INTERPOLATION THEOREM BETWEEN B_0^p AND BMO

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Abstract. C. Fefferman and E. M. Stein proved the interpolation theorem between L^p and BMO. The purpose of this paper is to consider the analogue of the above theorem of C. Fefferman and E. M. Stein, i.e. the interpolation theorem between B_0^p and BMO, by means of the properties of the sharp function $f^\sharp$ and the duality that the space CMO^p is the dual space to $HA^{p'}$, $1/p + 1/p' = 1$.

1 Introduction In [B], A. Beurling showed that

$$B^p = \left\{ f \in L^p_{loc}(\mathbf{R}^n) : \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x)|^p dx \right)^{1/p} < \infty \right\},$$

where $1 < p < \infty$ and where $B(0,R)$ is the open ball in $\mathbf{R}^n$, having center 0 and radius $R > 0$, is the dual of the so-called Beurling algebra $A^{p'}$, $1/p + 1/p' = 1$. Also, Y. Chen and K. Lau [CL] and J. Garcia-Cuerva [G] defined the spaces CMO^p, which are similar to John-Nirenberg's BMO, and developed the H^1-theory analogue concerning the Hardy spaces HA^p associted with A^p, $1 < p < \infty$, on $\mathbf{R}^1$ and $\mathbf{R}^n$, respectively. In particular, the 'grand maximal function' characterization, the atomic decomposition, and HA^p-$CMO^{p'}$ duality corresponding to Fefferman-Stein's H^1-BMO duality were obtained.

By regarding B^p as an L^p analogue, K. Matsuoka calculated the interpolation space between B^p-spaces and also the related interpolation spaces, using the complex method in [M_1] and the real method in [M_2]. His results are, e.g.,

$$(B^{p_0}, B^{p_1})_{[\theta]} = (B^{p_0}, B^{p_1})^{[\theta]} = B^p \quad \text{(equal norms)}$$

and

$$(B^{p_0}, B^{p_1})_{\theta,p} = B^p \quad \text{(equivalent norms)},$$

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where $1 < p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ (cf. [M$_3$]). On the other hand, C. Fefferman and E. M. Stein [FS] gave the interpolation space between L^p and BMO, i.e.

$$(L^{p_0}, BMO)_{[\theta]} = L^p,$$

where $1 < p_0 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0$ (cf. [J]). They deduced this result from the properties of the sharp function $f^\sharp$.

In this paper, we will show the interpolation theorem between the subspace B_0^p of B^p and BMO, which is the analogue of the interpolation theorem between L^p and BMO obtained by C. Fefferman and E. M. Stein, using the sharp function $f^\sharp$ and HA^p-$CMO^{p'}$ duality.

2 Preliminaries

First, we will recall the definitions of the space B^p, the subspace B_0^p of B^p and the Beurling algebra A^p (see [CL] and [G]).

Definition 2.1 *Let $1 < p < \infty$, and let*

$$B^p = \left\{ f \in L^p_{loc}(\mathbf{R}^n) : \|f\|_{B^p} = \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x)|^p dx \right)^{1/p} < \infty \right\}, \tag{2.1}$$

where $B(0,R)$ is the open ball in $\mathbf{R}^n$, having center 0 and radius $R > 0$, and

$$B_0^p = \left\{ f \in B^p : \lim_{R \to \infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x)|^p dx = 0 \right\}. \tag{2.2}$$

Also let

$$A^p = \left\{ f : \|f\|_{A^p} = \inf_{\omega \in \Omega} \left(\int_{\mathbf{R}^n} |f(x)|^p \omega(x)^{-(p-1)} dx \right)^{1/p} < \infty \right\}, \tag{2.3}$$

where Ω is the class of functions ω on $\mathbf{R}^n$ such that ω's are positive, radial, nonincreasing with respect to $|x|$, and

$$\omega(0) + \int_{\mathbf{R}^n} \omega(x) dx = 1.$$

Here we note that B^p, B_0^p and A^p are Banach spaces, and that the space $C_c^\infty(\mathbf{R}^n)$ of those C^∞ functions having compact support on $\mathbf{R}^n$ is dense in B_0^p and A^p for $1 < p < \infty$ (see Proposition 1.3 of [G]). Note also that

$$L^1 \cap L^{p_1} \supset A^{p_1} \supset A^{p_2} \qquad \text{and} \qquad B^{p_1} \supset B^{p_2} \supset L^\infty \tag{2.4}$$

if $1 < p_1 < p_2 < \infty$.

The following result is a basic duality theorem (see [B], [CL] and [G] for details).

Theorem 2.2 *Let $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then,*

$$(A^p)^* = B^{p'}, \tag{2.5}$$

and the duality is given by

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f(x)g(x)dx \quad \left(f \in A^p, \quad g \in B^{p'}\right). \tag{2.6}$$

And also

$$(B_0^p)^* = A^{p'}. \tag{2.7}$$

Next, we state the definitions of the function of central mean oscillation of order p and the Hardy space associated to A^p, which are due to [CL] and [G].

Definition 2.3 *Let $1 < p < \infty$. A function $f \in L^p_{loc}(\mathbf{R}^n)$ will be said to belong to a class of functions of central mean oscillation of order p, CMO^p, if*

$$\|f\|_{CMO^p} = \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x) - m_R(f)|^p dx \right)^{1/p} < \infty, \tag{2.8}$$

where $B(0,R)$ is the open ball in $\mathbf{R}^n$, having center 0 and radius $R > 0$, and

$$m_R(f) = \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x)dx. \tag{2.9}$$

Then, by identifying functions which differ by a constant almost everywhere, it follows that $CMO^p \supset B^p \supset B_0^p$, which inclusions are proper, and that CMO^p is a Banach space. Moreover,

$$CMO^{p_1} \supset CMO^{p_2} \tag{2.10}$$

if $1 < p_1 < p_2 < \infty$.

Definition 2.4 *For $1 < p < \infty$, we shall define the Hardy space HA^p associated to A^p and the norm $\|\cdot\|_{HA^p}$ as*

$$HA^p = \{f \in A^p : f \text{ real}, \quad f^* \in A^p\}, \tag{2.11}$$

where f^* is the nontangential maximal function of the Poisson integral of f, i.e. for every $x \in \mathbf{R}^n$,

$$\begin{aligned} f^*(x) &= \sup_{|y-x|<t} |(f * P_t)(y)| \\ &= \sup_{|y-x|<t} \left| c_n \int_{\mathbf{R}^n} f(y-x') \frac{t}{(t^2 + |x'|^2)^{(n+1)/2}} dx' \right|, \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}, \end{aligned} \tag{2.12}$$

and

$$\|f\|_{HA^p} = \|f^*\|_{A^p}, \tag{2.13}$$

respectively.

Then, with this norm, HA^p becomes a Banach space. Moreover,

$$H^1 \cap A^{p_1} \supset HA^{p_1} \supset HA^{p_2} \tag{2.14}$$

if $1 < p_1 < p_2 < \infty$.

In the following, the theorem describes the duality corresponding to Fefferman-Stein's H^1-BMO duality, adapted to HA^p-space (see [CL] and [G]).

Theorem 2.5 *Let $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then,*

$$(HA^p)^* = CMO^{p'}, \tag{2.15}$$

and the duality is given by

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f(x)g(x)dx \qquad \left(f \in HA^p, \quad g \in CMO^{p'}\right). \tag{2.16}$$

In the remainder of this section, we make some comments about the Hardy-Littlewood maximal function and the grand maximal function.

Definition 2.6 *For any function f on $\mathbf{R}^n$, the Hardy-Littlewood maximal function Mf is defined by*

$$(Mf)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|dy \qquad (x \in \mathbf{R}^n), \tag{2.17}$$

where the supremum is taken over all open balls $B \subset \mathbf{R}^n$ containing x.

Then, the following maximal theorem, which is the analogue of the Hardy-Littlewood maximal theorem, was shown in [CL] and [G].

Theorem 2.7 *Suppose $1 < p < \infty$. If $f \in B^p$, then $Mf \in B^p$ and*

$$\|Mf\|_{B^p} \le C_p \|f\|_{B^p}, \tag{2.18}$$

where C_p is a constant depending only on n and p.

Definition 2.8 *Let N be a positive integer, and let*

$$\mathcal{A}_N = \left\{ \phi \in \mathcal{S}(\mathbf{R}^n) : \sup_{|\alpha| \le N, |\beta| \le N} \|\phi\|_{\alpha,\beta} = \sup_{|\alpha| \le N, |\beta| \le N} \left\{ \sup_{x \in \mathbf{R}^n} \left| x^\alpha \partial_x^\beta \phi(x)\right| \right\} \le 1 \right\}, \tag{2.19}$$

where $\mathcal{S}(\mathbf{R}^n)$ is the Schwarz class and α and β are n-tuples of natural numbers. Then, for any function f on $\mathbf{R}^n$, the grand maximal function $\mathcal{M}_N f$ is defined by

$$(\mathcal{M}_N f)(x) = \sup_{\phi \in \mathcal{A}_N} \left\{ \sup_{|y-x|<t} |(f * \phi_t)(y)| \right\} \qquad (x \in \mathbf{R}^n), \tag{2.20}$$

where $\phi_t(x) = t^{-n}\phi(x/t)$, $t > 0$. If N is sufficiently large and then N is fixed, we use the notation $\mathcal{M}$ to stand for $\mathcal{M}_N$.

Then, in [G], the following characterization of HA^p was proved.

Theorem 2.9 *Let $1 < p < \infty$ and f be a real function on $\mathbf{R}^n$. Then, the following conditions are equivalent:*

(i) $f \in HA^p$;

(ii) $\mathcal{M}f \in A^p$.

Besides,

$$\|f\|_{HA^p} \approx \|\mathcal{M}f\|_{A^p}. \tag{2.21}$$

3 The sharp function In this section, we will consider the property of the sharp function which enables us to interpolate between B_0^p and BMO.

Definition 3.1 *For any $f \in L^1_{loc}(\mathbf{R}^n)$, let*

$$f_B = \frac{1}{|B|}\int_B f(y)dy \tag{3.1}$$

be the mean value of f in a open ball $B \subset \mathbf{R}^n$. Then, the sharp function $f^\sharp$ is defined by

$$f^\sharp(x) = \sup_{x \in B} \frac{1}{|B|}\int_B |f(y) - f_B|dy \qquad (x \in \mathbf{R}^n), \tag{3.2}$$

where the supremum is taken over all open balls $B \subset \mathbf{R}^n$ containing x.

Then, applying Theorem 2.7, we obviously get the following theorem (see [M_4]).

Theorem 3.2 *Suppose $1 < p < \infty$. If $f \in B^p$, then $f^\sharp \in B^p$ and*

$$\|f^\sharp\|_{B^p} \leq C_p\|f\|_{B^p}, \tag{3.3}$$

where C_p is a constant depending only on n and p.

Combining the sharp function with the grand maximal function, the following duality inequality is obtained (see [St, p. 147]).

Proposition 3.3 *The duality inequality*

$$\left|\int_{\mathbf{R}^n} f(x)g(x)dx\right| \leq c\int_{\mathbf{R}^n} f^\sharp(x)(\mathcal{M}g)(x)dx \tag{3.4}$$

holds whenever $g \in H^1$ and f is bounded.

As a consequence of Proposition 3.3, we can now prove the following inequality (cf. [M_4]).

Theorem 3.4 *Suppose $1 < p < \infty$. If $f \in B_0^{p_0}$, for some $1 < p_0 < p$, and $f^\sharp \in B^p$, then $f \in CMO^p$ and*

$$\|f\|_{CMO^p} \le C_p \|f^\sharp\|_{B^p}, \tag{3.5}$$

where C_p is a constant depending only on n and p.

proof. The proof of this theorem is similar to that of Theorem 2 of [St, p. 148].

Given $f \in B_0^{p_0}$, there exists a sequence $\{f_k\} \subset C_c^\infty(\mathbf{R}^n)$ such that $f_k \to f$ in B^{p_0} norm. Hence, the maximal theorem 2.7 shows that ${f_k}^\sharp \to f^\sharp$ in B^{p_0} norm. Then, using Proposition 3.3 and (2.14), for any $g \in HA^{{p_0}'}$, where $1/p_0 + 1/{p_0}' = 1$,

$$\left| \int_{\mathbf{R}^n} f_k(x) g(x) dx \right| \le c \int_{\mathbf{R}^n} {f_k}^\sharp(x) (\mathcal{M}g)(x) dx.$$

Further, by applying Theorem 2.2 and (2.21), a passage to the limit gives

$$\left| \int_{\mathbf{R}^n} f(x) g(x) dx \right| \le c \int_{\mathbf{R}^n} f^\sharp(x) (\mathcal{M}g)(x) dx,$$

whenever $f \in B_0^{p_0}$ and $g \in HA^{{p_0}'}$. Therefore, it follows from (2.14), Theorem 2.2 and (2.21) that

$$\left| \int_{\mathbf{R}^n} f(x) g(x) dx \right| \le c_{p'} \|f^\sharp\|_{B^p} \cdot \|g\|_{HA^{p'}},$$

where $1/p + 1/p' = 1$. Thus, by the Hahn-Banach theorem and Theorem 2.5, we obtain

$$\|f\|_{CMO^p} = \sup_{\|g\|_{HA^{p'}} \le 1} \left| \int_{\mathbf{R}^n} f(x) g(x) dx \right| \le C_p \|f^\sharp\|_{B^p}.$$

This completes the proof.

4 Interpolation theorem

We first state the real interpolation space between B^p and L^∞, which was obtained in [M$_2$].

Theorem 4.1 *Suppose $1 < p_0 < \infty$ and $0 < \theta < 1$. Then*

$$(B^{p_0}, L^\infty)_{\theta,p} = B^p \quad \textit{(equivalent norms)}, \tag{4.1}$$

where $1/p = (1-\theta)/p_0$.

Next, we note that this theorem implies the following interpolation theorem (Corollary 4.2).

Corollary 4.2 *Suppose $1 < p_0 < \infty$, and let T be a sublinear operator such that*

$$T : B^{p_0} \to B^{p_0}$$

and

$$T : L^\infty \to L^\infty$$

boundedly. Then, for every p *with* $p_0 < p < \infty$*,*

$$T : B^p \to B^p$$

boundedly.

We are now in a position to prove the following inerpolation theorem between B_0^p and BMO. This theorem is the analogous result of the interpolation theorem between L^p and BMO due to C. Fefferman and E. M. Stein, which is an extension of the Marcinkiewicz interpolation theorem (see Theorem 3.7 of Chapter II of [GR]).

Theorem 4.3 *Suppose* $1 < p_0 < \infty$*, and let* T *be a linear operator such that*

$$T : B^{p_0} \to B_0^{p_0}$$

and

$$T : L^\infty \to BMO$$

boundedly. Then, for every p *with* $p_0 < p < \infty$*,*

$$T : B^p \to CMO^p$$

boundedly.

proof. Considering the sublinear operator $f \to (Tf)^\sharp$, it follows from Theorem 3.2 and the definition of BMO that this is bounded in B^{p_0} and also in L^∞. Hence, by Corollary 4.2, we have that for every p with $p_0 < p < \infty$, it is bounded in B^p.

Now, for every p with $p_0 < p < \infty$, let $f \in B^p$. Then, in view of the assumption and (2.4), $Tf \in B_0^{p_0}$. Moreover, by applying the above assertion just proved, $(Tf)^\sharp \in B^p$. Thus, using Theorems 3.4 and 3.2, $Tf \in CMO^p$ and

$$\|Tf\|_{CMO^p} \le C_p \|(Tf)^\sharp\|_{B^p} \le C_p \|f\|_{B^p}.$$

This concludes the proof of Theorem 4.3.

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