)\}}(\Gamma) = 0$. Since every weak RBS is ∂-polar, the assertion of Theorem 8 is established via the argument on the explicit representation of solution $u(x) = -\log P_{\delta_x}$ exp $(-\langle \bar{X}_\tau, f\rangle)$ to the problem (1), where $\bar{X}_\tau(B) := X_\tau(R_+ \times B)$, $\forall B \in \mathcal{B}(\mathbf{R}^d)$ with the first exit time τ from D (cf. [Dy91], [Dy94], [D99b]).

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