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A UNIFIED APPROACH TO ANALYTICAL SOLUTION OF A HYDROMAGNETIC FREE CONVECTION FLOW

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ABSTRACT. Free convection flow near an infinite vertical plate of an electrically conducting fluid of Prandtl number unity has been considered when the flow takes place under the combined influence of heat flux at the boundary and an externally applied magnetic field. For the impulsive and accelerated motions of the boundary, unified closed form analytical solutions have been obtained for the fluid velocity and the skin friction corresponding to the cases of magnetic field fixed relative to the fluid or to the boundary. As a special case, an approximation of the solution in terms of exponential functions is also discussed.

1. Introduction. Laminar free convection at a vertical wall is of interest in many applications such as cooling of nuclear reactors, heat exchangers, solar energy collectors and crystal growth, among others. As the magnetic field is known to be an efficient mechanism through which flow and heat transfer in electrically conducting fluids can be controlled, the flow and heat transfer features of such media also find applications in the design of magnetohydrodynamic (MHD) generators, MHD flow meters and MHD pumps. When an electrically conducting fluid flows under the influence of an externally applied magnetic field, the Navier-Stokes equations governing the resulting MHD flow are highly coupled and non-linear. General features of the flow are thus describable only in terms of numerical solutions. However, exact solutions of the Navier-Stokes equations can be obtained for some special cases when the effects of the quadratic convection terms are negligible. In this paper, we consider one such problem in which the fluid flow is assumed to be generated by the motion, in its own plane, of an infinite vertical plate bounding the fluid as well as heat flux at the plate.

The boundary layer flows of fluids past flat plates have been discussed extensively in literature due largely to their relative simplicity of analyses coupled with their utility as idealized models for the studies of more complicated fluid-body interactions. When the fluid flow takes place near regular boundaries such as the vertical plate considered here, it can be shown that the governing equations reduce to linear partial differential equations and can be analyzed exactly, in many cases. The solutions of the resulting equations are still dependent on the type of initial and boundary conditions to be imposed. For instance, Gebhart *et al.* [1] have given a detailed account of the implications of different types of boundary conditions for convection flows. The solutions of hydromagnetic free convection flows near plates of infinite extent have been discussed by several authors (see, e.g., [2 – 6]) under different physical conditions. The objective of the present work is to present a unified closed form solution for the unsteady flow of an electrically conducting fluid

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under the influence of an external magnetic field when the flow takes place near an infinite vertical wall which is subject to continuous motion as well as heat flux. Since the influence of the magnetic field on the flow features is known to be not uniform in the sense that the velocity profiles may not be similar for the cases of magnetic lines of force being fixed relative to the fluid or to the moving boundary, we shall consider these two cases and obtain the corresponding solutions. The closed form exact solutions have been presented for two types of plate motion: (i) impulsive and (ii) accelerated. The solutions have been presented for the special class of fluids of Prandtl number, Pr, equal to unity. This corresponds to those fluids whose velocity and thermal boundary layer thicknesses are of the same order of magnitude. This simplifying assumption on Pr has enabled us to present the exact solutions for the boundary layer velocity variable in terms of reasonably simple analytical expressions involving exponential and complemetary error functions. The solution of the unsteady flow problem has been obtained using Laplace transforms. Furthermore, using an approximation of the complementary error function [7], it is indicated that the transient solution can be presented in an exponential form. This solution, though approximate, has been shown to possess the qualitative features of the original solution.

In Section 2, the governing equations have been presented with the relevant initial and boundary conditions. These have been solved exactly for the two special types of the boundary motion mentioned above. The solutions for the boundary layer velocity variables have been used to calculate the skin frictions. The effects of magnetic field and buoyancy force on the boundary layer flow and skin friction have been discussed in Section 3. Following [7], an approximation of the exact solution in terms of exponential functions has also been discussed.

2. Governing Equations and Solution. The physical situation corresponds to that of the unsteady two-dimensional flow of an electrically conducting fluid of Prandtl number equal to unity past an infinite vertical flat plate which is assumed to be non-conducting. With respect to an arbitrary origin O on this planar wall, the axis Ox' is taken along the wall in the upward direction and the axis Oy' is taken perpendicular to it into the fluid. For times $t' \leq 0$, the plate and the fluid medium are at rest and at the constant temperature T'_∞. At time $t' > 0$, the plate is set into motion with a velocity proportional to ${t'}^n$, and simultaneously, heat is also supplied to the plate at a constant rate. The flow takes place under the influence of an external magnetic field of constant strength $(0, B_y, 0)$ applied in the y' direction. We assume that magnetic Reynolds number is very small which corresponds to negligible induced magnetic field compared to externally applied one. Two different flow situations will be considered here with respect to the magnetic force. The first corresponds to the case when the magnetic lines of force are fixed relative to the fluid, and the other when these are fixed relative to the boundary. These two cases will, however, be embedded into a single momentum equation. As is common in flow problems past flat plates of the type considered here, we further assume that the convective and pressure gradient terms in the momentum and energy equations are negligible. Moreover, as a result of the boundary layer approximations on the flow variables, the physical variables become functions of the time variable t' and the space variable y' only. Thus, when the magnetic field is fixed relative to the fluid, the usual momentum and thermal boundary layer equations can be written in the form

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial {y'}^2} + g\beta(T' - T'_\infty) - \frac{\sigma B_y^2}{\rho} u' \tag{1}$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho c_p} \frac{\partial^2 T'}{\partial {y'}^2} \tag{2}$$

where u' is the velocity in the x' direction, T' is the temperature of the fluid, g the acceleration due to gravity, β the volumetric coefficient of thermal expansion, ν the kinematic viscosity, ρ the density, k the thermal conductivity and c_p is the specific heat of the fluid at constant pressure.

When the magnetic field is fixed relative to the boundary moving with velocity Ut'^n, where U is a constant, the momentum equation (1) takes the form

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2} + g\beta(T' - T'_\infty) - \frac{\sigma B_y^2}{\rho}\left(u' - Ut'^n\right) \tag{3}$$

Equations (1) and (3) can be combined into the single equation

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2} + g\beta(T' - T'_\infty) - \frac{\sigma B_y^2}{\rho}\left(u' - \lambda Ut'^n\right) \tag{4}$$

where

$$\lambda = \begin{cases} 0, & \text{if } B_y \text{ is fixed relative to the fluid} \\ 1, & \text{if } B_y \text{ is fixed relative to the plate.} \end{cases} \tag{5}$$

Equations (2) and (3) are to be supplemented by the initial and boundary conditions

$$\begin{aligned} u' = 0, \quad T' = 0, \quad &\text{for} \quad y' \geq 0 \quad \text{and} \quad t' \leq 0 \\ u' = Ut'^n, \quad \frac{\partial T'}{\partial y'} = -\frac{q}{k} \quad &\text{at} \quad y' = 0 \quad \text{for} \quad t' > 0 \\ u' \to 0, \quad T' \to T'_\infty \quad &\text{as} \quad y' \to \infty \quad \text{for} \quad t' > 0 \end{aligned} \tag{6}$$

where q is the heat flux per unit area at the plate. The flow described by equations (2), (4) and (6) is quite general in that the initial motion of the vertical plate is given by a power law in the time variable. As our objective is to obtain the solution in terms of certain non-dimensional parameters characterising the magnetic and buoyancy forces, it is necessary to non-dimensionalise the governing equations. This would necessitate solving the problem for specific values of n which, in turn, introduces separate non-dimensionalisations of the variables. Herein, we propose to obtain the explicit solutions for two types of boundary motion which are commonly discussed in literature, namely, impulsive and accelerated motions. These correspond to $n = 0$ and $n = 1$, respectively. As noted before, the solution will be presented for the case $\mathrm{Pr} = 1$.

Impulsive Motion. In this case, we introduce the non-dimensional quantities

$$\begin{aligned} y &= Uy'/\nu, \quad t = U^2 t'/\nu, \quad u = u'/U \\ T &= kU(T' - T'_\infty)/(\nu q), \quad \mathrm{Pr} = c_p \nu \rho / k \\ m_0 &= \sigma \nu B_y^2/(\rho U^2), \quad G_0 = qg\beta\nu^2/(kU^4) \end{aligned} \tag{7}$$

In the above, Pr is the Prandtl number, and m_0 and G_0 are the magnetic and buoyancy parameters, respectively. Using equation (7), equations (2) and (4) can be expressed, when $Pr = 1$, in the dimensionless forms

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} \tag{8}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m_0(u - \lambda) + G_0 T \tag{9}$$

The initial and boundary conditions become

$$
\begin{aligned}
u = 0, \quad T = 0 \quad &\text{for} \quad y \geq 0 \quad \text{and} \quad t \leq 0 \\
u = 1, \quad \partial T/\partial y = -1 \quad &\text{at} \quad y = 0 \quad \text{for} \quad t > 0 \\
u \to 0, \quad T \to 0 \quad &\text{as} \quad y \to \infty \quad \text{for} \quad t > 0
\end{aligned} \tag{10}
$$

In order to solve equation (9) subject to the conditions (10), we have to first obtain the temperature distribution from equation (8). The exact solutions will be obtained using Laplace transforms. Defining the transform variables

$$\bar{u}(y,s) = \int_0^\infty u \, \exp(-st)\, \mathrm{d}t, \quad \overline{T}(y,s) = \int_0^\infty T \, \exp(-st)\, \mathrm{d}t$$

and taking the Laplace transforms of equations (8) and (9) will result in a set of (ordinary) differential equations for the transformed functions in the independent variable y. On solving them with the corresponding transformed boundary conditions, we obtain

$$\overline{T}(y,s) = s^{-3/2} \exp\left(-y\sqrt{s}\,\right) \tag{11}$$

$$
\begin{aligned}
\bar{u}(y,s) &= \frac{G_0}{m_0}\left[\overline{T}(y,s) - s^{-3/2} \exp\left(-y\sqrt{s+m_0}\,\right)\right] \\
&\quad + \lambda\left[\frac{m_0}{s\,(s+m_0\,)} + \frac{\exp\left(-y\sqrt{s+m_0}\,\right)}{s+m_0}\right] \\
&\quad + (1-\lambda)s^{-1}\exp\left(-y\sqrt{s+m_0}\,\right)
\end{aligned} \tag{12}
$$

On inversion of equations (11) and (12), we obtain [8, 9]

$$T(y,t) = 2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right) - y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) \tag{13}$$

$$u(y,t) = u_1^{(0)}(y,t) + u_2^{(0)}(y,t) + u_3^{(0)}(y,t) + u_4^{(0)}(y,t) \tag{14}$$

where

$$
\begin{aligned}
u_1^{(0)}(y,t) &= G_0\,T(y,t)/m_0 \\
u_2^{(0)}(y,t) &= -\frac{G_0}{2m_0\sqrt{\pi}}\int_0^t \frac{\varphi_-(y,\xi)+\varphi_+(y,\xi)}{\sqrt{t-\xi}}\,\mathrm{d}\xi \\
u_3^{(0)}(y,t) &= \lambda\left[1-\exp\left(-m_0t\right)+\exp\left(-m_0t\right)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] \\
u_4^{(0)}(y,t) &= \frac{1-\lambda}{2}\left[\varphi_-(y,t)+\varphi_+(y,t)\right] \\
\varphi_\mp(y,t) &= \exp(\mp y\sqrt{m_0}\,)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} \mp \sqrt{m_0t}\right)
\end{aligned}
$$

and erfc denotes the complementary error function defined by

$$\mathrm{erfc}(x) = 1-\mathrm{erf}(x), \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-\eta^2\right)\mathrm{d}\eta$$

In the above, we note that $u_2^{(0)}(y,t)$ is given in terms of a convolution integral. It can also be expressed in the form

$$u_2^{(0)}(y,t) = -\frac{G_0}{m_0\sqrt{\pi}}\int_0^{\sqrt{t}}\left[\phi_-(y,z,t)+\phi_+(y,z,t)\right]\mathrm{d}z \tag{15}$$

where

$$\phi_{\mp}(y,z,t) = \exp(\mp y\sqrt{m_0}\,)\ \mathrm{erfc}\left(\frac{y}{2\sqrt{t-z^2}} \mp \sqrt{m_0(t-z^2)}\right)$$

Equation (14) gives the velocity distribution in the boundary layer. An associated quantity of practical interest in the boundary layer flow is the skin friction. Denoting it by τ $\left(= -\,\partial u/\partial y|_{y=0}\right)$, we can write

$$\tau = \tau_1^{(0)} + \tau_2^{(0)} + \tau_3^{(0)} + \tau_4^{(0)} \tag{16}$$

where

$$\begin{aligned}
\tau_1^{(0)} &= G_0/m_0 \\
\tau_2^{(0)} &= \frac{2G_0}{\pi m_0}\int_0^{\sqrt{t}} \frac{f_1(z,t)-f_2(z,t)}{\sqrt{t-z^2}}\,\mathrm{d}z \\
\tau_3^{(0)} &= \frac{\lambda}{\sqrt{\pi t}}\exp\left(-m_0 t\right) \\
\tau_4^{(0)} &= (1-\lambda)\left[\sqrt{m_0}\left\{1-\mathrm{erfc}\left(\sqrt{m_0 t}\right)\right\} + \frac{\exp\left(-m_0 t\right)}{\sqrt{\pi t}}\right] \\
f_1(z,t) &= \sqrt{\pi m_0\left(t-z^2\right)}\left\{\mathrm{erfc}\left(\sqrt{m_0\left(t-z^2\right)}\right)-1\right\} \\
f_2(z,t) &= \exp\left\{m_0\left(z^2-t\right)\right\}
\end{aligned}$$

Accelerated Motion. In this case $u' = Ut'$ at $y' = 0$ for $t' > 0$, and the non-dimensional variables are defined by

$$\begin{aligned}
y &= y'(U/\nu^2)^{1/3}, \quad t = t'(U^2/\nu)^{1/3} \\
u &= u'/(\nu U)^{1/3}, \quad T = kU^{1/3}(T'-T'_\infty)/(q\nu^{2/3}) \\
m_1 &= \sigma\nu^{1/3}B_y^2/(\rho U^{2/3}), \quad G_1 = qg\beta\nu^{2/3}/(kU^{4/3})
\end{aligned} \tag{17}$$

As before, the dimensionless parameters m_1 and G_1 represent the effects of the magnetic and buoyancy forces, respectively, under the modified boundary motion. The momentum equation to be solved is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m_1(u-\lambda t) + G_1 T \tag{18}$$

The solution procedure runs parallel to the case of impulsive motion discussed above, and is not repeated, for brevity. In view of the boundary condition $u = t$ at $y = 0$, $\bar{u}(y,s)$ in this case assumes the form

$$\begin{aligned}
\bar{u}(y,s) = {} & \frac{G_1}{m_1}\left[\overline{T}(y,s) - s^{-3/2}\ \exp\left(-y\sqrt{s+m_1}\,\right)\right] \\
& + \frac{\lambda m_1}{(s+m_1)s^2}\left[1-\exp\left(-y\sqrt{s+m_1}\,\right)\right] \\
& + s^{-2}\exp\left(-y\sqrt{s+m_1}\,\right)
\end{aligned} \tag{19}$$

Taking the inverse transform of equation (19), it can be shown that

$$u(y,t) = u_1^{(1)}(y,t) + u_2^{(1)}(y,t) + u_3^{(1)}(y,t) + u_4^{(1)}(y,t) \tag{20}$$

where

$$\begin{aligned}
u_i^{(1)}(y,t) &= u_i^{(0)}(y,t), \quad (i=1,2) \\
u_3^{(1)}(y,t) &= \frac{\lambda}{m_1}\left[\frac{1}{2}\left\{\varphi_-(y,t)+\varphi_+(y,t)\right\}\right. \\
&\qquad \left. + \, m_1 t - 1 + \exp\left(-m_1 t\right) \operatorname{erf}\left(\frac{y}{2\sqrt{t}}\right)\right] \\
u_4^{(1)}(y,t) &= \frac{1-\lambda}{2}\left[\left(t-\frac{y}{2\sqrt{m_1}}\right)\varphi_-(y,t)\right. \\
&\qquad \left. + \left(t+\frac{y}{2\sqrt{m_1}}\right)\varphi_+(y,t)\right]
\end{aligned}$$

and it is understood that $u_i^{(1)}(y,t)$, $(i=1 \text{ to } 4)$ will have G_1 and m_1 in place of G_0 and m_0, respectively. The skin friction τ is given by

$$\tau = \tau_1^{(1)} + \tau_2^{(1)} + \tau_3^{(1)} + \tau_4^{(1)} \tag{21}$$

where, as in the case of velocity, $\tau_i^{(1)} = \tau_i^{(0)}$, $(i=1,2)$, and

$$\begin{aligned}
\tau_3^{(1)} &= \frac{\lambda}{\sqrt{m_1}}\left[1-\operatorname{erfc}\left(\sqrt{m_1 t}\right)\right] \\
\tau_4^{(1)} &= (1-\lambda)\left[\left(\frac{1+2m_1 t}{2\sqrt{m_1}}\right)\left\{1-\operatorname{erfc}\left(\sqrt{m_1 t}\right)\right\}\right. \\
&\qquad \left. + \sqrt{\frac{t}{\pi}}\,\exp(-m_1 t)\right]
\end{aligned}$$

3. Numerical Results. In this section, we have presented the computed values of the exact solutions obtained in the previous section for specific values of the magnetic and buoyancy parameters. For simplicity and uniformity of presentation, the parameters have been given common values for both impulsive and accelerated motions of the plate, although their definitions in equations (7) and (17), respectively, differ due to the non-dimensionalisation processes. We shall thus use them without the suffixes in this section. The computed results are shown in Figs. 1 and 2, and Tables 1 and 2. The computation of the velocity variables given by equations (14) and (20), and the skin frictions given by equations (16) and (21) requires the evaluation of the convolution integrals for the components $u_2(y,t)$ and $\tau_2(t)$ as well as complementary error functions. The integrals were evaluated using Simpson's rule. The arguments of the complementary error functions are real for the present problem. The complementary error function is defined in terms of an integral [see after equation (14)] which, in turn, can be expressed in terms of the infinite series

$$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{22}$$

A FORTRAN program was written to compute the boundary layer velocity and skin friction, and the velocity profiles in Figs. 1 and 2 are the Excel plots of these computed values. Figure 1 shows the temporal and spatial variations of the boundary layer velocity for the impulsive motion of the boundary while Fig. 2 shows the corresponding results for

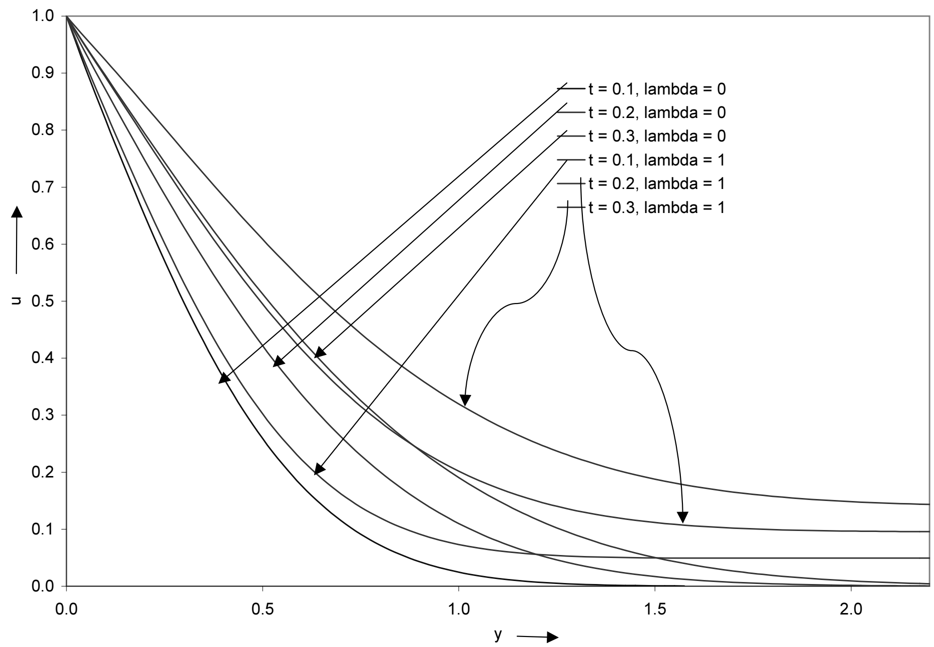

Figure 1: Velocity u due to impulsive start of the plate. ($G = 1.0$, $m = 0.5$)

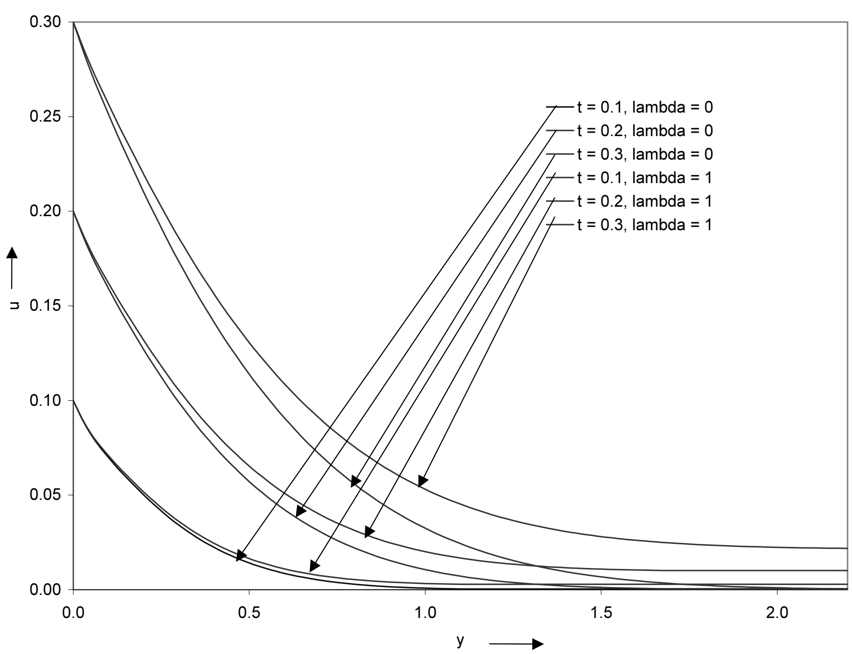

Figure 2: Velocity u due to accelerated start of the plate. ($G = 1.0$, $m = 0.5$)

the case of accelerated motion. The curves also show the effects of the magnetic field when it is fixed relative to the fluid ($\lambda = 0$) or fixed relative to the boundary ($\lambda = 1$). It is seen that the magnitude of the boundary layer velocity is always less in the former case than in the latter. The transient motion of the fluid is such that the velocity decreases from its initial value at the boundary to its free-stream stationary value through layers of different depths, and the boundary layer thicknesses depend on the initial conditions as well as the parameter values. As far as the temporal variations are concerned, the velocity increases gradually in the boundary layer until it attains its steady state profile.

The effect of the magnetic field on the fluid flow is shown in Table 1 for both types of motion. It is seen that when the magnetic field is fixed relative to the fluid, it has a diminishing effect on the fluid velocity, and the reverse occurs when it is fixed relative to the plate. Although not shown in the table, for brevity, this feature was seen to be valid for different values of G. The change in the velocity profiles is a consequence of the force exerted by the magnetic field on the fluid.

Table 1: Velocity u when $G = 1.0$, $t = 0.1$

y	u [Impulsive]				u [Accelerated]			
	$\lambda = 0$		$\lambda = 1$		$\lambda = 0$		$\lambda = 1$	
	m		m		m		m	
	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5
0.0	1.000	1.000	1.000	1.000	0.100	0.100	0.100	0.100
0.2	0.656	0.650	0.662	0.676	0.050	0.050	0.050	0.052
0.4	0.373	0.366	0.381	0.406	0.023	0.022	0.023	0.025
0.6	0.181	0.176	0.190	0.222	0.009	0.009	0.009	0.011
0.8	0.074	0.072	0.084	0.120	0.003	0.003	0.003	0.005
1.0	0.025	0.025	0.035	0.073	0.001	0.001	0.001	0.003

The influence of the magnetic field and the bouyancy force on the shear rate at the boundary is shown in Table 2. It may be noted here that the analytical expressions for the skin friction given by equation (16) cannot be used for accurate computation because the integral in τ_2 is undefined when $z = \sqrt{t}$. However, on integrating by parts, the sum of the first and second terms of τ in equation (16) can be written in the form

$$\begin{aligned} \tau_1^{(0)} + \tau_2^{(0)} &= \frac{2G}{\sqrt{\pi m_0}} \int_0^{\sqrt{t}} \left\{ \operatorname{erfc}\left(\sqrt{m_0\,(t - z^2)}\right) - 1 \right\} \, \mathrm{d}z \\ &\quad + \frac{4G}{\pi} \int_0^{\sqrt{t}} z f_2(z,t) \arcsin\left(z/\sqrt{t}\right) \, \mathrm{d}z \end{aligned} \tag{23}$$

In Table 2, we have presented two values for each τ. The values in the parentheses correspond to an exponential approximation of the complementary error function, which is explained later in this section. From Table 2, we note the opposite effects of the magnetic field on the skin friction depending on its mode of application as in the case of velocity. The skin friction when λ equals to zero is always larger than when it is equal to unity. This is true for both impulsive and accelerated motions. The effect of the magnetic field is to increase the skin friction when it is fixed relative to the fluid and the reverse occurs when it is fixed relative to the boundary. For the fluids of the type considered here (Pr = 1.0), the skin friction decreases with G. Also, it decreases with time for impulsive motion and increases for accelerated motion of the boundary.

Table 2: Skin friction τ

G	t	m	τ[Impulsive]		τ [Accelerated]	
			$\lambda = 0$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$
0.5	0.1	0.1	1.7771	1.7416	0.3332	0.3308
			(1.7777)	(1.7434)	(0.3288)	(0.3205)
		0.3	1.8126	1.7067	0.3356	0.3285
			(1.8115)	(1.7080)	(0.3326)	(0.3215)
		0.5	1.8479	1.6724	0.3381	0.3263
			(1.8459)	(1.6735)	(0.3357)	(0.3211)
	0.2	0.1	1.2370	1.1868	0.4582	0.4515
			(1.2387)	(1.1899)	(0.4540)	(0.4406)
		0.3	1.2870	1.1386	0.4651	0.4452
			(1.2867)	(1.1415)	(0.4628)	(0.4395)
		0.5	1.3364	1.0922	0.4720	0.4390
			(1.3355)	(1.0936)	(0.4707)	(0.4359)
1.0	0.1	0.1	1.7523	1.7168	0.3084	0.3060
			(1.7548)	(1.7204)	(0.3059)	(0.2975)
		0.3	1.7879	1.6819	0.3109	0.3038
			(1.7881)	(1.6846)	(0.3092)	(0.2981)
		0.5	1.8232	1.6477	0.3134	0.3016
			(1.8223)	(1.6499)	(0.3121)	(0.2975)
	0.2	0.1	1.1872	1.1371	0.4085	0.4018
			(1.1920)	(1.1433)	(0.4074)	(0.3940)
		0.3	1.2375	1.0891	0.4156	0.3957
			(1.2391)	(1.0930)	(0.4152)	(0.3919)
		0.5	1.2871	1.0430	0.4227	0.3898
			(1.2876)	(1.0456)	(0.4228)	(0.3880)

We close this section with a comment on a possible approximation of the solutions obtained in Section 2 in terms of elementary functions. We note that equations (14) and (20) for the velocity, and equations (16) and (21) for the skin friction involve both exponential and complementary error functions. As mentioned before, the latter is defined in terms of an integral which can be approximated at best by the infinite series in equation (22). However, if one is interested in reasonable accuracy of the results not sacrificing their qualitative features, it may be remarked that the velocity and skin friction can be expressed in terms of solely exponential functions, following an approximation for the complementary error function suggested by Heinz [7] in the form

$$\operatorname{erfc}(x) = \exp\left(ax + bx^2\right) + \varepsilon(x), \quad x \geq 0 \tag{24}$$

where $a = -1.0692$, $b = -0.8067$, $\varepsilon(x) \leq (4.5)10^{-3}$

As the above approximation is valid for positive arguments of $\operatorname{erfc}(x)$, this would restrict the direct application of the approximation to the values of velocity in the boundary layer subject to $(y/t) > 2\sqrt{m_i}$, $(i = 0, 1)$. The approximation of u in the complement of this region may be obtained using $\operatorname{erfc}(x) = 2 - \operatorname{erfc}(-x)$, $(x < 0)$. However, the region restriction does not apply to the approximation of skin friction. For comparison of the results using the approximation, we have shown in Table 2, within parentheses, the approximate values of the skin friction τ. It is seen that the approximation yields qualitatively acceptable results.

It is worth noting in this respect that there are several classical approximations of error functions developed for using in digital computers [8]. For instance, one of the widely used classical approximations of $\operatorname{erfc}(x)$ can be expressed for $x \geq 0$ in the form [8]

$$\operatorname{erfc}(x) = \left(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5\right) \exp\left(-x^2\right) + \varepsilon(x) \tag{25}$$

where

$$\begin{aligned}
z &= (1+cx)^{-1}, \quad c = 0.3275911\\
a_1 &= 0.254829592, \;\; a_2 = -0.284496736, \;\; a_3 = 1.421413741\\
a_4 &= -1.453152027, \;\; a_5 = 1.061405429, \;\; |\varepsilon(x)| \leq (1.5)\left(10^{-7}\right)
\end{aligned}$$

Although the approximation given by equation (25) is of better accuracy than equation (24), the analytical expression of $\operatorname{erfc}(x)$ in equation (25) is much more complicated than in equation (24), and is therefore inconvenient to use in the solutions of the type occurring in equations (14) and (20). On the other hand, using equation (24), one can express the solutions exclusively in terms of exponential functions, which are more convenient for analyzing the qualitative features of the fluid motion.

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