INTUITIONISTIC FUZZY IDEALS OF Γ–RINGS

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Abstract. We introduce the notion of an intuitionistic fuzzy ideal of a gamma-ring M, and then some related properties are investigated. Characterizations of intuitionistic fuzzy ideals are given. Given an intuitionistic fuzzy ideal $A = (\mu_A, \gamma_A)$ of a Γ-ring and a homomorphism f of Γ-rings, we construct a new intuitionistic fuzzy ideal $A^f = (\mu_A^f, \gamma_A^f)$.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [9], several researchers were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. Jun et al considered the fuzzification of ideals in Γ-rings [4, 5, 6]. In this paper, we introduce the notion of an intuitionistic fuzzy ideal of a gamma-ring M, and then some related properties are investigated. Characterizations of intuitionistic fuzzy ideals are given. Also, for any intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ and a mapping of Γ-rings, we define a new intuitionistic fuzzy set $A^f = (\mu_A^f, \gamma_A^f)$. Then we show that if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M' then $A^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy ideal of M for every homomorphism f from a Γ-ring M to a Γ-ring M'; and if $A^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy ideal of M then $A = \mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M' where f is a homomorphism from a Γ-ring M onto a Γ-ring M'.

2. Preliminaries

If $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ are additive abelian groups, and for all x, y, z in M and all α, β in Γ, the following conditions are satisfied

(1) $x\alpha y$ is an element of M,
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a *Γ-ring*.

Through this paper M denotes a Γ-ring, and 0_M denotes the zero element of M unless otherwise specified.

A subset A of M is called a *left* (resp. *right*) *ideal* of M if A is an additive subgroup of M and

$$M\Gamma A = \{x\alpha y \mid x \in M, \alpha \in \Gamma, y \in A\} \text{ (resp. } A\Gamma M)$$

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is contained in A. If A is both a left and a right ideal, then A is a *two-sided ideal*, or simply an *ideal* of M. By a *fuzzy set* μ in a non-emptyset X we mean a function $\mu : X \to [0,1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. A fuzzy set μ in M is called a *fuzzy left* (resp. *right*) ideal of M if

(FI1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$,
(FI2) $\mu(x\alpha y) \geq \mu(y)$ (resp. $\mu(x\alpha y) \geq \mu(x)$),

for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set μ in M is called a *fuzzy ideal* of M if μ is a both a fuzzy left and a right ideal of M. We note that μ is a fuzzy ideal of M if and only if

(FI1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$,
(FI2) $\mu(x\alpha y) \geq \mu(x) \vee \mu(y)$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

Definition 2.1 ([2]). Let R be a non-empty fixed set. An *intuitionistic fuzzy set* (IFS for short) A is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in R\}$$

where the functions $\mu_A : R \to [0,1]$ and $\gamma_A : R \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in R$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in R$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in R\}$.

Definition 2.2 ([1,2]). Let X be a non-empty set and let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in X. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(iii) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\}$,
(iv) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)) : x \in X\}$,
(v) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)) : x \in X\}$,
(vi) $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}$,
(vii) $\Diamond A = \{(x, 1 - \gamma_A(x), \gamma_A(x)) : x \in X\}$.

Definition 2.3 ([1,2]). Let $\{A_i : i \in \Lambda\}$ be an arbitrary family of IFSs in X. Then

(i) $\cap A_i = \{(x, \wedge\mu_{A_i}(x), \vee\gamma_{A_i}(x)) : x \in X\}$,
(ii) $\cup A_i = \{(x, \vee\mu_{A_i}(x), \wedge\gamma_{A_i}(x)) : x \in X\}$.

Definition 2.4 ([3]). Let f be a map from a set X to a set Y. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IFSs in X and Y respectively, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is an IFS in X defined by $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$.

3. Intuitionistic fuzzy ideals

We start by defining the notion of intuitionistic fuzzy ideals.

Definition 3.1. An IFS $A = (\mu_A, \gamma_A)$ in M is called an *intuitionistic fuzzy ideal* of M if

(IF1) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x-y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in M$,
(IF2) $\mu_A(x\alpha y) \geq \mu_A(x) \vee \mu_A(y)$ and $\gamma_A(x\alpha y) \leq \gamma_A(x) \wedge \gamma_A(y)$ for all $x, y \in M$ and for all $\alpha \in \Gamma$.

Example 3.2. If G and H are additive abelian groups and $M = Hom(G,H)$, $\Gamma = Hom(H,G)$, then M is a Γ-ring with the operations pointwise addition and composition of homomorphisms ([1]). Define a fuzzy set $\mu_A : M \to [0,1]$ by $\mu_A(0_M) = 0.5, \mu_A(f) = 0.3$ and $\gamma_A : M \to [0,1]$ by $\gamma_A(0_M) = 0.2, \gamma_A(f) = 0.4$ where f is any member of M with $f \neq 0_M$. Then an IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of Γ-ring M.

Lemma 3.3. *If an* IFS $A = (\mu_A, \gamma_A)$ *in* M *satisfies the condition* (IF1), *then*
(i) $\mu_A(0) \geq \mu_A(x)$ *and* $\gamma_A(0) \leq \gamma_A(x)$,
(ii) $\mu_A(-x) = \mu_A(x)$ *and* $\gamma_A(-x) = \gamma_A(x)$,
for all $x \in M$.

Proof. (i) We have that for any $x \in M$,

$$\mu_A(0) = \mu_A(x - x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$$

and

$$\gamma_A(0) = \gamma_A(x - x) \leq \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x).$$

(ii) By using (i) we get

$$\mu_A(-x) = \mu_A(0 - x) \geq \mu_A(0) \wedge \mu_A(x) = \mu_A(x)$$

and

$$\gamma_A(-x) = \gamma_A(0 - x) \leq \gamma_A(0) \vee \gamma_A(x) = \gamma_A(x)$$

for all $x \in M$. Since x is arbitrary, we conclude that $\mu_A(-x) = \mu_A(x)$ and $\gamma_A(-x) = \gamma_A(x)$ for all $x \in M$, ending the proof. □

Proposition 3.4. *If an* IFS $A = (\mu_A, \gamma_A)$ *in* M *satisfies the condition* (IF1), *then*
(i) $\mu_A(x - y) = \mu_A(0)$ *implies* $\mu_A(x) = \mu_A(y)$,
(ii) $\gamma_A(x - y) = \gamma_A(0)$ *implies* $\gamma_A(x) = \gamma_A(y)$,
for all $x, y \in M$.

Proof. (i) Let $x, y \in M$ be such that $\mu_A(x - y) = \mu_A(0)$. Then

$$\begin{aligned}\mu_A(x) &= \mu_A(x - y + y)\\ &\geq \mu_A(x - y) \wedge \mu_A(y)\\ &= \mu_A(0) \wedge \mu_A(y)\\ &= \mu_A(y) \qquad \text{[by Lemma 3.3(i)]}\end{aligned}$$

Similarly, $\mu_A(y) \geq \mu_A(x)$ and so $\mu_A(x) = \mu_A(y)$.
(ii) If $\gamma_A(x - y) = \gamma_A(0)$ for all $x, y \in M$, then

$$\begin{aligned}\gamma_A(x) &= \gamma_A(x - y + y)\\ &\leq \gamma_A(x - y) \vee \gamma_A(y)\\ &= \gamma_A(0) \vee \gamma_A(y)\\ &= \gamma_A(y) \qquad \text{[by Lemma 3.3(i)]}\end{aligned}$$

Similarly, we have $\gamma_A(y) \leq \gamma_A(x)$ and so $\gamma_A(x) = \gamma_A(y)$. □

Theorem 3.5. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are intuitionistic fuzzy ideals of M, then so is $A \cap B$.*

Proof. For any $x, y \in M$, we have that

$$\begin{aligned}(\mu_A \wedge \mu_B)(x-y) &= \mu_A(x-y) \wedge \mu_B(x-y)\\ &\geq (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y))\\ &= (\mu_A \wedge \mu_B)(x) \wedge (\mu_A \wedge \mu_B)(y),\end{aligned}$$

$$\begin{aligned}(\gamma_A \vee \gamma_B)(x-y) &= \gamma_A(x-y) \vee \gamma_B(x-y)\\ &\leq (\gamma_A(x) \vee \gamma_B(x)) \vee (\gamma_A(y) \vee \gamma_B(y))\\ &= (\gamma_A \vee \gamma_B)(x) \vee (\gamma_A \vee \gamma_B)(y),\end{aligned}$$

and if $x, y \in M$ and $\alpha \in \Gamma$, then we have that

$$\begin{aligned}(\mu_A \wedge \mu_B)(x\alpha y) &= \mu_A(x\alpha y) \wedge \mu_B(x\alpha y)\\ &\geq (\mu_A(x) \vee \mu_A(y)) \wedge (\mu_B(x) \vee \mu_B(y))\\ &= (\mu_A(x) \wedge \mu_B(x)) \vee (\mu_A(y) \wedge \mu_B(y))\\ &= (\mu_A \wedge \mu_B)(x) \vee (\mu_A \wedge \mu_B)(y)\end{aligned}$$

$$\begin{aligned}(\gamma_A \vee \gamma_B)(x\alpha y) &= \gamma_A(x\alpha y) \vee \gamma_B(x\alpha y)\\ &\leq (\gamma_A(x) \wedge \gamma_A(y)) \vee (\gamma_B(x) \wedge \gamma_B(y))\\ &= (\gamma_A(x) \vee \gamma_B(x)) \wedge (\gamma_A(y) \vee \gamma_B(y))\\ &= (\gamma_A \vee \gamma_B)(x) \wedge (\gamma_A \vee \gamma_B)(y).\end{aligned}$$

Hence $A \cap B$ is an intuitionistic fuzzy ideal of M. □

Theorem 3.6. *If $\{A_i\}_{i \in \Lambda}$ is a family of intuitionistic fuzzy ideals of M, then $\cap A_i$ is an intuitionistic fuzzy ideal of M.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}(\cap\mu_{A_i})(x-y) &= \wedge\mu_{A_i}(x-y) \geq \wedge(\mu_{A_i}(x) \wedge \mu_{A_i}(y))\\ &= (\wedge\mu_{A_i}(x)) \wedge (\wedge\mu_{A_i}(y)) = (\cap\mu_{A_i})(x) \wedge (\cap\mu_{A_i})(y),\end{aligned}$$

$$\begin{aligned}(\cup\gamma_{A_i})(x-y) &= \vee\gamma_{A_i}(x-y) \leq \vee(\gamma_{A_i}(x) \vee \gamma_{A_i}(y))\\ &= (\vee\gamma_{A_i}(x)) \vee (\vee\gamma_{A_i}(y)) = (\cup\gamma_{A_i})(x) \vee (\cup\gamma_{A_i})(y),\end{aligned}$$

$$\begin{aligned}(\cap\mu_{A_i})(x\alpha y) &= \wedge\mu_{A_i}(x\alpha y) \geq \wedge(\mu_{A_i}(x) \vee \mu_{A_i}(y))\\ &= (\cap\mu_{A_i}(x)) \vee (\cap\mu_{A_i}(y)),\end{aligned}$$

and

$$\begin{aligned}(\cup\gamma_{A_i})(x\alpha y) &= \vee\gamma_{A_i}(x\alpha y) \leq \vee(\gamma_{A_i}(x) \wedge \gamma_{A_i}(y))\\ &= (\cup\gamma_{A_i}(x)) \wedge (\cup\gamma_{A_i}(y)).\end{aligned}$$

Hence $\cap A_i$ is an intuitionistic fuzzy ideal of M. □

Theorem 3.7. *If an* IFS $A = (\mu_A, \gamma_A)$ *in* M *is an intuitionistic fuzzy ideal of* M*, then so is* $\Box A$.

Proof. It is sufficient to show that $\overline{\mu_A}$ satisfies the second condition of (IF1) and the second condition of (IF2). For any $x, y \in M$ and any $\alpha \in \Gamma$, we have

$$\begin{aligned}\overline{\mu_A}(x-y) &= 1 - \mu_A(x-y) \le 1 - \mu_A(x) \wedge \mu_A(y)\\ &= (1-\mu_A(x)) \vee (1-\mu_A(y)) = \overline{\mu_A}(x) \vee \overline{\mu_A}(y),\end{aligned}$$

$$\begin{aligned}\overline{\mu_A}(x\alpha y) = 1 - \mu_A(x\alpha y) \le 1 - \mu_A(x) \vee \mu_A(y) &= (1-\mu_A(x)) \wedge (1-\mu_A(y))\\ &= \overline{\mu_A}(x) \wedge \overline{\mu_A}(y)\end{aligned}$$

Therefore $\Box A$ is an intuitionistic fuzzy ideal of M. $\Box$

Definition 3.8. Let $A = (\mu_A, \gamma_A)$ be an IFS in M and let $\alpha \in [0,1]$. Then the sets

$$\mu_{A,\alpha}^{\ge} := \{x \in M : \mu_A(x) \ge \alpha\}$$

and

$$\gamma_{A,\alpha}^{\le} := \{x \in M : \gamma_A(x) \le \alpha\}$$

are called a *μ-level α-cut* and a *γ-level α-cut* of A, respectively.

Theorem 3.9. *If an* IFS $A = (\mu_A, \gamma_A)$ *in* M *is an intuitionistic fuzzy ideal of* M*, then the μ-level α-cut* $\mu_{A,\alpha}^{\ge}$ *and γ-level α-cut* $\gamma_{A,\alpha}^{\le}$ *of* A *are ideals of* M *for every* $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0,1]$.

Proof. Let $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0,1]$ and let $x, y \in \mu_{A,\alpha}^{\ge}$. Then $\mu_A(x) \ge \alpha$ and $\mu_A(y) \ge \alpha$. It follows from the first condition of (IF1) that

$$\mu_A(x-y) \ge \mu_A(x) \wedge \mu_A(y) \ge \alpha \text{ so that } x - y \in \mu_{A,\alpha}^{\ge}.$$

If $x, y \in \gamma_{A,\alpha}^{\le}$, then $\gamma_A(x) \le \alpha$ and $\gamma_A(y) \le \alpha$, and so

$$\gamma_A(x-y) \le \gamma_A(x) \vee \gamma_A(y) \le \alpha.$$

Hence we have $x - y \in \gamma_{A,\alpha}^{\le}$. Now let $x \in M, \beta \in \Gamma$ and $y \in \mu_{A,\alpha}^{\ge}$. Then $\mu_A(x\alpha y) \ge \mu_A(x) \vee \mu_A(y) \ge \mu_A(y) \ge \alpha$ and so $x\alpha y \in \mu_{A,\alpha}^{\ge}$ (resp. $y\alpha x \in \mu_{A,\alpha}^{\ge}$). If $y \in \gamma_{A,\alpha}^{\le}$, then $\gamma_A(x\beta y) \le \gamma_A(x) \wedge \gamma_A(y) \le \gamma_A(y) \le \alpha$ and thus $x\beta y \in \gamma_{A,\alpha}^{\le}$ (resp. $x\beta x \in \gamma_{A,\alpha}$). Therefore $\mu_{A,\alpha}^{\ge}$ and $\gamma_{A,\alpha}^{\le}$ are ideals of M. $\Box$

Theorem 3.10. *Let* $A = (\mu_A, \gamma_A)$ *be an* IFS *in* M *such that the non-empty sets* $\mu_{A,\alpha}^{\ge}$ *and* $\gamma_{A,\alpha}^{\le}$ *are ideals of* M *for all* $\alpha \in [0,1]$*. Then* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal of* M.

Proof. Let $\alpha \in [0,1]$ and suppose that $\mu_{A,\alpha}^{\ge}(\neq \emptyset)$ and $\gamma_{A,\alpha}^{\le}(\neq \emptyset)$ are ideals of M. We must show that $A = (\mu_A, \gamma_A)$ satisfies the conditions (IF1)-(IF2). If the first condition of (IF1) is false, then there exist $x_0, y_0 \in M$ such that $\mu_A(x_0 - y_0) < \mu_A(x_0) \wedge \mu_A(y_0)$. Taking

$$\alpha_0 := \frac{1}{2}(\mu_A(x_0 - y_0) + \mu_A(x_0) \wedge \mu_A(y_0)),$$

we have $\mu_A(x_0 - y_0) < \alpha_0 < \mu_A(x_0) \wedge \mu_A(y_0)$. It follows that $x_0, y_0 \in \mu^{\geq}_{A,\alpha_0}$ and $x_0 - y_0 \notin \mu^{\geq}_{A,\alpha_0}$, which is a contradiction. Assume that the second condition of (IF1) does not hold. Then $\gamma_A(x_0 - y_0) > \gamma_A(x_0) \vee \gamma_A(y_0)$ for some $x_0, y_0 \in M$. Let $\beta_0 := \frac{1}{2}(\gamma_A(x_0 - y_0) + \gamma_A(x_0) \vee \gamma_A(y_0))$. Then $\gamma_A(x_0 - y_0) > \beta_0 > \gamma_A(x_0) \vee \gamma_A(y_0)$ and so $x_0, y_0 \in \gamma^{\leq}_{A,\beta_0}$ but $x_0 - y_0 \notin \gamma^{\leq}_{A,\beta_0}$. This is a contradiction. Now if the first condition of (IF2) is not true then there exist $x_0, y_0 \in M$ and $\zeta \in \Gamma$, such that $\mu_A(x_0\zeta y_0) < \mu_A(x_0) \wedge \mu_A(y_0)$. Putting $\gamma_0 := \frac{1}{2}(\mu_A(x_0\zeta y_0) + \mu_A(x_0) \wedge \mu_A(y_0))$, then $\mu_A(x_0\zeta y_0) < \gamma_0 < \mu_A(x_0) \wedge \mu_A(y_0)$. It follows that $x_0, y_0 \in \mu^{\geq}_{A,\gamma_0}$ and $x_0\zeta y_0 \notin \mu^{\geq}_{A,\gamma_0}$, a contradiction. Finally suppose that the second condition of (IF2) does not hold. Then $\gamma_A(x_0\zeta y_0) > \gamma_A(x_0) \vee \gamma_A(y_0)$ for some $x_0, y_0 \in M$ and $\zeta \in \Gamma$. Selecting $\delta_0 := \frac{1}{2}(\gamma_A(x_0\zeta y_0) + \gamma_A(x_0) \vee \gamma_A(y_0))$, we get $\gamma_A(x_0\zeta y_0) > \delta_0 > \gamma_(x_0) \vee \gamma_(y_0)$ and so $x_0, y_0 \in \gamma^{\leq}_{A,\delta_0}$, but $x_0\zeta y_0 \notin \gamma^{\leq}_{A,\delta_0}$. This is impossible and we are done. □

Theorem 3.11. *Let H be an ideal of M and let $A = (\mu_A, \gamma_A)$ be an* IFS *in M defined by*

$$\mu_A(x) := \begin{cases} \alpha_0 & \textit{if } x \in H, \\ \alpha_1 & \textit{otherwise}, \end{cases} \qquad \gamma_A(x) := \begin{cases} \beta_0 & \textit{if } x \in H, \\ \beta_1 & \textit{otherwise}, \end{cases}$$

for all $x \in M$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M and $\mu^{\geq}_{A,\alpha_0} = H = \gamma^{\leq}_{A,\beta_0}$.

Proof. Let $x, y \in M$. If any one of x and y does not belong to H, then

$$\mu_A(x - y) \geq \alpha_1 = \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(x - y) \leq \beta_1 = \gamma_A(x) \vee \gamma_A(y).$$

Also, let $x, y \in M$ and $\alpha \in \Gamma$. If $y \notin H$, then $\mu_A(x\alpha y) \geq \alpha_1 = \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x\alpha y) \leq \beta_1 = \gamma_A(x) \vee \gamma_A(y)$. Assume that $y \in H$. Since H is an ideal of M, it follows that $x\alpha y \in H$. Hence $\mu_A(x\alpha y) = \alpha_0 = \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x\alpha y) = \beta_0 = \gamma_A(x) \vee \gamma_A(y)$. Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M. Obviously $\mu^{\geq}_{A,\alpha_0} = H = \gamma^{\leq}_{A,\beta_0}$. □

Corollary 3.12. *Let χ_H be the characteristic function of an ideal H of M. Then the* IFS *$\tilde{H} = (\chi_H, \widetilde{\chi_H})$ is an intuitionistic fuzzy ideal of M.*

Theorem 3.13. *If an* IFS *$A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M, then*

$$\mu_A(x) := \sup\{\alpha \in [0,1] \mid x \in \mu^{\geq}_{A,\alpha}\} \textit{ and } \gamma_A(x) := \inf\{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\}$$

for all $x \in M$.

Proof. Let $\delta := \sup\{\alpha \in [0,1] \mid x \in \mu^{\geq}_{A,\alpha}\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in \mu^{\geq}_{A,\alpha}$. It follows that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leq \mu_A(x)$ since ε is arbitrary. We now show that $\mu_A(x) \leq \delta$. Let $\mu_A(x) = \beta$. Then $x \in \mu^{\geq}_{A,\beta}$ and so

$$\beta \in \{\alpha \in [0,1] \mid x \in \mu^{\geq}_{A,\alpha}\}.$$

Hence $\mu_A(x) = \beta \leq \sup\{\alpha \in [0,1] \mid x \in \mu^{\geq}_{A,\alpha}\} = \delta$. Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] \mid x \in \mu^{\geq}_{A,\alpha}\}.$$

Now let $\eta = \inf\{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\}$. Then

$$\inf\{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\} < \eta + \varepsilon \text{ for any } \varepsilon < 0,$$

and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in \gamma^{\leq}_{A,\alpha}$. Since $\gamma_A(x) \leq \alpha$ and ε is arbitrary, it follows that $\gamma_A(x) \leq \eta$. To prove $\gamma_A(x) \geq \eta$, let $\gamma_A(x) = \zeta$. Then $x \in \gamma^{\leq}_{A,\zeta}$ and thus $\zeta \in \{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\}$. Hence

$$\inf\{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\} \leq \zeta, \text{ i.e., } \eta \leq \zeta = \gamma_A(x).$$

Consequently

$$\gamma_A(x) = \eta = \inf\{\alpha \in [0,1] \mid x \in \gamma^{\leq}_{A,\alpha}\}.$$

This completes the proof. □

Theorem 3.14. *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M if and only if the fuzzy sets μ_A and $\overline{\gamma_A}$ are fuzzy ideals of M.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of M. Then clearly μ_A is a fuzzy ideal of M. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \overline{\gamma_A}(x-y) &= 1 - \gamma_A(x-y) \\ &\geq 1 - \gamma_A(x) \vee \gamma_A(y) \\ &= (1-\gamma_A(x)) \wedge (1-\gamma_A(y)) \\ &= \overline{\gamma_A}(x) \wedge \overline{\gamma_A}(y), \text{ and} \end{aligned}$$

$$\begin{aligned} \overline{\gamma_A}(x\alpha y) &= 1 - \gamma_A(x\alpha y) \\ &\geq 1 - \gamma_A(x) \wedge \gamma_A(y) \\ &= (1-\gamma_A(x)) \vee (1-\gamma_A(y)) \\ &= \overline{\gamma_A}(y) \vee \overline{\gamma_A}(y). \end{aligned}$$

Hence $\overline{\gamma_A}$ is a fuzzy ideal of M.

Conversely suppose that μ_A and $\overline{\gamma_A}$ are fuzzy ideals of M. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} 1 - \gamma_A(x-y) &= \overline{\gamma_A}(x-y) \geq \overline{\gamma_A}(x) \wedge \overline{\gamma_A}(y) \\ &= (1-\gamma_A(x)) \wedge (1-\gamma_A(y)) \\ &= 1 - \gamma_A(x) \vee \gamma_A(y), \text{ and} \end{aligned}$$

$$\begin{aligned} 1 - \gamma_A(x\alpha y) &= \overline{\gamma_A}(x\alpha y) \geq \overline{\gamma_A}(x) \vee \overline{\gamma_A}(y) \\ &= (1-\gamma_A(x)) \vee (1-\gamma_A(y)) \\ &= 1 - \gamma_A(x) \wedge \gamma_A(y), \end{aligned}$$

which imply that $\gamma_A(x-y) \leq \gamma_A(x) \vee \gamma_A(y)$ and $\gamma_A(x\alpha y) \leq \gamma_A(x) \wedge \gamma_A(y)$. This completes the proof. □

Corollary 3.15. *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M if and only if $\Box A = (\mu_A, \overline{\mu_A})$ and $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ are intuitionistic fuzzy ideals of M.*

Proof. It is straightforward by Theorem 3.14. $\Box$

A mapping f from a Γ-ring M to a Γ-ring M' is called a *homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Theorem 3.16. *Let $f : M \to M'$ be a homomorphism of Γ-rings. If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy ideal of M', then the preimage $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy ideal of M.*

Proof. Assume that $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy ideal of M' and let $x, y \in M$. Then

$$\begin{aligned} f^{-1}(\mu_B)(x-y) &= \mu_B(f(x-y)) \\ &= \mu_B(f(x)-f(y)) \\ &\geq \mu_B(f(x)) \wedge \mu_B(f(y)) \\ &= f^{-1}(\mu_B(x)) \wedge f^{-1}(\mu_B(y)), \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_B)(x-y) &= \gamma_B(f(x-y)) \\ &= \gamma_B(f(x)-f(y)) \\ &\leq \gamma_B(f(x)) \vee \gamma_B(f(y)) \\ &= f^{-1}(\gamma_B(x)) \vee f^{-1}(\gamma_B(y)). \end{aligned}$$

Also, for any $x, y \in M$ and $\alpha \in \Gamma$ we have

$$\begin{aligned} f^{-1}(\mu_B)(x\alpha y) &= \mu_B(f(x\alpha y)) \\ &= \mu_B(f(x)\alpha f(y)) \\ &\geq \mu_B(f(x)) \vee \mu_B(f(y)) \\ &= f^{-1}(\mu_B(x)) \vee f^{-1}(\mu_B(y)), \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(\gamma_B)(x\alpha y) &= \gamma_B(f(x\alpha y)) \\ &= \gamma_B(f(x)\alpha f(y)) \\ &\leq \gamma_B(f(x)) \wedge \gamma_B(f(y)) \\ &= f^{-1}(\gamma_B(x)) \wedge f^{-1}(\gamma_B(y)). \end{aligned}$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic fuzzy ideal of M. $\Box$

Let $f : M \to M'$ be a homomorphism of Γ-rings. For any IFS $A = (\mu_A, \gamma_A)$ in M', we define a new IFS $A^f = (\mu_A^f, \gamma_A^f)$ in M by

$$\mu_A^f(x) := \mu_A(f(x)) \text{ and } \gamma_A^f(x) := \gamma_A(f(x))$$

for all $x \in M$.

Theorem 3.17. *Let $f : M \to M'$ be a homomorphism of Γ-rings. If an IFS $A = (\mu_A, \gamma_A)$ in M' is an intuitionistic fuzzy ideal of M', then an IFS $A^f = (\mu_A^f, \gamma_A^f)$ in M is an intuitionistic fuzzy ideal of M.*

Proof. Let $x, y \in M$. Then

$$\begin{aligned}\mu_A^f(x-y) &= \mu_A(f(x-y))\\ &= \mu_A(f(x)-f(y))\\ &\geq \mu_A(f(x)) \wedge \mu_A(f(y))\\ &= \mu_A^f(x) \wedge \mu_A^f(y),\end{aligned}$$

and

$$\begin{aligned}\gamma_A^f(x-y) &= \gamma_A(f(x-y))\\ &= \gamma_A(f(x)-f(y))\\ &\leq \gamma_A(f(x) \vee \gamma_A(f(y))\\ &= \gamma_A^f(x) \vee \gamma_A^f(y).\end{aligned}$$

Also, let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu_A^f(x\alpha y) &= \mu_A(f(x\alpha y))\\ &= \mu_A(f(x)\alpha f(y))\\ &\geq \mu_A(f(x)) \vee \mu_A(f(y))\\ &= \mu_A^f(x) \vee \mu_A^f(y),\end{aligned}$$

and

$$\begin{aligned}\gamma_A^f(x\alpha y) &= \gamma_A(f(x\alpha y))\\ &= \gamma_A(f(x)\alpha f(y))\\ &\leq \gamma_A(f(x)) \wedge \gamma_A(f(y))\\ &= \gamma_A^f(x) \wedge \gamma_A^f(y).\end{aligned}$$

Hence $A^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy ideal of M. □

If we strengthen the condition of f, then we can construct the converse of Theorem 3.17 as follows.

Theorem 3.18. *Let $f : M \to M'$ be an epimorphism of Γ-rings and let $A = (\mu_A, \gamma_A)$ be an IFS in M'. If $A^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy ideal of M, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M'.*

Proof. Let $x, y \in M'$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in M$. It follows that

$$\begin{aligned}\mu_A(x-y) =&\mu_A(f(a)-f(b))\\ &\mu_A(f(a-b)) = \mu_A^f(a-b)\\ &\geq \mu_A^f(a) \wedge \mu_A^f(b)\\ &= \mu_A(f(a)) \wedge \mu_A(f(b))\\ &= \mu_A(x) \wedge \mu_A(y)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x-y) &= \gamma_A(f(a)-f(b)) \\ &= \gamma_A(f(a-b)) = \gamma_A^f(a-b) \\ &\leq \gamma_A^f(a) \wedge \gamma_A^f(b) \\ &= \gamma_A(f(a)) \wedge \gamma_A(f(b)) \\ &= \gamma_A(x) \wedge \gamma_A(y).\end{aligned}$$

Also, let $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu_A(x\alpha y) &= \mu_A(f(a)\alpha f(b)) \\ &= \mu_A(f(a\alpha b)) = \mu_A^f(a\alpha b) \\ &\geq \mu_A^f(a) \vee \mu_A^f(b) \\ &= \mu_A(f(a)) \vee \mu_A(f(b)) \\ &= \mu_A(x) \vee \mu_A(y),\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x\alpha y) &= \gamma_A(f(a)\alpha f(b)) \\ &= \gamma_A(f(a\alpha b)) = \gamma_A^f(a\alpha b) \\ &\leq \gamma_A^f(a) \wedge \gamma_A^f(b) \\ &= \gamma_A(f(a)) \wedge \gamma_A(f(b)) \\ &= \gamma_A(x) \wedge \gamma_A(y).\end{aligned}$$

This completes the proof. □

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