FUZZY HYPERBCK–IDEALS OF HYPERBCK–ALGEBRAS

YOUNG BAE JUN AND XIAO LONG XIN

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ABSTRACT. The fuzzification of the notion of a (weak, strong, reflexive) hyperBCK-ideal is considered, and relations among them and some related properties are given.

1. Introduction

The study of BCK-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras (see [1, 2, 7]). The hyperstructure theory (called also multialgebras) was introduced in 1934 by Marty [6] at the 8th congress of Scandinavian Mathematiciens. In [5], Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyperBCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a (weak, strong, reflexive) hyperBCK-ideal, and gave relations among them. In this paper we consider the fuzzification of the notion of a (weak, strong, reflexive) hyperBCK-ideal, gave relations among them and investigate some related properties.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCK-algebra if it satisfies: for all $x,y,z \in X$,

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$.

Note that an algebra $(X,*,0)$ of type (2,0) is a BCK-algebra if and only if

(i) $((y*z)*(x*z))*(y*x)=0$,
(ii) $((z*x)*y)*((z*y)*x)=0$,
(iii) $(x*y)*x=0$,
(iv) $x*y=0$ and $y*x=0$ imply that $x=y$,

for all $x,y \in X$ (see [7]). Note that the identity $x*(x*(x*y))=x*y$ holds in a BCK-algebra. A non-empty subset I of a BCK-algebra X is called an ideal of X if $0 \in I$, and $x*y \in I$ and $y \in I$ imply $x \in I$ for all $x,y \in X$.

A *fuzzy set* μ in a set X is a function $\mu : X \to [0,1]$. A fuzzy set μ in a set X is said to satisfy the **inf** (resp. **sup**) *property* if for any subset T of X there exists $x_0 \in T$ such that $\mu(x_0) = \inf\limits_{x \in T} \mu(x)$ (resp. $\mu(x_0) = \sup\limits_{x \in T} \mu(x)$).

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Let H be a non-empty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 2.1 (Jun et al. [5]). By a *hyperBCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfing the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Example 2.2 (Jun et al. [5]). (1) Let $(H, *, 0)$ be a BCK-algebra and define a hyperoperation "$\circ$" on H by $x \circ y = \{x * y\}$ for all $x, y \in H$. Then H is a hyperBCK-algebra.

(2) Define a hyperoperation "$\circ$" on $H := [0, \infty)$ by

$$x \circ y := \begin{cases} [0, x] & \text{if } x \le y \\ (0, y] & \text{if } x > y \ne 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then H is a hyperBCK-algebra.

(3) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Then H is a hyperBCK-algebra.

Proposition 2.3 (Jun et al. [5]). *In a hyperBCK-algebra H, the condition* (HK3) *is equivalent to the condition:*

(2-1) $x \circ y \ll \{x\}$ *for all* $x, y \in H$.

In any hyperBCK-algebra H, the following hold (see Jun et al. [5]):

(2-2) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ and $0 \circ 0 \ll \{0\}$ for all $x, y \in H$,
(2-3) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$,
(2-4) $0 \circ 0 = \{0\}$,
(2-5) $0 \ll x$,
(2-6) $x \ll x$,
(2-7) $A \ll A$,
(2-8) $A \subseteq B$ implies $A \ll B$,
(2-9) $0 \circ x = \{0\}$,
(2-10) $0 \circ A = \{0\}$,
(2-11) $A \ll \{0\}$ implies $A = \{0\}$,
(2-12) $A \circ B \ll A$,
(2-13) $x \in x \circ 0$,
(2-14) $x \circ 0 \ll \{y\}$ implies $x \ll y$,
(2-15) $y \ll z$ implies $x \circ z \ll x \circ y$,
(2-16) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,
(2-17) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsetes A, B and C of H.

Definition 2.4 (Jun et al. [5])**.** Let I be a non-empty subset of a hyperBCK-algebra H. Then I is said to be a *hyperBCK-ideal* of H if

(HI1) $0 \in I$,

(HI2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.5 (Jun et al. [4])**.** A hyperBCK-ideal I of H is said to be *reflexive* if $x \circ x \subseteq I$ for all $x \in H$.

Definition 2.6 (Jun et al. [4])**.** Let I be a non-empty subset of H. Then I is called a *strong hyperBCK-ideal* of a hyperBCK-algebra H if it satisfies (HI1) and

(SHI) $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Note that every strong hyperBCK-ideal of a hyperBCK-algebra is a hyperBCK-ideal (see [4, Theorem 3.8]).

Definition 2.7 (Jun et al. [5])**.** Let I be a non-empty subset of a hyperBCK-algebra H. Then I is called a *weak hyperBCK-ideal* of H if it satisfies (HI1) and

(WHI) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

3. Fuzzy HyperBCK-ideals

In what follows, H shall mean a hyperBCK-algebra unless specified otherwise.

Definition 3.1. A fuzzy set μ in H is called a *fuzzy hyperBCK-ideal* of H if

(i) $x \ll y$ implies $\mu(y) \leq \mu(x)$,

(ii) $\mu(x) \geq \min\{\inf\limits_{a \in x \circ y} \mu(a), \mu(y)\}$,

for all $x, y \in H$.

Example 3.2. Let H be the hyperBCK-algebra in Example 2.2(3). Define a fuzzy set μ in H by $\mu(0) = 1$, $\mu(1) = 0.3$ and $\mu(2) = 0$. It is easily verified that μ is a fuzzy hyperBCK-ideal of H.

Definition 3.3. A fuzzy set μ in H is called a *fuzzy strong hyperBCK-ideal* of H if

$$\inf_{a \in x \circ x} \mu(a) \geq \mu(x) \geq \min\{\sup_{b \in x \circ y} \mu(b), \mu(y)\}$$

for all $x, y \in H$.

Definition 3.4. A fuzzy set μ in H is called a *fuzzy s-weak hyperBCK-ideal* of H if

(i) $\mu(0) \geq \mu(x)$ for all $x \in H$,

(ii) for every $x, y \in H$ there exists $a \in x \circ y$ such that $\mu(x) \geq \min\{\mu(a), \mu(y)\}$.

Definition 3.5. A fuzzy set μ in H is called a *fuzzy weak hyperBCK-ideal* of H if

$$\mu(0) \geq \mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}$$

for all $x, y \in H$.

Let μ be a fuzzy s-weak hyperBCK-ideal of H and let $x, y \in H$. Then there exists $a \in x \circ y$ such that $\mu(x) \geq \min\{\mu(a), \mu(y)\}$. Since $\mu(a) \geq \inf\limits_{b \in x \circ y} \mu(b)$, it follows that

$$\mu(x) \geq \min\{\inf_{b \in x \circ y} \mu(b), \mu(y)\}.$$

Hence every fuzzy s-weak hyperBCK-ideal is a fuzzy weak hyperBCK-ideal. It is not easy to find an example of a fuzzy weak hyperBCK-ideal which is not a fuzzy s-weak hyperBCK-ideal. But we have the following proposition.

Proposition 3.6. *Let μ be a fuzzy weak hyperBCK-ideal of H. If μ satisfies the* **inf** *property, then μ is a fuzzy s-weak hyperBCK-ideal of H.*

Proof. Since μ satisfies the **inf** property, there exists $a_0 \in x \circ y$ such that $\mu(a_0) = \inf_{a \in x \circ y} \mu(a)$. It follows that

$$\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\} = \min\{\mu(a_0), \mu(y)\},$$

ending the proof. □

Note that, in a finite hyperBCK-algebra, every fuzzy set satisfies **inf** (also **sup**) property. Hence the concept of fuzzy weak hyperBCK-ideals and fuzzy s-weak hyperBCK-ideals coincide in a finite hyperBCK-algebra.

Proposition 3.7. *Let μ be a fuzzy strong hyperBCK-ideal of H and let $x, y \in H$. Then*

(i) $\mu(0) \geq \mu(x)$,

(ii) $x \ll y$ *implies* $\mu(y) \leq \mu(x)$,

(iii) $\mu(x) \geq \min\{\mu(a), \mu(y)\}$ *for all* $a \in x \circ y$.

Proof. (i) Since $0 \in x \circ x$ for all $x \in H$, we have

$$\mu(0) \geq \inf_{a \in x \circ x} \mu(a) \geq \mu(x),$$

which proves (i).

(ii) Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$ and so $\sup_{b \in x \circ y} \mu(b) \geq \mu(0)$. If follows from (i) that

$$\mu(x) \geq \min\{\sup_{b \in x \circ y} \mu(b), \mu(y)\} \geq \min\{\mu(0), \mu(y)\} = \mu(y).$$

(iii) Let $x, y \in H$. Since

$$\mu(x) \geq \min\{\sup_{b \in x \circ y} \mu(b), \mu(y)\} \geq \min\{\mu(a), \mu(y)\}$$

for all $a \in x \circ y$, we conclude that (iii) is true. □

Corollary 3.8. *If μ is a fuzzy strong hyperBCK-ideal of H, then*

$$\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}$$

for all $x, y \in H$.

Proof. Since $\mu(a) \geq \inf_{b \in x \circ y} \mu(b)$ for all $a \in x \circ y$, the result is by Proposition 3.7(iii). □

Corollary 3.9. *Every fuzzy strong hyperBCK-ideal is both a fuzzy s-weak hyperBCK-ideal (and hence a fuzzy weak hyperBCK-ideal) and a fuzzy hyperBCK-ideal.*

Proof. Straightforward. □

Proposition 3.10. *Let μ be a fuzzy hyperBCK-ideal of H and let $x, y \in H$. Then*

(i) $\mu(0) \geq \mu(x)$,

(ii) *if μ satisfies the* **inf** *property, then* $\mu(x) \geq \min\{\mu(a), \mu(y)\}$ *for some* $a \in x \circ y$.

Proof. (i) Since $0 \ll x$ for each $x \in H$, we have $\mu(x) \leq \mu(0)$ by Definition 3.1(i) and hence (i) holds.

(ii) Since μ satisfies the **inf** property, there is $a_0 \in x \circ y$ such that $\mu(a_0) = \inf_{a \in x \circ y} \mu(a)$. Hence

$$\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\} = \min\{\mu(a_0), \mu(y)\},$$

which implies that (ii) is true. □

Corollary 3.11. (i) *Every fuzzy hyperBCK-ideal of H is a fuzzy weak hyperBCK-ideal of H.*

(ii) *If μ is a fuzzy hyperBCK-ideal of H satisfying* **inf** *property, then μ is a fuzzy s-weak hyperBCK-ideal of H.*

Proof. Straightforward. $\square$

The following example shows that the converse of Corollary 3.9 and Corollary 3.11(i) may not be true.

Example 3.12. (1) Consider the hyperBCK-algebra H in Example 2.2(3). Define a fuzzy set μ in H by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x = 2. \end{cases}$$

Then we can see that μ is a fuzzy hyperBCK-ideal of H and hence it is also a fuzzy weak hyperBCK-ideal of H. But μ is not a fuzzy strong hyperBCK-ideal of H since $\min\{\sup_{a \in 2\circ 1} \mu(a), \mu(1)\} = \min\{\mu(1), \mu(1)\} = \mu(1) = \frac{1}{2} > 0 = \mu(2)$.

(2) Consider the hyperBCK-algebra H in Example 2.2(3). Define a fuzzy set μ in H by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = 2, \\ 0 & \text{if } x = 1. \end{cases}$$

Then μ is a fuzzy weak hyperBCK-ideal of H but it is not a fuzzy hyperBCK-ideal of H since $1 \ll 2$ but $\mu(1) \not\geq \mu(2)$.

Theorem 3.13. *If μ is a fuzzy strong hyperBCK-ideal of H, then the set $\mu_t := \{x \in H \mid \mu(x) \geq t\}$ is a strong hyperBCK-ideal of H when $\mu_t \neq \emptyset$ for $t \in [0,1]$.*

proof. Let μ be a fuzzy strong hyperBCK-ideal of H and $\mu_t \neq \emptyset$ for $t \in [0,1]$. Then there is $a \in \mu_t$ and so $\mu(a) \geq t$. By Proposition 3.7(i), $\mu(0) \geq \mu(a) \geq t$ and so $0 \in \mu_t$. Let $x, y \in H$ be such that $(x \circ y) \cap \mu_t \neq \emptyset$ and $y \in \mu_t$. Then there exists $a_0 \in (x \circ y) \cap \mu_t$ and hence $\mu(a_0) \geq t$. By Definition 3.3, we have $\mu(x) \geq \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\} \geq \min\{\mu(a_0), \mu(y)\} \geq \min\{t, t\} = t$ and so $x \in \mu_t$. It follows that μ_t is a strong hyperBCK-ideal of H. $\square$

Lemma 3.14 ([3, Proposition 3.7])**.** *Let A be a subset of a hyperBCK-algebra H. If I is a hyperBCK-ideal of H such that $A \ll I$, then A is contained in I.*

Theorem 3.15. *Let μ be a fuzzy set in H satisfying the* **sup** *property. If the set $\mu_t := \{x \in H \mid \mu(x) \geq t\} (\neq \emptyset)$ is a strong hyperBCK-ideal of H for all $t \in [0,1]$, then μ is a fuzzy strong hyperBCK-ideal of H.*

Proof. Assume that $\mu_t (\neq \emptyset)$ is a strong hyperBCK-ideal of H for all $t \in [0,1]$. Then there exists $x \in \mu_t$ and hence $x \circ x \ll x \in \mu_t$. Using Lemma 3.14, we have $x \circ x \subseteq \mu_t$. Thus for each $a \in x \circ x$, we have $a \in \mu_t$ and hence $\mu(a) \geq t$. It follows that $\inf_{a \in x \circ x} \mu(a) \geq t = \mu(x)$. Moreover let $x, y \in H$ and put $k = \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\}$. By hypothesis, μ_k is a strong hyperBCK-ideal of H. Since μ satisfies the **sup** property, there is $a_0 \in x \circ y$ such that $\mu(a_0) = \sup_{a \in x \circ y} \mu(a)$. Thus $\mu(a_0) = \sup_{a \in x \circ y} \mu(a) \geq \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\} = k$ and so $a_0 \in \mu_k$. This shows that $a_0 \in x \circ y \cap \mu_k$ and hence $x \circ y \cap \mu_k \neq \emptyset$. Combining $y \in \mu_k$ and noticing that μ_k is a strong hyperBCK-ideal of H we get $x \in \mu_k$. Hence $\mu(x) \geq k = \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\}$. Therefore μ is a fuzzy strong hyperBCK-ideal of H. $\square$

Example 3.16. (1) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Then H is a hyperBCK-algebra. Moreover we can see that $I_1 := \{0, 1\}$ and $I_2 := \{0, 2\}$ are strong hyperBCK-ideals of H. Define μ by $\mu(0) = 1, \mu(1) = 0.3$ and $\mu(2) = 0.2$. Then we can see that

$$\mu_t = \begin{cases} H & \text{if } 0 \leq t \leq 0.2, \\ \{0, 1\} & \text{if } 0.2 < t \leq 0.3, \\ \{0\} & \text{if } 0.3 < t \leq 1. \end{cases}$$

Since $\{0\}, \{0, 1\}$ and H are strong hyperBCK-ideals of H, It follows from Theorem 3.15 that μ is a fuzzy strong hyperBCK-ideal of H.

(2) Consider the hyperBCK-algebra H as in Example 2.2(2). We can see that there exist only two strong hyperBCK-ideals $\{0\}$ and H itself (see [4, Example 3.6(2)]). Define a fuzzy set μ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \in (1, \infty). \end{cases}$$

Then $\mu_1 = [0, 1]$ is not a strong hyperBCK-ideal of H and so μ is not a fuzzy strong hyperBCK-ideal of H by Theorem 3.13.

Theorem 3.17. *Let μ be a fuzzy set in H. Then μ is a fuzzy hyper-BCK-ideal of H if and only if $\mu_t := \{x \in H \mid \mu(x) \geq t\}$ is a hyperBCK-ideal of H whenever $\mu_t \neq \emptyset$ for $t \in [0, 1]$.*

Proof. Let μ be a fuzzy hyperBCK-ideal of H and assume $\mu_t \neq \emptyset$ where $t \in [0, 1]$. Then there exists $a \in \mu_t$ and hence $\mu(a) \geq t$. By Proposition 3.10(i), $\mu(0) \geq \mu(a) \geq t$ and so $0 \in \mu_t$. Let $x, y \in H$ be such that $x \circ y \ll \mu_t$ and $y \in \mu_t$. Thus for any $a \in x \circ y$, there exists $a_0 \in \mu_t$ such that $a \ll a_0$ and so $\mu(a_0) \leq \mu(a)$. Hence $\mu(a) \geq t$ for all $a \in x \circ y$. It follows that $\inf_{a \in x \circ y} \mu(a) \geq t$ so that $\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\} \geq \min\{t, \mu(y)\} \geq t$, which shows that $x \in \mu_t$ and μ_t is a hyperBCK-ideal of H.

Conversely assume that for each $t \in [0, 1]$, $\mu_t (\neq \emptyset)$ is a hyperBCK-ideal of H. Let $x \ll y$ and $t = \mu(y)$. Then $y \in \mu_t$, and thus $x \ll \mu_t$. It follows from Lemma 3.14 that $x \in \mu_t$ and hence $\mu(x) \geq t = \mu(y)$. Moreover let $x, y \in H$ and put $t = \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}$. Then $y \in \mu_t$, and for each $a \in x \circ y$ we have $\mu(a) \geq \inf_{b \in x \circ y} \mu(b) \geq \min\{\inf_{b \in x \circ y} \mu(b), \mu(y)\} = t$ and hence $a \in \mu_t$, which follows that $x \circ y \subseteq \mu_t$. Thus $x \circ y \ll \mu_t$. Combining $y \in \mu_t$ and μ_t being a hyperBCK-ideal of H, we get $x \in \mu_t$ and so $\mu(x) \geq t = \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}$, which shows that μ is a fuzzy hyperBCK-ideal of H. $\square$

Theorem 3.18. *Let μ be a fuzzy set in H. Then μ is a fuzzy weak hyperBCK-ideal of H if and only if the set $\mu_t := \{x \in H \mid \mu(x) \geq t\}$ is a weak hyperBCK-ideal of H whenever $\mu_t \neq \emptyset$ for $t \in [0, 1]$.*

Proof. The proof is similat to the proof of Theorem 3.17. $\square$

For any subset $I \subset H$ we define a fuzzy set μ_I in H by

$$\mu_I(x) := \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.19. *Let I be a subset of H. Then*

(i) *I is a strong hyperBCK-ideal of H if and only if μ_I is a fuzzy strong hyperBCK-ideal of H.*

(ii) *I is a hyperBCK-ideal of H if and only if μ_I is a fuzzy hyper-BCK-ideal of H.*

(iii) *I is a weak hyperBCK-ideal of H if and only if μ_I is a fuzzy weak hyperBCK-ideal of H.*

Proof. Let I be a subset of H. Then clearly μ_I is a fuzzy set in H satisfying **inf** and **sup** property.

(i) Let I be a strong hyperBCK-ideal of H. Note that μ_I satisfies

$$(\mu_I)_t = \begin{cases} I & \text{if } 0 < t \leq 1, \\ H & \text{if } t = 0. \end{cases}$$

Then for each $t \in [0,1]$, $(\mu_I)_t$ is a strong hyperBCK-ideal of H. By Theorem 3.15, μ_I is a fuzzy strong hyperBCK-ideal of H. Conversely let μ_I be a fuzzy strong hyperBCK-ideal of H. Then $(\mu_I)_1 = I$ is a strong hyperBCK-ideal of H by Theorem 3.13. Hence (i) is true.

(ii) and (iii) are similar to (i). □

Using Theorems 3.13, 3.15, 3.17, 3.18, 3.19 and Corollaries 3.9 and 3.11, we have the following corollary.

Corollary 3.20. (i) Every strong hyperBCK-ideal of H is a hyper-BCK-ideal of H;

(ii) Every hyperBCK-ideal of H is a weak hyperBCK-ideal of H.

Definition 3.21. Let μ be a fuzzy set in H. Then μ is called a *fuzzy reflexive hyperBCK-ideal* of H if it satisfies

(i) $\inf_{a \in x \circ x} \mu(a) \geq \mu(y)$,

(ii) $\mu(x) \geq \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\}$,

for all $x, y \in H$.

Theorem 3.22. *Every fuzzy reflexive hyperBCK-ideal of H is a fuzzy strong hyperBCK-ideal of H.*

Proof. Straightforward. □

Theorem 3.23. *If μ is a fuzzy reflexive hyperBCK-ideal of H, then μ_t is a reflexive hyperBCK-ideal of H whenever $\mu_t \neq \emptyset$ for $t \in [0,1]$.*

Proof. Let μ be a fuzzy reflexive hyperBCK-ideal of H and $\mu_t \neq \emptyset$ for $t \in [0,1]$. Then there exists $a \in H$ such that $a \in \mu_t$ and so $\mu(a) \geq t$. By Theorem 3.22, μ is a fuzzy strong hyperBCK-ideal and moreover a hyperBCK-ideal of H. It follows from Theorem 3.17 that μ_t is a hyperBCK-idael of H. Moreover for each $x \in H$ and $c \in x \circ x$, $\mu(c) \geq \inf_{b \in x \circ x} \mu(b) \geq \mu(a) \geq t$ by Definition 3.21(i). Hence $c \in \mu_t$ for each $c \in x \circ x$, which shows that $x \circ x \subseteq \mu_t$. Therefore μ_t is a reflexive hyperBCK-ideal of H. □

Theorem 3.24. *Let μ is a fuzzy set in H satisfying the* **sup** *property. If $\mu_t (\neq \emptyset)$ is a reflexive hyperBCK-ideal of H for all $t \in [0,1]$, then μ is a fuzzy reflexive hyperBCK-ideal of H.*

Proof. Let $\mu_t (\neq \emptyset)$ be a reflexive hyperBCK-ideal of H for all $t \in [0,1]$. By [4, Theorem 3.6], $\mu_t (\neq \emptyset)$ is a strong hyperBCK-ideal of H for all $t \in [0,1]$. Using Theorem 3.15, we have that μ is a fuzzy strong hyperBCK-ideal of H and thus it satisfies the condition (ii) of the Definition 3.21. Now we show that μ satisfies (i) in the Definition 3.21. Let $x, y \in H$ and $\mu(y) = t$. Since $\mu_t (\neq \emptyset)$ is a reflexive hyperBCK-ideal of H, it follows that $x \circ x \subseteq \mu_t$. Thus for each $c \in x \circ x$, we have $c \in \mu_t$, and so $\mu(c) \geq t$. This shows that $\inf_{c \in x \circ x} \mu(c) \geq t = \mu(y)$. Therefore μ is a fuzzy reflexive hyperBCK-ideal of H. $\square$

Theorem 3.25. *Let μ be a fuzzy strong hyperBCK-ideal of H satisfying the* **sup** *property. Then μ is a fuzzy reflexive hyperBCK-ideal of H if and only if $\inf_{a \in x \circ x} \mu(a) \geq \mu(0)$ for all $x \in H$.*

Proof. Let μ be a fuzzy reflexive hyperBCK-ideal of H. Then, by Definition 3.21(i), we have $\inf_{a \in x \circ x} \mu(a) \geq \mu(0)$. Conversely, assume $\inf_{a \in x \circ x} \mu(a) \geq \mu(0)$ for all $x \in H$. Since μ is a fuzzy hyperBCK-ideal of H, we get $\mu(0) \geq \mu(y)$ for all $y \in H$ and hence $\inf_{a \in x \circ x} \mu(a) \geq \mu(y)$ for all $x, y \in H$. Moreover put $t = \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\}$ where $x, y \in H$. Using Theorem 3.13 and 3.22, the set μ_t is a strong hyperBCK-ideal of H. Since μ satisfies the **sup** property, there exists $a_0 \in x \circ y$ such that $\mu(a_0) = \sup_{a \in x \circ y} \mu(a)$ and so $\mu(a_0) \geq t$, i.e., $a_0 \in \mu_t$. This shows that $x \circ y \cap \mu_t \neq \emptyset$. It follows from $y \in \mu_t$ that $x \in \mu_t$ since μ_t is a strong hyperBCK-ideal of H. Therefore $\mu(x) \geq t = \min\{\sup_{a \in x \circ y} \mu(a), \mu(y)\}$, and hence μ is a fuzzy reflexive hyperBCK-ideal of H. $\square$

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Y. B. Jun
Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea
e-mail: ybjun@nongae.gsnu.ac.kr

X. L. Xin
Department of Mathematics
Northwest University
Xian 710069, P. R. China