CONNECTIONS ON $\mathbb{A}$-FRAME BUNDLES

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ABSTRACT. We study the frame bundle $F(\ell)$ of a vector bundle ℓ with fibre type a projective finitely generated module over a (topological) algebra $\mathbb{A}$. The topological-algebraic structure of $\mathbb{A}$ is crucial in our considerations. In fact, if $\mathbb{A}$ is a Q-algebra, $F(\ell)$ is a smooth principal bundle and its connections correspond bijectively to $\mathbb{A}$-connections on ℓ, as in the case of Banach bundles. If $\mathbb{A}$ is not a Q-algebra, then $F(\ell)$ is only a topological principal bundle. However, it can be provided with sheaf-theoretic entities, legitimately called connections, which essentially describe the connections of ℓ. As a result, the geometry of our bundles can be reduced to a topological-algebraic context embodying all the previous cases and giving an example of the effectiveness of the methods of the "abstract differential geometry" initiated in [10] for "vector sheaves" and further applied and extended for "principal sheaves" in [25], here combined with topological algebra theory.

1. Introduction

In pure mathematics as well as in theoretical physics, some of the vector spaces dealt with have the additional structure of a module over a topological algebra $\mathbb{A}$. The consideration of this additional topological-algebraic structure lead to a successful treatment of a number of problems, e.g. in operator theory [3], in theoretical physics [21], in differential topology [11], to name but a few of them. In particular, one frequently encounters manifolds and vector bundles, whose models are *projective finitely generated* $\mathbb{A}$-*modules* (for some applications of this aspect in differential geometry, mechanics and PDEs, see also [4, 20, 22]). In the sequel, such manifolds and bundles are called $\mathbb{A}$-*manifolds* and $\mathbb{A}$-*bundles*, respectively.

The structure and classification of the *topological* $\mathbb{A}$-bundles have been studied in a series of papers by A. Mallios (see [9] and the references therein), whereas *differentiable* $\mathbb{A}$-bundles have been considered by the second author, within an appropriate differential framework ([14, 15, 16]). In both cases, $\mathbb{A}$ is a locally m-convex algebra (in effect, even arbitrary topological algebras, in general, have been employed by A. Mallios, loc.cit), thus abstracting all the aforementioned examples and applications.

An object of prime interest over a differentiable $\mathbb{A}$-bundle, being, as it actually is, of a differential-geometric character, is, undoubtedly, that of a connection. It has already been proven ([16]) that $\mathbb{A}$-bundles admit linear connections, which, in contrast to the general infinite-dimensional case (cf. [1, 26]), are equivalent to covariant $\mathbb{A}$-derivations (see [15]). On the other hand, it is often convenient to view a vector bundle as one associated with its bundle of frames and to reduce linear connections on the former to connections on the latter.

The purpose of this paper is to study the previous reduction in the context of $\mathbb{A}$-bundles, with $\mathbb{A}$ a suitable, not necessarily normed topological algebra. Unfortunately, a major difficulty arises here: If P is the fibre type of a non-Banachable infinite dimensional vector

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bundle, the general linear group of P is not necessarily a Lie group, not even a topological one (for instance, if P is not normable; see [12, p. 369] and [28, p. 144, (1.4)]). Thus the frame bundle of such a vector bundle may not exist, let alone have connections.

However, in our context, the module structure of the models allows us to partially overcome this difficulty. In fact, for an $\mathbb{A}$-bundle ℓ, the structural group of the frame bundle $F(\ell)$ is the group $GL_{\mathbb{A}}(P)$ of $\mathbb{A}$-linear automorphisms of P, which is always a topological group. If, in addition, $\mathbb{A}$ *is a Q-algebra* (i.e., the set of the invertible elements of $\mathbb{A}$ is open), $GL_{\mathbb{A}}(P)$ is proved to be a Lie group and $F(\ell)$ is a differentiable principal bundle (Theorems 3.5 and 3.6, below). Moreover, each linear connection ∇ on ℓ induces a 0-cochain of local forms (ω_α) satisfying the analogue of the well known compatibility condition and vice-versa. In turn, the same family globalizes to a principal connection form ω on $F(\ell)$ and the bijective correspondence between the linear connections of ℓ and the connections on $F(\ell)$ is established (Theorem 4.3).

Nevertheless, in the most important and frequently met examples of $\mathbb{A}$-bundles, $\mathbb{A}$ is either the algebra $\mathcal{C}(X)$ of continuous functions on a topological space X, or the algebra $\mathcal{C}^\infty(X)$ of differentiable functions over a smooth manifold. In both cases, $\mathbb{A}$ is *not* a Q-algebra, unless X is compact (as a matter of fact, the compactness of X is the necessary and sufficient condition, the previous functional algebras to be Q; see [7, 10]). Therefore, in the general (non-compact) case, one has to treat $F(\ell)$ only as a topological bundle. In this case, (ω_α) cannot be globalized to a connection form on $F(\ell)$, as before. Yet, singling out some of the continuous sections of $F(\ell)$ (which play the rôle of “differentiable” sections of $F(\ell)$), we obtain an appropriate *sheaf of germs of sections $\mathcal{F}(E)$ of the topological bundle* $F(\ell)$ and we provide it with a global morphism D (with values in a sheaf of forms), fully determined by (and determining) (ω_α). As a result (see Theorem 5.3), linear connections on ℓ are in a bijective correspondence with the morphisms D, which can legitimately be called connections of $\mathcal{F}(E)$ (cf. [25], in conjunction with [10]).

In brief, we conclude that linear connections on ℓ (defined within a smooth context) are completely determined by sheaf-theoretic objects (defined in a topological-algebraic context) and vice-versa. In particular, in the case of Q-algebras, the previous sheaf morphisms coincide, within a bijection, with ordinary connections on the differential bundle of frames $F(\ell)$.

2. Preliminaries

Although several aspects of the theory of topological $\mathbb{A}$-bundles are developed for general locally convex (or even topological) algebras [9], in the differential framework, in order to obtain a convenient differentiation method, additional assumptions on the algebra $\mathbb{A}$ are necessary (see [14]). Thus, throughout this paper, $\mathbb{A}$ is a *commutative locally m-convex* (abr. *lmc*) *algebra with unit* (for the relevant terminology see [7]) and $\mathcal{P}(\mathbb{A})$ is the category of *projective finitely generated $\mathbb{A}$-modules*. We recall that, by definition, for every $M \in \mathcal{P}(\mathbb{A})$, there are $M_1 \in \mathcal{P}(\mathbb{A})$ and $m \in \mathbb{N}$, so that $M \oplus M_1 = \mathbb{A}^m$. Obviously, $\mathbb{A} \in \mathcal{P}(\mathbb{A})$. Since the objects of $\mathcal{P}(\mathbb{A})$ will be the models of the manifolds and bundles considered here, we should fix a topology and a method of differentiation on them.

Let M, M_1, m be as above. The product topology on $\mathbb{A}^m$ induces the relative topology τ_M on M. It turns out that τ_M does not depend on M_1 or m, and it is the unique topology making M a topological $\mathbb{A}$-module and any $\mathbb{A}$-multilinear map on M continuous (for details see [8, 13]).

In the sequel, for any M, $N \in \mathcal{P}(\mathbb{A})$, 0_M denotes the zero element of M, $\mathcal{N}(x)$ the family of open neighbourhoods of $x \in M$ and $L_{\mathbb{A}}(M, N)$ the set of $\mathbb{A}$-linear maps $M \to N$. We apply the following differentiation method due to Vu Xuan Chi [27], originally defined for modules over a topological ring:

Let $x \in M$ and $W \in \mathcal{N}(x)$. A mapping $f : W \to N$ is said to be $\mathbb{A}$-*differentiable at* x, if there exists $Df(x) \in L_{\mathbb{A}}(M, N)$, such that *the remainder of* f *at* x

$$\phi(h) := f(x+h) - f(x) - Df(x)(h)$$

satisfies the condition

$$\forall V \in \mathcal{N}(0_N)\ \ \exists U \in \mathcal{N}(0_M)\ :\ \forall B \in \mathcal{N}(0_{\mathbb{A}})\ \ \exists A \in \mathcal{N}(0_{\mathbb{A}}) :$$

$$a \in A\ \Rightarrow\ \phi(aU) \subset aBV.$$

This differentiation, applied to our case, where the structure of the modules is richer than the one considered in [27], has a number of fundamental properties, missing both in the context of [27] and in that of locally convex spaces. In particular, (i) $\mathbb{A}$-differentiability implies continuity; (ii) the composition and the evaluation mappings are $\mathbb{A}$-differentiable and (iii) the chain rule holds for every order of differentiation (for the proofs we refer to [14], where an analogous differentiation is introduced for *-algebras). An infinitely $\mathbb{A}$-differentiable map will be called $\mathbb{A}$-*smooth*.

Following the standard pattern, we obtain the category $Man(\mathbb{A})$ of $\mathbb{A}$-*manifolds*, modelled on the objects of $\mathcal{P}(\mathbb{A})$, and $\mathbb{A}$-*smooth morphisms*. We obtain the tangent spaces $T(X, x)$, $x \in X \in Man(\mathbb{A})$, by considering classes of equivalent $\mathbb{A}$-*curves*, that is, $\mathbb{A}$-smooth maps defined on open neighbourhoods of $0_{\mathbb{A}}$ with range in X. If X is modelled on M and (U, ϕ) is a chart at x, then the bijection

$$\bar{\phi} : T(X, x) \to M : [(\alpha, x)] \mapsto D(\phi \circ \alpha)(0)$$

provides an $\mathbb{A}$-module structure on $T(X, x)$. Furthermore, the tangent bundle TX of X is an $\mathbb{A}$-manifold. If $f : X \to Y$ is $\mathbb{A}$-smooth, the differential $df : TX \to TY$ is also $\mathbb{A}$-smooth and its restrictions on the tangent spaces are $\mathbb{A}$-linear maps.

3. $\mathbb{A}$-BUNDLES AND THEIR FRAME BUNDLES

Let $X, E \in Man(\mathbb{A})$, $\pi : E \to X$ be $\mathbb{A}$-smooth and $P \in \mathcal{P}(\mathbb{A})$. Moreover, let

$$E_x := \pi^{-1}(x) \in \mathcal{P}(\mathbb{A}); \quad \forall x \in X,$$

and assume that there exists an open covering $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ of X and a family of (*trivializing*) $\mathbb{A}$-diffeomorphisms

$$\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times P, \quad \alpha \in I,$$

such that $pr_1 \circ \tau_\alpha = \pi$ and each restriction $\tau_{\alpha x} : E_x \to \{x\} \times P;\ x \in U_\alpha$, is an $\mathbb{A}$-module isomorphism. We call the triplet $\ell = (E, \pi, X)$ an $\mathbb{A}$-*bundle*.

One trivially checks that the tangent bundle $\ell_X := (TX, \pi_X, X)$ of any $\mathbb{A}$-manifold X is an $\mathbb{A}$-bundle.

It is worthy to note that in the framework of infinite dimensional vector bundles, even in the Banach case, we need one more assumption, that is, the differentiability of the transition functions (cf. condition (VB.3) in [6]). In our context, although the underlying vector spaces of the models are infinite-dimensional locally convex spaces, the condition under discussion is guaranteed by the properties of the projective finitely generated $\mathbb{A}$-modules. In fact, we have

Lemma 3.1. *Let* M, N, $P \in \mathcal{P}(\mathbb{A})$, $U \subseteq M$ *open and* $f : U \times N \to P$ *an* $\mathbb{A}$-*smooth map, so that the partial maps* f_x, $x \in U$, *are* $\mathbb{A}$-*linear. Then, the map*

$$F : U \to L_{\mathbb{A}}(N, P) : x \mapsto f_x$$

is $\mathbb{A}$*-smooth.*

Proof. Let $N_1, P_1 \in \mathcal{P}(\mathbb{A})$ and $n, p \in \mathbb{N}$, with $N \oplus N_1 = \mathbb{A}^n$ and $P \oplus P_1 = \mathbb{A}^p$. We consider the $\mathbb{A}$-smooth extension of f

$$\overline{f} : U \times (N \oplus N_1) \to P \oplus P_1 : (x, y, y_1) \mapsto (f(x, y), 0),$$

which induces the map $\overline{F} : U \to L_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p) : x \mapsto \overline{f}_x$. It is clear that $L_{\mathbb{A}}(N, P)$ is a direct factor of $L_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p)$. If pr denotes the respective projection, then $F = pr \circ \overline{F}$ is $\mathbb{A}$-smooth if and only if $\overline{F}$ is $\mathbb{A}$-smooth, the other component of $\overline{F}$ vanishing. Since $L_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p)$ is $\mathbb{A}$-isomorphic with the $\mathbb{A}$-module $\mathcal{M}_{n \times p}(\mathbb{A})$ of $n \times p$ matrices with entries in $\mathbb{A}$, it suffices to prove that

$$U \to \mathcal{M}_{n \times p}(\mathbb{A}) : x \mapsto (a_{ij}(x)) := (pr_j \circ \overline{f}_x(e_i))$$

is $\mathbb{A}$-smooth. This is a consequence of the $\mathbb{A}$-smoothness of the maps

$$\overline{f}_i : U \to \mathbb{A}^p : x \mapsto \overline{f}(x, e_i) = (a_{i1}(x), \ldots, a_{ip}(x)); \quad i = 1, \ldots, n. \qquad \square$$

As a result of the preceding lemma, we obtain

Proposition 3.2. *Let* $\ell = (E, \pi, X)$ *be an* $\mathbb{A}$*-bundle of fibre type* $P \in \mathcal{P}(\mathbb{A})$*, with trivializing covering* $\{(U_\alpha, \tau_\alpha)\}_{\alpha \in I}$*. Then, the transition functions*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to L_{\mathbb{A}}(P) := L_{\mathbb{A}}(P, P) : x \mapsto \tau_{\alpha x} \circ \tau_{\beta x}^{-1} \tag{3.1}$$

are $\mathbb{A}$*-smooth.*

The previous transition functions take values, in effect, in the group $GL_{\mathbb{A}}(P)$ of invertible elements of the algebra $L_{\mathbb{A}}(P)$. The latter, being an object of $\mathcal{P}(\mathbb{A})$, admits the canonical topology and the algebra multiplication

$$L_{\mathbb{A}}(P) \times L_{\mathbb{A}}(P) \to L_{\mathbb{A}}(P) : (f, g) \mapsto g \circ f \tag{3.2}$$

is continuous, as an $\mathbb{A}$-bilinear map. On the other hand, the *inversion is also continuous* (see [7, Lemma 6.3]), thus $GL_{\mathbb{A}}(M)$, topologized with the relative topology induced by the canonical topology of $L_{\mathbb{A}}(P)$, is a *topological group*. Therefore, $(g_{\alpha\beta})$ determines a *topological principal bundle* $F(\ell) = (F(E), GL_{\mathbb{A}}(P), X, p)$, called *the frame bundle of* ℓ. We summarize the previous considerations in

Theorem 3.3. *If* ℓ *is an* $\mathbb{A}$*-bundle of fibre type* $P \in \mathcal{P}(\mathbb{A})$*, then* $F(\ell)$ *is a topological principal bundle with structural group* $GL_{\mathbb{A}}(P)$*.*

Assume now that $\mathbb{A}$ is a *Q-algebra*, i.e., the group $\mathbb{A}^\circ$ of invertible elements of $\mathbb{A}$ is open. This property is inherited to the algebra $L_{\mathbb{A}}(P)$, for every $P \in \mathcal{P}(\mathbb{A})$ ([24, Corollary 1.2] and [8, Theorem 1.1]). The composition

$$GL_{\mathbb{A}}(P) \times GL_{\mathbb{A}}(P) \to GL_{\mathbb{A}}(P) : (f, g) \mapsto g \circ f,$$

being now the restriction of an $\mathbb{A}$-smooth map on an open subset is also $\mathbb{A}$-smooth. But, the differentiability of the inversion can not be deduced from that of the multiplication (viz. composition), since this deduction is based on the inverse mapping theorem, which is not valid in general. However, the differentiability in question is obtained in a straightforward way described in the next

Lemma 3.4. *Let* $\mathbb{A}$ *be a commutative lmc Q-algebra with unit and* $P \in \mathcal{P}(\mathbb{A})$*. Then the inversion* α *in* $GL_{\mathbb{A}}(P)$ *is* $\mathbb{A}$*-smooth.*

Proof. Let $f \in GL_{\mathbb{A}}(P)$. Setting $D\alpha(f)(h) := -f \circ h \circ f$, for every $h \in L_{\mathbb{A}}(P)$, we form the remainder of α at f

$$\phi(h) := \alpha(f + h) - \alpha(f) - D\alpha(f)(h).$$

It is not hard to show that $\phi(h) = f^{-1} \circ \psi(h) \circ f^{-1}$, where $\psi(h) = h \circ (f+h)^{-1} \circ h$.

We prove that ψ is infinitesimal: Let $\mathbf{0}$ be the zero element of $L_{\mathbb{A}}(P)$ and $V \in \mathcal{N}(\mathbf{0})$. The continuity of the composition at $(\mathbf{0}, f^{-1}, \mathbf{0})$ implies the existence of $U_1 \in \mathcal{N}(\mathbf{0})$, $V_1 \in \mathcal{N}(f^{-1})$ with $U_1 \circ V_1 \circ U_1 \subseteq V$. Since $V_1 \in \mathcal{N}(f^{-1})$ and α is continuous, there exists $V_2 \in \mathcal{N}(f)$ with $V_2^{-1} \subseteq V_1$. The continuity of the $\mathbb{A}$-module operations also determine $A_1 \in \mathcal{N}(0_{\mathbb{A}})$ and $U_2 \in \mathcal{N}(\mathbf{0})$, with $A_1 U_2 \subseteq V_2 - f \in \mathcal{N}(\mathbf{0})$. We set $U := U_1 \cap U_2$, and, for $B \in \mathcal{N}(0_{\mathbb{A}})$, $A := A_1 \cap B$. Then, for any $a \in A$ and $h \in U$, we have

$$\begin{aligned} \psi(ah) &= ah \circ (f+ah)^{-1} \circ ah = a^2 h \circ (f+ah)^{-1} \circ h \\ &\in aAU_1 \circ (f + A_1U_2)^{-1} \circ U_1 \subseteq aBU_1 \circ V_2^{-1} \circ U_1 \\ &\subseteq aBU_1 \circ V_1 \circ U_1 \subseteq aBV, \end{aligned}$$

thus proving the assertion. Since an $\mathbb{A}$-linear combination of infinitesimal mappings is infinitesimal, ϕ is also infinitesimal, by which we complete the proof. □

As a result, we obtain

Theorem 3.5. *Let $\mathbb{A}$ be a commutative lmc Q-algebra with unit. Then, $GL_{\mathbb{A}}(P)$ is a Lie group, for every $P \in \mathcal{P}(\mathbb{A})$.*

Applying now a standard reasoning we conclude that the following holds true

Theorem 3.6. *For any $\mathbb{A}$-bundle ℓ of fibre type $P \in \mathcal{P}(\mathbb{A})$, with $\mathbb{A}$ as in Theorem 3.5, the corresponding bundle of frames $F(\ell)$ is a differentiable principal bundle.*

4. Connections on $\mathbb{A}$-bundles

In this section we introduce connections on $\mathbb{A}$-bundles and, in the case of a Q-algebra, we study the interplay between them and their counterparts on the corresponding (differentiable) bundles of frames.

For any $\mathbb{A}$-bundle $\ell = (E, \pi, X)$, the set of $\mathbb{A}$-*smooth sections of* ℓ will be denoted by $\Gamma(X, E)$.

Definition 4.1. An $\mathbb{A}$-*connection on* $\ell = (E, \pi, X)$ is an $\mathbb{A}$-bilinear map

$$\nabla : \Gamma(X, TX) \times \Gamma(X, E) \to \Gamma(X, E) : (\xi, s) \to \nabla_\xi s$$

satisfying the following properties:

i) $\nabla_{f\xi} s = f \cdot \nabla_\xi s$,

ii) $\nabla_\xi (fs) = f \cdot \nabla_\xi s + (df \circ \xi) \cdot s$,

for every $\xi \in \Gamma(X, TX)$, $s \in \Gamma(X, E)$ and every $\mathbb{A}$-smooth $f : X \to \mathbb{A}$.

In this definition we essentially identify a connection with a covariant derivation. This well known result for finite dimensional bundles, (derived from the existence of bases in the models), is not true in the infinite dimensional case, even for Banach bundles (cf. [1, 26]). Here, although bases do not exist, this property of finite dimensional bundles is recovered by showing that ∇ is equivalent with a family of (generalized) Christoffel symbols, through an embedding of the given $\mathbb{A}$-bundle in one with bases and an extension of ∇ to a suitable map (which is not a connection!).

For our aim, we say that $X \in Man(\mathbb{A})$ *admits* $\mathbb{A}$-*bump functions*, if, for every open $U \subseteq X$ and $x \in U$, there is an open $V \subseteq X$, with $x \in V \subseteq \overline{V} \subseteq U$, and an $\mathbb{A}$-smooth $f : X \to \mathbb{A}$, so that $f|_{\overline{V}} = 1$, $f|_{X \setminus U} = 0$. The existence of $\mathbb{A}$-bump functions is ensured for every $\mathbb{A}$-manifold, if $\mathbb{A}$ coincides with the algebra $\mathcal{C}(X)$ of continuous functions on a completely regular Hausdorff topological space X or the algebra $\mathcal{C}^\infty(X)$ of smooth functions on a compact manifold X (see [17] and [18], respectively). The general case remains still an open problem.

Also, before proving our result, we fix some further notations. For an $\mathbb{A}$-bundle $\ell = (E,\pi,X)$, we consider an atlas $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in I}$ of X and a trivializing covering $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in I}$ of ℓ, over the same open covering $\mathcal{U} := \{U_\alpha\}_{\alpha\in I}$ of X. For every vector field $\xi \in \Gamma(X,TX)$, its local expression $\overline{\phi}_\alpha \circ \xi \circ \phi_\alpha^{-1}$ will be simply denoted by ξ_α. Similarly, for every section $\sigma \in \Gamma(X,E)$, the principal part of its local expression $pr_2 \circ \tau_\alpha \circ \sigma \circ \phi_\alpha^{-1}$ will be denoted by σ_α.

Theorem 4.2. *Let X be an $\mathbb{A}$-manifold modelled on $M \in \mathcal{P}(\mathbb{A})$ admitting $\mathbb{A}$-bump functions and let $\ell = (E,\pi,X)$ be an $\mathbb{A}$-bundle of fibre type $P \in \mathcal{P}(\mathbb{A})$. Then ℓ has an $\mathbb{A}$-connection ∇, if and only if, there is a family of $\mathbb{A}$-smooth maps (:generalized Christoffel symbols)*

$$\Gamma_\alpha : U_\alpha \to L^2_{\mathbb{A}}(M\times P,P) \equiv L_{\mathbb{A}}(M, L_{\mathbb{A}}(P)),\ \alpha \in I,$$

satisfying the compatibility condition:

$$\begin{aligned}\Gamma_\beta(x)(h,k) &= \\ &= [g_{\beta\alpha}(x)\circ\Gamma_\alpha(x) - D(g_{\beta\alpha}\circ\phi_\alpha^{-1})(\phi_\alpha(x))](\bar\phi_\alpha\circ\bar\phi_\beta^{-1}(h), g_{\alpha\beta}(x)(k)),\end{aligned}\tag{4.1}$$

for every $h\in M$, $k \in P$, $\alpha,\beta\in I$ and $x\in U_\alpha\cap U_\beta$.

Proof. The family $\{\Gamma_\alpha\}_{\alpha\in I}$ defines an $\mathbb{A}$-connection [1, 26]. Conversely, let ∇ be an $\mathbb{A}$-connection of ℓ. Since M, $P\in\mathcal{P}(\mathbb{A})$, there exist N, $Q\in\mathcal{P}(\mathbb{A})$ and m, $p\in\mathbb{N}$, such that $M\oplus N\simeq\mathbb{A}^m$ and $P\oplus Q\simeq\mathbb{A}^p$. Consider the trivial $\mathbb{A}$-bundle $\ell_1 = (N\times Q, p_1, N)$. Then the cartesian product $\ell\times\ell_1 = (F := E\times N\times Q,\ \pi\times p_1,\ X\times N)$ is an $\mathbb{A}$-bundle of fibre type $\mathbb{A}^p$ and the base space $Y := X\times N \in Man(\mathbb{A})$ is modelled on $\mathbb{A}^m$.

Let now Ω denote the zero section of ℓ_1. Also, let $p_X : X\times N\to X$ and $p_E : E\times N\times Q\to E$ be the canonical projections and consider the mapping

$$\widetilde{\nabla} : \Gamma(Y,TY)\times\Gamma(Y,F)\to\Gamma(Y,F) : (\Xi,S)\to\nabla_\xi s\times\Omega,$$

where $\xi(x) := dp_X(\Xi(x,0))$ and $s(x) := p_E(S(x,0))$, for every $x\in X$. $\widetilde{\nabla}$ is *not* an $\mathbb{A}$-connection. If $\{e_i\}_{1\le i\le m}$ and $\{e_j\}_{1\le j\le p}$ are the canonical bases of $\mathbb{A}^m$ and $\mathbb{A}^p$, p_M and p_N the canonical projections of $\mathbb{A}^m$ to M and N, and p_P and p_Q are the canonical projections of $\mathbb{A}^p$ to P and Q, then, for every $\alpha\in I$, the families $\{\partial_i\}_{1\le i\le m}$ and $\{\eta_j\}_{1\le j\le p}$, with

$$\partial_i(x,y) := (\overline{\phi}_\alpha^{-1}(p_M(e_i)),\ p_N(e_i))\,;\quad (x,y)\in U\times N,$$

$$\eta_j(x,y) := (\tau_{\alpha x}^{-1}(p_P(e_j)),\ p_Q(e_j))\,;\quad (x,y)\in U\times N,$$

are (local) frames of TY and F, respectively. The existence of $\mathbb{A}$-bump functions results in the definition of local $\widetilde{\nabla}$'s, therefore, of a family of $\mathbb{A}$-smooth maps $\widetilde{\Gamma}^k_{\alpha ij} : V_\alpha := U_\alpha\times N\to\mathbb{A}$, given by

$$\widetilde{\nabla}_{\partial_i}\eta_j = \sum_k \widetilde{\Gamma}^k_{\alpha ij}\eta_k.$$

We define the mappings

$$\widetilde{\Gamma}_\alpha : \phi_\alpha(U_\alpha)\times N\to L^2_{\mathbb{A}}(\mathbb{A}^m,\mathbb{A}^p;\mathbb{A}^p) : (h,y)\mapsto\widetilde{\Gamma}_\alpha(h,y) :$$

$$\widetilde{\Gamma}_\alpha(h,y)(a,b) = \sum_{i,j,k} a_i b_j \widetilde{\Gamma}^k_{\alpha ij}(x,y)e_k$$

for every $(a,b)\in\mathbb{A}^m\times\mathbb{A}^p$ and $(h=\phi(x),y)\in\phi_\alpha(U_\alpha)\times N$, and

$$\Gamma_\alpha : \phi_\alpha(U_\alpha)\to L^2_{\mathbb{A}}(M,P;P) : h\mapsto\Gamma_\alpha(h) :$$

$$\Gamma_\alpha(h)(a,b) = p_P \circ \widetilde{\Gamma}_\alpha(h,0)(a,b)\,; \quad (h,a,b) \in \phi_\alpha(U_\alpha) \times M \times P.$$

The Γ_α's are the required generalized Christoffel symbols of ∇. In fact, one can prove that

$$(\nabla_\xi s)_\alpha(h) = Ds_\alpha(h)(\xi_\alpha(h)) \,+\, \Gamma_\alpha(h)(\xi_\alpha(h), s_\alpha(h)).$$

for every $\xi \in \Gamma(X, TX)$, $s \in \Gamma(X, E)$, $\alpha \in I$. The compatibility condition is a matter of straightforward (although tedious) calculations (see also [1]). □

A similar procedure proves that an $\mathbb{A}$-connection is equivalent with a splitting of a certain short exact sequence of $\mathbb{A}$-bundles (see [15]), again a fact known for finite dimensional bundles (as well as, e.g. in the context of abstract differential geometry [10]) but not valid in the infinite dimensional framework.

Suppose now that $\mathbb{A}$ is a Q-algebra and let $\Lambda^1(U_\alpha, L_{\mathbb{A}}(P))$ denote the $\mathbb{A}$-module of $L_{\mathbb{A}}(P)$-valued smooth (in $Man(\mathbb{A})$) 1-forms on U_α ($\alpha \in I$). The Christoffel symbols $\{\Gamma_\alpha\}_{\alpha\in I}$ induce the 0-cochain of 1-forms $(\omega_\alpha)_{\alpha\in I}$ defined by

$$\omega_{\alpha,x}(v)\cdot h := \Gamma_\alpha(x)(\bar{\phi}_\alpha(v), h), \tag{4.2}$$

for every $x \in U_\alpha$, $v \in T_xB$ and $h \in P$.

We check that (4.1) and (4.2) yield the compatibility condition:

$$\omega_\beta = Ad(g_{\alpha\beta}^{-1})\cdot\omega_\alpha + g_{\alpha\beta}^{-1}\cdot dg_{\alpha\beta}\ . \tag{4.3}$$

Therefore, following the classical method (see e.g. [19, 23]), we obtain a global 1-form $\omega \in \Lambda^1(F(E), L_{\mathbb{A}}(P))$, given by the relations

$$\omega|_{\pi^{-1}(U_\alpha)} = Ad(g_\alpha^{-1})\cdot\pi^*\omega_\alpha \,+\, g_\alpha^{-1}\cdot dg_\alpha, \tag{4.4}$$

where $g_\alpha : \pi^{-1}(U_\alpha) \to GL_{\mathbb{A}}(P)$ is the $\mathbb{A}$-smooth map defined by the equality $p = s_\alpha(\pi(p))\cdot g_\alpha(p)$, for every $p \in \pi^{-1}(U_\alpha)$; here (s_α) are the natural sections of $F(\ell)$ with respect to $\mathcal{U}$. Then, ω is a connection form on the frame bundle $F(\ell)$ (see ibid. and [5]) with local connection forms the given ω_α's.

Conversely, starting with a connection form ω, condition (4.3), along with (4.2), implies (4.1) which, in turn, determines an $\mathbb{A}$-connection on E.

Therefore, we have proved

Theorem 4.3. *If $\mathbb{A}$ is a unital commutative lmc Q-algebra, $\mathbb{A}$-connections on $\ell = (E, \pi, X)$ correspond bijectively to connections on $F(\ell) = (F(E), GL_{\mathbb{A}}(P), X, p)$.*

5. Connections on topological frame bundles

The problem now is to see what can be said if $\mathbb{A}$ is not necessarily a Q-algebra. In this case, our framework lacks the tools (e.g. implicit function theorem) to prove that $GL_{\mathbb{A}}(P)$ is a manifold, and, consequently, a Lie group. Therefore, although the local forms (4.2) exist, (4.4) is meaningless and (ω_α) does not globalize to a connection form ω on $F(E)$. However, to each ∇, we shall correspond a global object generalizing the notion of a (principal) connection and living in an appropriate sheaf.

To this end we note that the principal parts of the (continuous) local sections of the topological bundle $F(\ell)$ take values in $GL_{\mathbb{A}}(P)$, which is embedded in $L_{\mathbb{A}}(P) \in \mathcal{P}(\mathbb{A})$. We say that a map $g : U \to GL_{\mathbb{A}}(P)$ ($U \subseteq X$ open) is $\mathbb{A}$-*smooth*, if the embeddings of g and g^{-1} in $L_{\mathbb{A}}(P)$, where $g^{-1}(x) := g(x)^{-1}$, with $x \in U$, are $\mathbb{A}$-smooth. Accordingly, a local section of $F(\ell)$ will be called $\mathbb{A}$-*smooth*, if its principal parts are $\mathbb{A}$-smooth. We denote by $\Gamma(U, F(E))$ the set of $\mathbb{A}$-smooth sections of $F(\ell)$ over an open $U \subseteq X$ and by $\mathcal{F}(E)$ the corresponding sheaf, generated by the presheaf $U \mapsto \Gamma(U, F(E))$; thus

$$\mathcal{F}(E)(U) \cong \Gamma(U, F(E))\,; \qquad U \subseteq X \text{ open}\ .$$

Similarly, $\mathcal{GL}(P)$ denotes the sheaf of germs of smooth $GL_{\mathbb{A}}(P)$-valued maps on X, i.e.

$$\mathcal{GL}(P)(U) \cong \mathcal{C}^{\infty}(U, GL_{\mathbb{A}}(P)), \tag{5.1}$$

the smoothness here being also meant in the generalized sense defined above. The local structure of $F(E)$ along with (5.1), induce the (local) isomorphisms

$$\widetilde{\Phi}_{\alpha} : \mathcal{F}(E)|_{U_{\alpha}} \xrightarrow{\cong} \mathcal{GL}(P)|_{U_{\alpha}}, \quad \alpha \in I \tag{5.2}$$

which are $\mathcal{GL}(P)|_{U_{\alpha}}$-equivariant. The corresponding transition transformations coincide, in virtue of (5.2), with the cocycle $(g_{\alpha\beta})$ of E, viewed now as an element of $Z^1(\mathcal{U}, \mathcal{GL}(P))$. Therefore, in A. Grothendieck's [2] terminology, $\mathcal{F}(E)$ is a *principal sheaf of structure type* $\mathcal{GL}(P)$ *and with structure sheaf* $\mathcal{GL}(P)$. If $\mathbb{A}$ is a Q-algebra, $\mathcal{F}(E)$ coincides with the sheaf of germs of ordinary smooth sections of $F(\ell)$. The isomorphisms (5.2) determine the *natural sections*

$$\sigma_{\alpha} := \widetilde{\Phi}_{\alpha}^{-1} \circ \mathbf{1}|_{U_{\alpha}},$$

of $\mathcal{F}(E)$, where $\mathbf{1}$ is the global unit section of $\mathcal{GL}(P)$.

Besides, we denote by $\widetilde{\Omega}^1 := \Omega^1_X(L_{\mathbb{A}}(P))$ the sheaf of germs of $L_{\mathbb{A}}(P)$-valued smooth 1-forms on X; thus

$$\widetilde{\Omega}^1(U) := \Omega^1_X(L_{\mathbb{A}}(P))(U) \cong \Lambda^1(U, L_{\mathbb{A}}(P)); \qquad U \subseteq X \text{ open }. \tag{5.3}$$

Finally, we define the morphism of sheaves of sets $\partial : \mathcal{G}(P) \to \widetilde{\Omega}^1$ induced by the sheafification of the ordinary operator of *total* (or *logarithmic*) *differentiation* $\partial(g) := g^{-1} \cdot dg$, for every $g \in \mathcal{C}^{\infty}(U, GL_{\mathbb{A}}(P))$ and $U \subseteq X$ open. Clearly, $\mathcal{GL}(P)$ acts naturally on $\widetilde{\Omega}^1$. Namely, for any $g \in \mathcal{GL}(P)(U)$ and $\theta \in \widetilde{\Omega}^1(U)$, the 1-form $Ad(g) \cdot \theta$ is given by

$$(Ad(g) \cdot \theta)_x(u) := g(x) \circ \theta_x(u) \circ g(x)^{-1},$$

if $x \in U$ and $u \in T_xX$. Therefore, we immediately check that

$$\partial(g \cdot h) = Ad(h^{-1}) \cdot \partial(g) + \partial(h); \quad g, h \in \mathcal{GL}(P)(U). \tag{5.4}$$

With the previous notations we prove the following basic result

Theorem 5.1. *Under the identification (5.3), the 0-cochain* $(\omega_{\alpha}) \in C^0(\mathcal{U}, \widetilde{\Omega}^1)$ *of local forms determines a unique morphism of sheaves of sets* $D : \mathcal{F}(E) \to \widetilde{\Omega}^1$ *satisfying*

$$D(s \cdot g) = Ad(g^{-1}) \cdot D(s) + \partial(g), \tag{5.5}$$

for every $s \in \mathcal{F}(E)(U)$, $g \in \mathcal{GL}(P)(U)$ *and* $U \subseteq X$ *open. Moreover,*

$$D(\sigma_{\alpha}) = \omega_{\alpha}, \quad \alpha \in I.$$

Proof. Let $u \in \mathcal{F}(E)$ be an arbitrarily chosen element with $p(u) = x$, where p is the projection of $\mathcal{F}(E)$. If $x \in U_{\alpha} \in \mathcal{U}$, then we set

$$D(u) := Ad(h_{\alpha}^{-1}) \cdot \omega_{\alpha}(x) + \partial(h_{\alpha}), \tag{5.6}$$

for an $h_{\alpha} \in \mathcal{GL}(P)(U_{\alpha})$, uniquely determined by $u = \sigma_{\alpha}(x) \cdot h_{\alpha}$.

D is well defined. Indeed, if $x \in U_{\alpha} \cap U_{\beta}$, we see that $h_{\beta} = g_{\alpha\beta}(x)^{-1} \cdot h_{\alpha}$. Thus, in virtue of (4.3) and (5.4), we check that

$$\begin{aligned}
Ad(h_{\beta}^{-1}) \cdot \omega_{\beta}(x) + \partial(h_{\beta}) &= Ad(h_{\alpha}^{-1} \cdot g_{\alpha\beta}(x)) \cdot (Ad(g_{\alpha\beta}^{-1}) \cdot \omega_{\alpha} + \partial(g_{\alpha\beta})) \\
&\quad + \partial(g_{\alpha\beta}(x)^{-1} \cdot h_{\alpha}) \\
&= Ad(h_{\alpha}^{-1}) \cdot \omega_{\alpha} + Ad(h_{\alpha}^{-1} \cdot g_{\alpha\beta}(x)) \cdot \partial(g_{\alpha\beta}) \\
&\quad - Ad(g_{\alpha}^{-1}) \cdot Ad(g_{\alpha\beta}(x))\partial(g_{\alpha\beta}(x)) + \partial(h_{\alpha}) \\
&= Ad(h_{\alpha}^{-1}) \cdot \omega_{\alpha} + \partial(h_{\alpha}).
\end{aligned}$$

The continuity of D at an arbitrary $u_\circ \in \mathcal{F}(E)$ is proved as follows: let $p(u_\circ) = x_\circ \in \mathcal{U}_\alpha$. By the local structure of sheaves, there are open neighbourhoods V and U of $u_\circ$ and $x_\circ$, respectively, such that $p|_V$ is a homeomorphism. Setting $\sigma := (p|_V)^{-1}$ and $W := \sigma(U \cap U_\alpha)$, we see that the local sections σ and σ_α satisfy the equality

$$\sigma(x) = \sigma_\alpha(x) \cdot h(x)\,; \quad x \in U \cap U_\alpha,$$

for a uniquely determined $h \in \mathcal{GL}(P)(U \cap U_\alpha)$. Hence, by (5.6),

$$D(\sigma(x)) = Ad(h(x)^{-1}) \cdot \omega_\alpha(x) \,+\, \partial(h(x))\,; \quad x \in U \cap U_\alpha,$$

from which we deduce that $D|_W = [Ad(h^{-1}) \cdot \omega_\alpha \,+\, \partial] \circ p|_W$, thus proving the continuity. □

Conversely, we have

Proposition 5.2. *Let $D : \mathcal{F}(E) \to \widetilde{\Omega}^1$ be a morphism of sheaves of sets satisfying (5.5). Then D determines a (unique) $\mathbb{A}$-connection ∇.*

Proof. Define the forms $\omega_\alpha := D(\sigma_\alpha) \in \tilde{\Omega}^1(U_\alpha)$, $\alpha \in I$. Then

$$\omega_\beta = D(s_\beta \cdot g_{\alpha\beta}) = Ad(g_{\alpha\beta}^{-1}) \cdot \omega_\alpha \,+\, \partial(g_{\alpha\beta}),$$

which, by the identifications (5.3), coincides with the compatibility condition (4.3). Then (4.2) determines the (generalized) Christoffel symbols of an $\mathbb{A}$-connection, as a consequence of Theorem 4.2. □

We call D a *generalized connection of* $\mathcal{F}(E)$, as a consequence of the next theorem, which summarizes the main results of the paper.

Theorem 5.3. *Let $\mathbb{A}$ be an arbitrary commutative lmc algebra with unit and let $\ell = (E, \pi, X)$ be an $\mathbb{A}$-bundle. Besides, let $F(\ell)$ denote the bundle of frames of ℓ and $\mathcal{F}(E)$ the sheaf of germs of sections of $F(\ell)$. Then, there exists a bijective correspondence between $\mathbb{A}$-connections ∇ on E and generalized connections D on $\mathcal{F}(E)$.*

If, in particular, $\mathbb{A}$ is a Q-algebra, then both of the previous connections correspond bijectively to a global connection (form) ω on the bundle of frames $F(\ell)$.

¿From the previous discussion we see that generalized connections D on a *topological object*, namely $\mathcal{F}(E)$, describe, through appropriate isomorphisms, the connections of E in all of their equivalent forms, as well as, in case of a Q-algebra, the corresponding connections of the (smooth) frame bundle $F(\ell)$. Therefore, this sheaf-theoretic approach provides a convenient technique for enlarging certain aspects of Differential Geometry to a non-smooth context (cf. also [10, 25]).

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