WEAKLY CONTINUOUS, WEAKLY $\vartheta-$CONTINUOUS, SUPER-CONTINUOUS AND TOPOLOGIES ON FUNCTION SPACES *

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Abstract. In this paper we study the notions of weakly continuity, weakly $\vartheta-$continuity and super-continuity. We also define topologies on the sets of all forms of continuous functions mentioned above. These results generalize some basic results of R. Arens, D. Dugundji and A. Di Concilio (see [1], [5], [2], [3] and [8]).

1 Introduction

Let Y, Z be topological spaces and let f be a map of Y into Z. Then f is *$\vartheta-$continuous* (respectively, *super-continuous*) *at* $y \in Y$ if for every open neighbourhood V of $f(y)$ there exists an open neighbourhood U of y such that $f(Cl(U)) \subseteq Cl(V)$ (respectively, $f(Int(Cl(U))) \subseteq V$). (Let Y be a space, then by $Cl(A)$ (respectively, $Int(A)$) we denote the closure (respectively, the interior) of A in Y). The map f is *$\vartheta-$continuous* (respectively, *super-continuous*) on Y if it is $\vartheta-$continuous (respectively, super-continuous) at each point of Y. (See for example [6], [9] and [16]). In what follows by $\Theta(Y,Z)$ (respectively, by $SUC(Y,Z)$) we denote the set of all $\vartheta-$continuous (respectively, super-continuous) maps of Y into Z. If τ is a topology on the set $\Theta(Y,Z)$ (respectively, $SUC(Y,Z)$), then the corresponding topological space is denoted by $\Theta_\tau(Y,Z)$ (respectively, by $SUC_\tau(Y,Z)$).

A map f of a space Y into a space Z is called *weakly continuous* (respectively, *weakly $\vartheta-$continuous*) at $y \in Y$ if for every open neighbourhood V of $f(y)$ there exists an open neighbourhood U of y such that $f(U) \subseteq Cl(V)$ (respectively, $f(Int(Cl(U))) \subseteq Cl(V)$). The map f is *weakly continuous* (respectively, *weakly $\vartheta-$continuous*) on Y if it is weakly continuous (respectively, weakly $\vartheta-$continuous) at each point of Y. (See for example [15], [4] and [17]). In what follows by $WC(Y,Z)$ (respectively, by $W\Theta(Y,Z)$) we denote the set of all weakly continuous (respectively, of all weakly $\vartheta-$continuous) maps of Y into Z. If τ is a topology on the set $WC(Y,Z)$ (respectively, on the set $W\Theta(Y,Z)$) then the corresponding topological space is denoted by $WC_\tau(Y,Z)$ (respectively, by $W\Theta_\tau(Y,Z)$).

Obviously, by the above definitions the following implications hold:

super-continuity $\Rightarrow$ continuity $\Rightarrow$ $\vartheta-$continuity $\Rightarrow$ weakly $\vartheta-$continuity $\Rightarrow$ weakly continuity.

Let Y be a topological space. A point $y \in Y$ is in the *$\vartheta-$closure* of a subset A of the space Y, $y \in Cl_\vartheta(A)$, if each open subset V about y satisfies $A \cap Cl(V) \neq \emptyset$. A is *$\vartheta-$closed* if $Cl_\vartheta(A) = A$. If $f : Y \to Z$ is weakly continuous and $A \subseteq Z$ a $\vartheta-$closed subsset of Z, then $f^{-1}(A)$ is closed in Y. (See [17] and [13]).

Let X be a space and $F : X \times Y \to Z$ be a $\vartheta-$continuous map. Then by F_x, where $x \in X$, we denote the $\vartheta-$continuous map of Y into Z, for which $F_x(y) = F(x,y)$, for every

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$y \in Y$. By $\widehat{F}$ we denote the map of X into the set $\Theta(Y,Z)$, for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let X be a space and $F : X \times Y \to Z$ be a continuous map. then by F_x, where $x \in X$, we denote the continuous map of Y into Z, for which $F_x(y) = F(x,y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $C(Y,Z)$ (by $C(Y,Z)$ we denote the set of all continuous maps of Y into Z) for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let G be a map of the space X into the set $\Theta(Y,Z)$ or into the set $C(Y,Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z, for which $\widetilde{G}(x,y) = G(x)(y)$, for every $(x,y) \in X \times Y$.

A topology τ on $C(Y,Z)$ (respectively, on $\Theta(Y,Z)$) is called *splitting* (respectively, $\vartheta-$*splitting*) if for every space X, the continuity (respectively, the $\vartheta-$continuity) of a map $F : X \times Y \to Z$ implies the continuity (respectively, the $\vartheta-$continuity) of the map $\widehat{F} : X \to C_\tau(Y,Z)$ (respectively, of the map $\widehat{F} : X \to \Theta_\tau(Y,Z)$). (See [5], [1], [14] and [2]).

A topology τ on $C(Y,Z)$ (respectively, on $\Theta(Y,Z)$) is called *jointly continuous* (respectively, $\vartheta-$*jointly continuous*) if for every space X, the continuity (respectively, the $\vartheta-$continuity) of a map $G : X \to C_\tau(Y,Z)$ (respectively, of a map $G : X \to \Theta_\tau(Y,Z)$) implies the continuity (respectively, the $\vartheta-$continuity) of the map $\widetilde{G} : X \times Y \to Z$. (See [5], [1] and [2]).

Let X be a set. A *net* in X is a map $S : \Lambda \to X$ of a directed set Λ into X. The net S is also denoted by $\{s_\lambda,\ \lambda \in \Lambda\}$, where $s_\lambda = S(\lambda)$.

Let $\mathcal{P}(Y)$ be the set of all subsets of a space Y. If Λ is a directed set, then by $\overline{\lim_{\Lambda}}(A_\lambda)$, where $A_\lambda \subseteq Y$, we denote the *upper limit* of the net $\{A_\lambda, \lambda \in \Lambda\}$ in $\mathcal{P}(Y)$, that is, the set of all points y of Y such that for every $\lambda_0 \in \Lambda$ and for every open neighbourhood U of y in Y there exists an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap U \neq \emptyset$. (For the notion of upper limit and its applications see for example [1] and [10]).

A net $\{y_\lambda : \lambda \in \Lambda\}$ in a space Y *converges* (respectively, $\vartheta-$*converges*) to y and write $y_\lambda \to y$ (respectively, $y_\lambda \xrightarrow{\vartheta} y$) if for each neighbourhood U of y there is some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $y_\lambda \in U$ (respectively, $y_\lambda \in Cl(U)$). Evidently any net which converges to y $\vartheta-$converges to y. (See [11] and [2]).

A function f is continuous (respectively, $\vartheta-$continuous) at $y \in Y$ if and only if whenever $y_\lambda \to y$ (respectively, $y_\lambda \xrightarrow{\vartheta} y$) in Y, then $f(y_\lambda) \to f(y)$ (respectively, $f(y_\lambda) \xrightarrow{\vartheta} f(y)$) in Z. (See [11] and [2]).

A net $\{f_\mu, \mu \in M\}$ in $C(Y,Z)$ (respectively, in $\Theta(Y,Z)$) *continuously converges* (respectively, $\vartheta-$*continuously converges*) to $f \in \Theta(Y,Z)$ if for any net $\{y_\lambda : \lambda \in \Lambda\}$ in Y such that $y_\lambda \to y$ (respectively, $y_\lambda \xrightarrow{\vartheta} y$) the net $\{f_\mu(\lambda), (\mu,\lambda) \in M \times \Lambda\}$ converges (respectively, $\vartheta-$converges) to $f(y)$ in Z, that is $f_\mu(y_\lambda) \to\ f(y)$ (respectively, $f_\mu(y_\lambda) \xrightarrow{\vartheta} f(y)$) in Z. (See [7], [12], [1] and [2]).

A net $\{f_\mu, \mu \in M\}$ in $C(Y,Z)$ (respectively, in $\Theta(Y,Z)$) continuously converges (respectively, $\vartheta-$continuously converges) to $f \in C(Y,Z)$ (respectively, to $f \in \Theta(Y,Z)$) if for any $y \in Y$ and any neighbourhood V of $f(y)$ there is a μ_0 and an open neighbourhood U of y such that $f_\mu(U) \subseteq V$ (respectively, $f_\mu(Cl(U)) \subseteq Cl(V)$), for every $\mu \in M$, $\mu \geq \mu_0$. (See [1] and [2]).

By $\mathcal{C}^*$ (respectively, $\mathcal{C}^*_\vartheta$) we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $C(Y,Z)$ (respectively, in $\Theta(Y,Z)$) which continuously converges (respectively, $\vartheta-$continuously converges) to $f \in C(Y,Z)$ (respectively, to $f \in \Theta(Y,Z)$). If τ is a topology on $C(Y,Z)$ (respectively, on $\Theta(Y,Z)$) then by $\mathcal{C}(\tau)$ (respectively, by $(\mathcal{C}(\tau))_\vartheta$) we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $C(Y,Z)$ (respectively, in $\Theta(Y,Z)$) which converges (respectively, $\vartheta-$converges) to $f \in C(Y,Z)$ (respectively, to

$f \in \Theta(Y,Z)$) in the topology τ.

The following criteria are given in [1] and [2]:

(1) A topology τ on $C(Y,Z)$ (respectively, on $\Theta(Y,Z)$) is splitting (respectively, $\vartheta-$splitting) if and only if $\mathcal{C}^* \subseteq \mathcal{C}(\tau)$ (respectively, $\mathcal{C}^*_\vartheta \subseteq (\mathcal{C}(\tau))_\vartheta$).

(2) A topology τ on $C(Y,Z)$ (respectively, on $\Theta(Y,Z)$) is jointly continuous (respectively, $\vartheta-$jointly continuous) if and only if $\mathcal{C}(\tau) \subseteq \mathcal{C}^*$. (respectively, $(\mathcal{C}(\tau))_\vartheta \subseteq \mathcal{C}^*_\vartheta$).

Throughout this paper the word space means "topological space".

2 Weakly $\vartheta-$continuous functions

1. DEFINITION. A net $\{y_\lambda, \lambda \in \Lambda\}$ in a topological space Y *weakly $\vartheta-$converges to* $y \in Y$ and write $y_\lambda \overset{w-\vartheta}{\to} y$ if for every neighbourhood U of y there is some $\lambda_0 \in \Lambda$ such that $y_\lambda \in Int(Cl(U))$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Evidently any net in Y which converges to $y \in Y$ weakly $\vartheta-$converges to y. Also any net which weakly $\vartheta-$converges to y $\vartheta-$converges to y.

2. THEOREM. A map f of a space Y into a space Z is weakly $\vartheta-$continuous at $y \in Y$ if and only if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which weakly $\vartheta-$converges to y, that is $y_\lambda \overset{w-\vartheta}{\to} y$ we have that the net $\{f(y_\lambda), \lambda \in \Lambda\}$ in Z $\vartheta-$converges to $f(y)$, that is $f(y_\lambda) \overset{\vartheta}{\to} f(y)$.

PROOF. Let us suppose that f is weakly $\vartheta-$continuous at $y \in Y$ and let $\{y_\lambda, \lambda \in \Lambda\}$ be a net in Y such that $y_\lambda \overset{w-\vartheta}{\to} y$. Then for every open neighbourhood V of $f(y)$ in Z there exists an open neighbourhood U of y in Y such that $f(Int(Cl(U))) \subseteq Cl(V)$. Since $y_\lambda \overset{w-\vartheta}{\to} y$. There exists an element $\lambda_0 \in \Lambda$ such that $y_\lambda \in Int(Cl(U))$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Thus, $f(y_\lambda) \in Cl(V)$, for every $\lambda \geq \lambda_0$, $\lambda \in \Lambda$ and therefore the net $\{f(y_\lambda), \lambda \in \Lambda\}$ in Z $\vartheta-$converges to $f(y)$, that is $f(y_\lambda) \overset{\vartheta}{\to} f(y)$.

Conversely, if the map f is not weakly $\vartheta-$continuous at $y \in Y$, then for some open neighbourhood V of $f(y)$ we have:

$$f(Int(Cl(U))) \nsubseteq Cl(V),$$

for every open neighbourhood U of y in Y. Thus, for every open neighbourhood U of y we can find $y_U \in Int(Cl(U))$ such that $f(y_U) \notin Cl(V)$. Let $\mathcal{N}(y)$ be the set of all open neighbourhoods U of y in Y. The set $\mathcal{N}(y)$ with the relation of inverse inclusion (that is, $U_1 \leq U_2$ if and only if $U_2 \subseteq U_1$) form a directed set. Clearly, the net $\{y_U, U \in \mathcal{N}(y)\}$ weakly $\vartheta-$converges to y in Y but the $\{f(y_U), U \in \mathcal{N}(y)\}$ does not $\vartheta-$converge to $f(y)$ in Z. Hence the map f is weakly $\vartheta-$continuous at $y \in Y$.

3. DEFINITION. A net $\{f_\mu, \mu \in M\}$ in $W\Theta(Y,Z)$ *weakly $\vartheta-$continuously converges to* $f \in W\Theta(Y,Z)$ if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which weakly $\vartheta-$converges to $y \in Y$ we have that the net $\{f_\mu(y_\lambda), (\lambda,\mu) \in \Lambda \times M\}$ $\vartheta-$converges to $f(y)$ in Z.

4. THEOREM. A net $\{f_\mu, \mu \in M\}$ in $W\Theta(Y,Z)$ weakly $\vartheta-$continuously converges to $f \in W\Theta(Y,Z)$ if and only if for every $y \in Y$ and for every open neighbourhood V of $f(y)$ in Z there exist an element $\mu_0 \in M$ and an open neighbourhood U of y in Y such that

$$f_\mu(Int(Cl(U)))) \subseteq Cl(V),$$

for every $\mu \geq \mu_0$, $\mu \in M$.

PROOF. Let $y \in Y$ and let V be an open neighbourhood of $f(y)$ in Z such that for every $\mu \in M$ and for every open neighbourhood U of $y \in Y$ there exists $\mu' \geq \mu$, $\mu' \in M$ such that

$$f_{\mu'}(Int(Cl(U)))) \nsubseteq Cl(V).$$

Then for every open neighbourhood U of y in Y we can choose a point $y_U \in Int(Cl(U))$ such that $f_{\mu'}(y_U) \notin Cl(V)$. Clearly, the net $\{y_U, U \in \mathcal{N}(y)\}$ weakly $\vartheta-$converges to y but the net $\{f_\mu(y_U), (U,\mu) \in \mathcal{N}(y) \times M\}$ does not $\vartheta-$converge to $f(y)$ in Z.

Conversely, let $\{y_\lambda, \lambda \in \Lambda\}$ be a net in $W\Theta(Y,Z)$ which weakly $\vartheta-$converges to y in Y and let V be an arbitrary open neighbourhood of $f(y)$ in Z. By assumption there exist an open neighbourhood U of y in Y and an element $\mu_0 \in M$ such that $f_\mu(Int(Cl(U))) \subseteq Cl(V)$, for every $\mu \geq \mu_0$, $\mu \in M$. Since the net $\{y_\lambda, \lambda \in \Lambda\}$ weakly $\vartheta-$converges to y in Y. There exists $\lambda_0 \in \Lambda$ such that $y_\lambda \in Int(Cl(U))$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Let $(\lambda_0, \mu_0) \in \Lambda \times M$. Then for every $(\lambda, \mu) \in \Lambda \times M$, $(\lambda,\mu) \geq (\lambda_0, \mu_0)$ we have $f_\mu(y_\lambda) \in f_\mu(Int(Cl(U))) \subseteq Cl(V)$. Thus the net $\{f_\mu(y_\lambda), (\lambda,\mu) \in \Lambda \times M\}$ $\vartheta-$converges to $f(y)$ in Z.

5. DEFINITION. Let $\mathcal{P}(Y)$ be the set of all subsets of a space Y. If Λ is a directed set, then by $w - \vartheta - \overline{\lim_{\Lambda}}(A_\lambda)$, where $A_\lambda \subseteq Y$, we denote the *weakly $\vartheta-$upper limit* of the net $\{A_\lambda, \lambda \in \Lambda\}$ in $\mathcal{P}(Y)$, that is, the set of all points y of Y such that for every $\lambda_0 \in \Lambda$ and for every open neighbourhood U of y in Y there exists an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap Int(Cl(U)) \neq \emptyset$.

6. THEOREM. If a net $\{f_\lambda, \lambda \in \Lambda\}$ in $W\Theta(Y,Z)$ weakly $\vartheta-$continuously converges to $f \in W\Theta(Y,Z)$, then

$$w - \vartheta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K),$$

for every $\vartheta-$closed subset K of Z.

PROOF. Let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $W\Theta(Y,Z)$, which weakly $\vartheta-$continuously converges to f and let K be an arbitrary $\vartheta-$closed subset of Z. Let $y \in w - \vartheta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$ and let W be an arbitrary open neighbourhood of $f(y)$ in Z. Since the net $\{f_\lambda,\ \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to f, there exist an open neighbourhood V of y in Y and an element $\lambda_0 \in \Lambda$ such that $f_\lambda(Int(Cl(V))) \subseteq Cl(W)$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. (See Theorem 4). On the other hand, there exists an element $\lambda \in \Lambda$, $\lambda \geq \lambda_0$ such that $Int(Cl(V)) \cap f_\lambda^{-1}(K) \neq \emptyset$. Hence, $f_\lambda(Int(Cl(V))) \cap K \subseteq Cl(W) \cap K \neq \emptyset$. This means that $f(y) \in \mathrm{Cl}_\vartheta(K) = K$. Thus, $y \in f^{-1}(K)$.

7. THEOREM. Let $\{f_\lambda, \lambda \in \Lambda\}$ be a net in $W\Theta(Y,Z)$ such that

$$w - \vartheta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K), \tag{1}$$

for every closed subset K of Z. Then the net $\{f_\lambda, \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to $f \in W\Theta(Y,Z)$.

PROOF. Let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $W\Theta(Y,Z)$ and $f \in W\Theta(Y,Z)$ such that relation (1) holds for every closed subset K of Z. We prove that the net $\{f_\lambda,\ \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to f. Let $y \in Y$ and W be an open neighbourhood of $f(y)$ in Z. Since $y \notin f^{-1}(K)$, where $K = Z \setminus W$ we have $y \notin w - \vartheta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$. This means that there exists an element $\lambda_0 \in \Lambda$ and an open neighbourhood V of y in Y such that $f_\lambda^{-1}(K) \cap Int(Cl(V)) = \emptyset$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Then we have $Int(Cl(V)) \subseteq Y \setminus f_\lambda^{-1}(K) = f_\lambda^{-1}(Z \setminus K) \subseteq f_\lambda^{-1}(W) \subseteq f_\lambda^{-1}(Cl(W))$ and therefore $f_\lambda(Int(Cl(V))) \subseteq Cl(W)$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$, that is the net $\{f_\lambda,\ \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to f.

8. THEOREM. The following propositions are true:

(1) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $W\Theta(Y,Z)$ such that $f_\lambda = f$, for every $\lambda \in \Lambda$, then the net $\{f_\lambda, \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to $f \in W\Theta(Y,Z)$.

(2) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $W\Theta(Y,Z)$ which weakly $\vartheta-$continuously converges to $f \in W\Theta(Y,Z)$ and $\{g_\mu, \mu \in M\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, then the net $\{g_\mu, \mu \in M\}$ weakly $\vartheta-$continuously converges to f.

PROOF. We shall verify only (2). Let $y \in Y$ and V be an open neighbourhood of $f(y)$ in Z. Then, there is $\lambda_0 \in \Lambda$ and an open neighbourhood U of y such that $f_\lambda(Int(Cl(U))) \subseteq Cl(V)$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

Since $\{g_\mu, \mu \in M\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, there is a map $N : M \to \Lambda$ such that:

(i) $g_\mu = f_{N(\mu)}$ and

(ii) for the element $\lambda_0 \in \Lambda$ there is $\mu_0 \in M$ such that if $\mu \geq \mu_0$, $\mu \in M$, then $N(\mu) \geq \lambda_0$.

By the above we have:

$$g_\mu(Int(Cl(U))) = f_{N(\mu)}(Int(Cl(U))) \subseteq Cl(V),$$

for every $\mu \geq \mu_0$, $\mu \in M$.

Thus, the net $\{g_\mu, \mu \in M\}$ weakly $\vartheta-$continuously converges to f.

3 Weakly continuous functions

1. THEOREM. A map f of a space Y into a space Z is weakly continuous at $y \in Y$ if and only if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which converges to y, that is $y_\lambda \to y$ we have that the net $\{f(y_\lambda), \lambda \in \Lambda\}$ in Z $\vartheta-$converges to $f(y)$, that is $f(y_\lambda) \overset{\vartheta}{\to} f(y)$.

The proof of this theorem is similar to the proof of Theorem 2.I.

2. DEFINITION. A net $\{f_\mu, \mu \in M\}$ in $WC(Y,Z)$ *weakly continuously converges to* $f \in WC(Y,Z)$ if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which converges to $y \in Y$ we have that the net $\{f_\mu(y_\lambda), (\lambda,\mu) \in \Lambda \times M\}$ $\vartheta-$converges to $f(y)$ in Z.

3. THEOREM. A net $\{f_\mu, \mu \in M\}$ in $WC(Y,Z)$ weakly continuously converges to $f \in WC(Y,Z)$ if and only if for every $y \in Y$ and for every open neighbourhood V of $f(y)$ in Z there exist an element $\mu_0 \in M$ and an open neighbourhood U of y in Y such that

$$f_\mu(U) \subseteq Cl(V),$$

for every $\mu \geq \mu_0$, $\mu \in M$.

The proof of this theorem is similar to the proof of Theorem 4.I.

4. THEOREM. If a net $\{f_\lambda, \lambda \in \Lambda\}$ in $WC(Y,Z)$ weakly continuously converges to $f \in WC(Y,Z)$, then

$$\overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K),$$

for every $\vartheta-$closed subset K of Z.

The proof of this theorem is similar to the proof of Theorem 6.I.

5. THEOREM. Let $\{f_\lambda, \lambda \in \Lambda\}$ be a net in $WC(Y,Z)$ such that

$$\overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K), \tag{1}$$

for every closed subset K of Z. Then the net $\{f_\lambda, \lambda \in \Lambda\}$ weakly continuously converges to $f \in WC(Y,Z)$.

The proof of this theorem is similar to the proof of Theorem 7.I.

6. THEOREM. The following propositions are true:

(1) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $WC(Y,Z)$ such that $f_\lambda = f$, for every $\lambda \in \Lambda$, then the $\{f_\lambda, \lambda \in \Lambda\}$ weakly continuously converges to $f \in WC(Y,Z)$.

(2) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $WC(Y,Z)$ which weakly continuously converges to $f \in WC(Y,Z)$ and $\{g_\mu, \mu \in M\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, then the net $\{g_\mu, \mu \in M\}$ weakly continuously converges to f.

The proof of this theorem is similar to the proof of Theorem 8.I.

4 Super continuous functions

1. THEOREM. A map f of a space Y into a space Z is super-continuous at $y \in Y$ if and only if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which weakly $\vartheta-$converges to y, that is $y_\lambda \overset{w-\vartheta}{\to} y$ we have that the net $\{f(y_\lambda), \lambda \in \Lambda\}$ in Z converges to $f(y)$, that is $f(y_\lambda) \to f(y)$.

The proof of this theorem is similar to the proof of Theorem 2.I.

2. DEFINITION. A net $\{f_\mu, \mu \in M\}$ in $SUC(Y,Z)$ *super continuously converges to* $f \in SUC(Y,Z)$ if for every net $\{y_\lambda, \lambda \in \Lambda\}$ in Y which weakly $\vartheta-$converges to $y \in Y$ we have that the net $\{f_\mu(y_\lambda), (\lambda, \mu) \in \Lambda \times M\}$ converges to $f(y)$ in Z.

3. THEOREM. A net $\{f_\mu, \mu \in M\}$ in $SUC(Y,Z)$ super continuously converges to $f \in SUC(Y,Z)$ if and only if for every $y \in Y$ and for every open neighbourhood V of $f(y)$ in Z there exist an element $\mu_0 \in M$ and an open neighbourhood U of y in Y such that

$$f_\mu(Int(Cl(U))) \subseteq V,$$

for every $\mu \geq \mu_0$, $\mu \in M$.

The proof of this theorem is similar to the proof of theorem 4.I.

4. THEOREM. A net $\{f_\lambda, \lambda \in \Lambda\}$ in $SUC(Y,Z)$ super continuously converges to $f \in SUC(Y,Z)$ if and only if

$$w - \vartheta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subseteq f^{-1}(K),$$

for every closed subset K of Z. The proof of this theorem is similar to the proof of theorems 6.I and 7.I.

5. THEOREM. The following propositions are true:

(1) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $SUC(Y,Z)$ such that $f_\lambda = f$, for every $\lambda \in \Lambda$, then the $\{f_\lambda, \lambda \in \Lambda\}$ super continuously converges to $f \in SUC(Y,Z)$.

(2) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $SUC(Y,Z)$ which almost strongly $\vartheta-$continuously converges to $f \in AS\Theta(Y,Z)$ and $\{g_\mu, \mu \in M\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, then the net $\{g_\mu, \mu \in M\}$ super continuously converges to f.

The proof of this theorem is similar to the proof of Theorem 8.I.

5 Function spaces

1. NOTATIONS. Let X be a space and $F : X \times Y \to Z$ be a weakly continuous map (respectively, a weakly $\vartheta-$continuous map). By F_x, where $x \in X$, we denote the weakly continuous map (respectively, the weakly $\vartheta-$continuous map) of Y into Z, for which $F_x(y) = F(x,y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $WC(Y,Z)$ (respectively, into the set $W\Theta(Y,Z)$) for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let X be a space and $F : X \times Y \to Z$ be a super-continuous map. By F_x, where $x \in X$, we denote the super-continuous map of Y into Z, for which $F_x(y) = F(x, y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $SUC(Y, Z)$ for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let G be a map of the space X into the set $WC(Y, Z)$ or into the set $W\Theta(Y, Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z, for which $\widetilde{G}(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$.

Let G be a map of the space X into the set $SUC(Y, Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z, for which $\widetilde{G}(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$.

2. DEFINITIONS. A topology τ on $WC(Y, Z)$ (respectively, on $W\Theta(Y, Z)$) is called *weakly splitting* (respectively, *weakly $\vartheta-$splitting*) if for every X, the weak continuity (respectively, the weak $\vartheta-$continuity) of a map $F : X \times Y \to Z$ implies the weak continuity (respectively, the weak $\vartheta-$continuity) of the map $\widehat{F} : X \to WC_\tau(Y, Z)$ (respectively, of the map $\widehat{F} : X \to W\Theta_\tau(Y, Z)$).

A topology τ on $SUC(Y, Z)$ is called *super splitting* if for every X, the super-continuity of a map $F : X \times Y \to Z$ implies the super-continuity of the map $\widehat{F} : X \to SUC_\tau(Y, Z)$.

If in the above it is assumed that the space X belongs to a given family $\mathcal{A}$ of spaces, then the topology τ is called *weakly $\mathcal{A}-$splitting*, *weakly $\mathcal{A}-\vartheta-$splitting* and *super $\mathcal{A}-$splitting*, respectively. If $\mathcal{A} = \{X\}$, then instead of "super $\mathcal{A}-$splitting" we write "super $X-$splitting".

Obviously, if $\mathcal{A}$ is the family of all spaces, then the notions weakly $\mathcal{A}-$splitting, weakly $\mathcal{A} - \vartheta-$splitting and super $\mathcal{A}-$splitting coincide with the notions weakly splitting, weakly $\vartheta-$splitting and super splitting, respectively.

A topology τ on $WC(Y, Z)$ (respectively, on $W\Theta(Y, Z)$) is called *weakly jointly continuous* (respectively, *weakly $\vartheta-$jointly continuous*) if for every X, the weak continuity (respectively, the weak $\vartheta-$continuity) of a map $G : X \to WC_\tau(Y, Z)$ (respectively, of a map $G : X \to W\Theta_\tau(Y, Z)$) implies the weak continuity (respectively, the weak $\vartheta-$continuity) of the map $\widetilde{G} : X \times Y \to Z$.

A topology τ on $SUC(Y, Z))$ is called *super jointly continuous* if for every space X, the super-continuity of a map $G : X \to SUC_\tau(Y, Z)$ implies the super-continuity of the map $\widetilde{G} : X \times Y \to Z$.

If in the above it is assumed that the space X belongs to a given family $\mathcal{A}$ of spaces, then the topology τ is called *weakly $\mathcal{A}-$jointly continuous*, *weakly $\mathcal{A} - \vartheta-$jointly continuous* and *super $\mathcal{A}-$jointly continuous*, respectively. If $\mathcal{A} = \{X\}$, then instead of "super $\mathcal{A}-$jointly continuous" we write "super $X-$jointly continuous".

Obviously, if $\mathcal{A}$ is the family of all spaces, then the notions weakly $\mathcal{A}-$jointly continuous, weakly $\mathcal{A} - \vartheta-$jointly continuous and super $\mathcal{A}-$jointly continuous coincide with the notions weakly jointly continuous, weakly $\vartheta-$jointly continuous and super jointly continuous, respectively.

2.1. REMARK. Clearly, the above notions of weakly splitting and weakly jointly continuous does not coincide with the notions of weakly splitting and weakly jointly continuous which were defined in [18].

3. THEOREM. The following propositions are true:

(1) If a topology τ on $W\Theta(Y, Z)$ is weakly $\vartheta-$jointly continuous, then the evaluation map $e : W\Theta_\tau(Y, Z) \times Y \to Z$ defined by $e(f, y) = f(y)$ is weakly $\vartheta-$continuous.

(2) If a topology τ on $WC(Y, Z)$ is weakly jointly continuous, then the evaluation map $e : WC_\tau(Y, Z) \times Y \to Z$ defined by $e(f, y) = f(y)$ is weakly continuous.

PROOF. (1) Clearly, the identity map $G \equiv 1 : W\Theta_\tau(Y,Z) \to W\Theta_\tau(Y,Z)$ is weakly $\vartheta-$continuous. Since the topology τ is weakly $\vartheta-$jointly continuous. The map $\widetilde{G} \equiv e : W\Theta_\tau(Y,Z) \times Y \to Z$ is weakly $\vartheta-$continuous.

Similarly, we can prove the Proposition (2).

4. THEOREM. The following propositions are true:

(1) If the evaluation map map $e : W\Theta_\tau(Y,Z) \times X \to Z$ is $\vartheta-$continuous, then the topology τ is weakly $\vartheta-$jointly continuous.

(2) If the evaluation map map $e : WC_\tau(Y,Z) \times X \to Z$ is $\vartheta-$continuous, then the topology τ is weakly jointly continuous.

PROOF. (1) Let X be a space, $G : X \to W\Theta_\tau(Y,Z)$ be a weakly $\vartheta-$continuous map and $1 : Y \to Y$ be the identity map. Clearly, the map $G \times 1 : X \times Y \to W\Theta_\tau(Y,Z) \times Y$ is weakly $\vartheta-$continuous. Also the map $e : W\Theta_\tau(Y,Z) \times Y \to Z$ is $\vartheta-$continuous. It is not difficult prove that the map $e \circ (G \times 1) : X \times Y \to Z$ is weakly $\vartheta-$continuous and $\widetilde{G} = e \circ (G \times 1)$.

Similarly we can prove the Proposition (2).

5. THEOREM. On the set $SUC(Y,Z)$ there exists the greatest super splitting topology.

PROOF. Let T be the set of all super splitting topologies on the set $SUC(Y,Z)$. Let $\tau = \vee T$. We prove that τ is the greatest super splitting topology. It is sufficient to prove that τ is an super splitting topology. Let $F : X \times Y \to Z$ be a super-continuous map. We prove that the map $\widehat{F} : X \to SUC_\tau(Y,Z)$ is super-continuous. Let $x \in X$ and let U be an open neighbourhood of $\widehat{F}(x)$ in $SUC_\tau(Y,Z)$. Since $\tau = \vee T$ we have that $U \in \tau'$, $\tau' \in T$. Also, since $\widehat{F} : X \to SUC_{\tau'}(Y,Z)$ is super continuous there exists an open neighbourhood V of X such that $\widehat{F}(Int(Cl(V))) \subseteq U$. Thus the map $\widehat{F}$ is super continuous and the topology τ is super splitting.

5.1. COROLLARY. Let $\mathcal{A}$ be an arbitrary family of spaces. Then on the set $SUC(Y,Z)$ there exists the greatest super $\mathcal{A} - \vartheta-$splitting topology, which is denoted by $\tau_s(\mathcal{A})$.

6. THEOREM. The following propositions are true:

(1) Let $\mathcal{A}_i$, $i \in I$, be a family of spaces and let $\mathcal{A} = \bigcup\{\mathcal{A}_i : i \in I\}$. Then, $\tau_s(\mathcal{A}) = \bigcap\{\tau_s(\mathcal{A}_i) : i \in I\}$.

(2) Let $\mathcal{A}_i$, $i \in I$, be a family of spaces and let $\mathcal{A} = \bigcap\{\mathcal{A}_i : i \in I\}$. If $\mathcal{A} \neq \emptyset$, then $\vee\{\tau_s(\mathcal{A}_i) : i \in I\} \subseteq \tau_s(\mathcal{A})$.

The proof of this theorem is clear.

7. NOTATIONS. By $\mathcal{C}_w^*$ (respectively, $\mathcal{C}_{w-\vartheta}^*$) we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $WC(Y,Z)$ (respectively, in $W\Theta(Y,Z)$) which weakly continuously converges (respectively, weakly $\vartheta-$continuously converges) to $f \in WC(Y,Z)$ (respectively, to $f \in W\Theta(Y,Z)$). If τ is a topology on $WC(Y,Z)$ (respectively, on $W\Theta(Y,Z)$) then by $(\mathcal{C}(\tau))_w$ (respectively, by $(\mathcal{C}(\tau))_{w-\vartheta}$) we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $WC(Y,Z)$ (respectively, in $W\Theta(Y,Z)$) which $\vartheta-$converges (respectively, $\vartheta-$converges) to $f \in WC(Y,Z)$ (respectively, to $f \in W\Theta(Y,Z)$) in the topology τ.

By $\mathcal{C}_{s-c}^*$ we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $SUC(Y,Z)$ which super continuously converges to $f \in SUC(Y,Z)$. If τ is a topology on $SUC(Y,Z)$ then by $(\mathcal{C}(\tau))_{s-c}$ we denote the class of all pairs $(\{f_\lambda, \lambda \in \Lambda\}, f)$, where $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $SUC(Y,Z)$ which converges to $f \in SUC(Y,Z)$.

8. THEOREM. A topology τ on $W\Theta(Y,Z)$ is weakly $\vartheta-$splitting if and only if

$$\mathcal{C}^*_{w-\vartheta} \subseteq (\mathcal{C}(\tau))_{w-\vartheta}.$$

PROOF. Let τ be an weakly $\vartheta-$splitting topology on $W\Theta(Y,Z)$ and let $(\{f_\lambda, \lambda \in \Lambda\}, f) \in \mathcal{C}^*_{w-\vartheta}$. We prove that the net $\{f_\lambda, \lambda \in \Lambda\}$ $\vartheta-$converges to f in the topology τ. Indeed, we consider the set $X \equiv \Lambda \cup \{\infty\}$, where ∞ is a symbol such that $\infty \geq \lambda$, for every $\lambda \in \Lambda$. Then we topologize $X = \Lambda \cup \{\infty\}$ defining any singleton $\{\lambda\}$, $\lambda \in \Lambda$ to be open and neighbourhoods of ∞ the sets $\{\lambda \in X : \lambda \geq \lambda_0$ for some $\lambda_0 \in \Lambda\}$ (See [1] and [2]). Let $F : X \times Y \to Z$ be a map, for which $F(\lambda, y) = f_\lambda(y)$, $\lambda \neq \infty$ and $F(\infty, y) = f(y)$, for every $y \in Y$. The map F is weakly $\vartheta-$continuous. Obviously $\widehat{F}(\lambda) = f_\lambda$ and $\widehat{F}(\infty) = f$. Since the topology τ is weakly $\vartheta-$splitting, the map $\widehat{F} : X \to W\Theta_\tau(Y,Z)$ is weakly $\vartheta-$continuous.

By the weakly $\vartheta-$continuity of $\widehat{F}$ we have that for every open neighbourhood U of f in $W\Theta_\tau(Y,Z)$, there exists an open neighbourhood V of ∞ in X such that

$$\widehat{F}(Int(Cl(V))) \subseteq Cl(U).$$

By the definition of the topology on X, there exists an element $\lambda_0 \in \Lambda$ such that $\lambda \in V \subseteq Int(Cl(V))$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Hence $\widehat{F}(\lambda) = f_\lambda \in Cl(U)$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$, that is the net $\{f_\lambda, \lambda \in \Lambda\}$ $\vartheta-$converges to f in the topology τ. Thus $\mathcal{C}^*_{w-\vartheta} \subseteq (\mathcal{C}(\tau))_{w-\vartheta}$.

Conversely, let τ be a topology on $W\Theta(Y,Z)$ such that $\mathcal{C}^*_{w-\vartheta} \subseteq (\mathcal{C}(\tau))_{w-\vartheta}$. We prove that the topology τ is weakly $\vartheta-$splitting.

Let X be an arbitrary space and let $F : X \times Y \to Z$ be a weakly $\vartheta-$continuous map. Consider the map $\widehat{F} : X \to W\Theta_\tau(Y,Z)$. We must prove that the map $\widehat{F}$ is weakly $\vartheta-$continuous. Let $\{x_\lambda, \lambda \in \Lambda\}$ be a net in X which weakly $\vartheta-$converges to x. We prove that the net $\{\widehat{F}(x_\lambda), \lambda \in \Lambda\}$ $\vartheta-$converges to $\widehat{F}(x)$, that is $\widehat{F}(x_\lambda) \xrightarrow{\vartheta} \widehat{F}(x)$. Let $\{y_\mu, \mu \in M\}$ be a net in Y which weakly $\vartheta-$converges to y in Y. Since the map F is weakly $\vartheta-$continuous and the net $\{(x_\lambda, y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ of $X \times Y$ weakly $\vartheta-$converges to (x,y) in $X \times Y$ we have $F(x_\lambda, y_\mu) \xrightarrow{\vartheta} F(x,y)$. This means that $F_{x_\lambda}(y_\mu) \xrightarrow{\vartheta} F_x(y)$. Thus the net $\{\widehat{F}(x_\lambda), \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to $\widehat{F}(x)$. By assumption the net $\{\widehat{F}(x_\lambda), \lambda \in \Lambda\}$ $\vartheta-$converges to $\widehat{F}(x)$. Thus the map $\widehat{F}$ is weakly $\vartheta-$continuous and the topology τ weakly $\vartheta-$splitting.

9. THEOREM. A topology τ on $WC(Y,Z)$ is weakly splitting if and only if

$$\mathcal{C}^*_{w} \subseteq (\mathcal{C}(\tau))_{w}.$$

The proof of this theorem is similar to the proof of Theorem 8.

10. THEOREM. A topology τ on $SUC(Y,Z)$ is super splitting if and only if

$$\mathcal{C}^*_{s-c} \subseteq (\mathcal{C}(\tau))_{s-c}.$$

The proof of this theorem is similar to the proof of Theorem 8.

11. THEOREM. A topology τ on $W\Theta(Y,Z)$ is weakly $\vartheta-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\vartheta} \subseteq \mathcal{C}^*_{w-\vartheta}.$$

PROOF. Let τ be a weakly $\vartheta-$jointly continuous map, X be the space which was defined in the proof of Theorem 8 and $(\{f_\lambda, \lambda \in \Lambda\}, f) \in (\mathcal{C}(\tau))_{w-\vartheta}$. Clearly, the map $G : X \to W\Theta_\tau(Y, Z)$, where $G(\lambda) = f_\lambda$ and $G(\infty) = f$ is weakly $\vartheta-$continuous. Thus, the map $\widetilde{G} : X \times Y \to Z$ is weakly $\vartheta-$continuous.

Now, we prove that $(\{f_\lambda, \lambda \in \Lambda\}, f) \in \mathcal{C}^*_{w-\vartheta}$. It is sufficient to prove that if $\{y_\mu, \mu \in M\}$ is a net in Y which weakly $\vartheta-$convergence to $y \in Y$, then the net $\{f_\lambda(y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ $\vartheta-$converges to $f(y)$. The net $\{\lambda, \lambda \in \Lambda\}$ in X weakly $\vartheta-$converges to ∞. Hence, the net $\{(\lambda, y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ weakly $\vartheta-$converges to (∞, y). Since the map $\widetilde{G}$ is weakly $\vartheta-$continuous the met $\{\widetilde{G}(\lambda, y_\mu) = G(\lambda)(y_\mu) = f_\lambda(y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ $\vartheta-$converges to $\widetilde{G}(\infty, y) = f(y)$.

Conversely, let τ be a topology on $W\Theta(Y, Z)$ such that

$$(\mathcal{C}(\tau))_{w-\vartheta} \subseteq \mathcal{C}^*_{w-\vartheta}.$$

We prove that the topology τ is weakly $\vartheta-$jointly continuous. Let X be an arbitrary space and let $G : X \to W\Theta_\tau(Y, Z)$ be a weakly $\vartheta-$continuous map. We prove that the map $\widetilde{G} : X \times Y \to Z$ is weakly $\vartheta-$continuous. Let $\{(x_\lambda, y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ be a net in $X \times Y$ which weakly $\vartheta-$converges to (x, y). We prove that the net $\{\widetilde{G}(x_\lambda, y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ in Z $\vartheta-$converges to $\widetilde{G}(x, y)$.

Since the net $\{x_\lambda, \lambda \in \Lambda\}$ weakly $\vartheta-$converges to x in X and the map G is weakly $\vartheta-$continuous. The net $\{G(x_\lambda), \lambda \in \Lambda\}$ $\vartheta-$converges to $G(x)$. Thus by assumption the net $\{G(x_\lambda), \lambda \in \Lambda\}$ weakly $\vartheta-$continuously converges to $G(x)$. Now, since the net $\{y_\mu, \mu \in M\}$ weakly $\vartheta-$converges to y the net $\{G(x_\lambda)(y_\mu) = \widetilde{G}(x_\lambda, y_\mu), (\lambda, \mu) \in \Lambda \times M\}$ $\vartheta-$converges to $G(x)(y) = \widetilde{G}(x, y)$. Hence the topology τ is weakly $\vartheta-$jointly continuous.

12. THEOREM. A topology τ on $WC(Y, Z)$ is weakly jointly continuous if and only if

$$(\mathcal{C}(\tau))_w \subseteq \mathcal{C}^*_w.$$

The proof of this theorem is similar to the proof of Theorem 11.

13. THEOREM. A topology τ on $SUC(Y, Z)$ is super jointly continuous if and only if

$$(\mathcal{C}(\tau))_{s-c} \subseteq \mathcal{C}^*_{s-c}.$$

The proof of this theorem is similar to the proof of Theorem 11.

14. COROLLARY. A topology τ on $W\Theta(Y, Z)$ is simultaneously, weakly $\vartheta-$splitting and weakly $\vartheta-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\vartheta} = \mathcal{C}^*_{w-\vartheta}.$$

15. COROLLARY. A topology τ on $WC(Y, Z)$ is simultaneously, weakly splitting and weakly jointly continuous if and only if

$$(\mathcal{C}(\tau))_w = \mathcal{C}^*_w.$$

16. COROLLARY. A topology τ on $SUC(Y, Z)$ is simultaneously, super splitting and super jointly continuous if and only if

$$(\mathcal{C}(\tau))_{s-c} = \mathcal{C}^*_{s-c}.$$

17. THE WEAKLY $\mathcal{A}-$EXPONENTIAL FUNCTION. Let $\mathcal{A}$ be an arbitrary family of spaces, let $X \in \mathcal{A}$ and let Y. Z be topological spaces. then the *weakly $\mathcal{A}-$exponential function*

$$e^{w-\mathcal{A}}_{XYZ} : WC(X \times Y, Z) \to WC(X, WC_\tau(Y, Z))$$

is defined by $e^{w-\mathcal{A}}_{XYZ}(F) = \widehat{F}$, for every $F \in WC(X \times Y, Z)$.

Clearly we have the following propositions:

(1) If τ is a weakly $\mathcal{A}-$splitting topology on $WC(Y, Z)$ then the weakly $\mathcal{A}-$exponential function $e^{w-\mathcal{A}}_{XYZ}$ is well defined.

(2) If $WC(X, WC_\tau(Y, Z)) \subseteq e^{w-\mathcal{A}}_{XYZ}(WC(X \times Y, Z))$, then the topology τ on $WC(Y, Z)$ is weakly $\mathcal{A}-$jointly continuous.

(3) If $e^{w-\mathcal{A}}_{XYZ}(WC(X \times Y, Z)) \subseteq WC(X, WC_\tau(Y, Z))$, then the topology τ is weakly $\mathcal{A}-$splitting.

18. THE WEAKLY $\mathcal{A}-\vartheta-$EXPONENTIAL FUNCTION. Let $\mathcal{A}$ be an arbitrary family of spaces, let $X \in \mathcal{A}$ and let Y. Z be topological spaces. Then the *weakly $\mathcal{A}-\vartheta-$exponential function*

$$e^{w-\mathcal{A}-\vartheta}_{XYZ} : W\Theta(X \times Y, Z) \to W\Theta(X, W\Theta_\tau(Y, Z))$$

is defined by $e^{w-\mathcal{A}-\vartheta}_{XYZ}(F) = \widehat{F}$, for every $F \in W\Theta(X \times Y, Z)$.

Clearly we have the following propositions:

(1) If τ is a weakly $\mathcal{A} - \vartheta-$splitting topology on $W\Theta(Y, Z)$ then the weakly $\mathcal{A} - \vartheta-$exponential function $e^{w-\mathcal{A}-\vartheta}_{XYZ}$ is well defined.

(2) If $W\Theta(X, W\Theta_\tau(Y, Z)) \subseteq e^{w-\mathcal{A}-\vartheta}_{XYZ}(W\Theta(X \times Y, Z))$, then the topology τ on $W\Theta(Y, Z)$ is weakly $\mathcal{A} - \vartheta-$jointly continuous.

(3) If $e^{w-\mathcal{A}-\vartheta}_{XYZ}(W\Theta(X \times Y, Z)) \subseteq W\Theta(X, W\Theta_\tau(Y, Z))$, then the topology τ is weakly $\mathcal{A} - \vartheta-$splitting.

19. THE SUPER $\mathcal{A}-$EXPONENTIAL FUNCTION. Let $\mathcal{A}$ be an arbitrary family of spaces, let $X \in \mathcal{A}$ and let Y. Z be topological spaces. Then the *super $\mathcal{A}-$exponential function*

$$e^{s-\mathcal{A}}_{XYZ} : SUC(X \times Y, Z) \to SUC(X, SUC_\tau(Y, Z))$$

is defined by $e^{s-\mathcal{A}}_{XYZ}(F) = \widehat{F}$, for every $F \in SUC(X \times Y, Z)$.

Clearly we have the following propositions:

(1) If τ is a super $\mathcal{A}-$splitting topology on $SUC(Y, Z)$ then the super $\mathcal{A}-$exponential function $e^{s-\mathcal{A}}_{XYZ}$ is well defined.

(2) If $SUC(X, SUC_\tau(Y, Z)) \subseteq e^{s-\mathcal{A}}_{XYZ}(SUC(X \times Y, Z))$, then the topology τ on $SUC(Y, Z)$ is super $\mathcal{A}-$jointly continuous.

(3) If $e^{s-\mathcal{A}}_{XYZ}(SUC(X \times Y, Z)) \subseteq SUC(X, SUC_\tau(Y, Z))$, then the topology τ is super $\mathcal{A}-$splitting.

20. NOTATIONS AND DEFINITIONS. Let Ω be a family of directed sets. Then for every $\Lambda \in \Omega$ we consider the set $\mathrm{Sp}(\Lambda) \equiv \Lambda \cup \{\infty\}$, where ∞ is a symbol such that $\infty \geq \lambda$, for every $\lambda \in \Lambda$. Then we topologize $\mathrm{Sp}(\Lambda) = \Lambda \cup \{\infty\}$ defining any singleton $\{\lambda\}$, $\lambda \in \Lambda$ to be open and neighbourhoods of ∞ the sets $\{\lambda \in \mathrm{Sp}(\Lambda) : \lambda \geq \lambda_0 \text{ for some } \lambda_0 \in \Lambda\}$ (See [1] and [2]). By $\mathrm{Sp}(\Omega)$ we denote the family of all spaces $\mathrm{Sp}(\Lambda)$, where $\Lambda \in \Omega$. Instead of "weakly $\mathrm{Sp}(\Omega)-$splitting", "weakly $\mathrm{Sp}(\Omega)-\vartheta-$splitting", "super $\mathrm{Sp}(\Omega)-$splitting", "weakly $\mathrm{Sp}(\Omega)-$jointly continuous", "weakly $\mathrm{Sp}(\Omega) - \vartheta-$jointly continuous" and "super $\mathrm{Sp}(\Omega)-$jointly continuous", we write "weakly $\Omega-$splitting", "weakly $\Omega - \vartheta-$splitting", "super $\Omega-$splitting", "weakly $\Omega-$jointly continuous", "weakly $\Omega - \vartheta-$jointly continuous" and "super $\Omega-$jointly continuous", respectively.

By $\mathcal{C}^*_{w-\Omega}$, $\mathcal{C}^*_{w-\vartheta-\Omega}$, $\mathcal{C}^*_{s-c-\Omega}$, $(\mathcal{C}(\tau))_{w-\Omega}$, $(\mathcal{C}(\tau))_{w-\vartheta-\Omega}$ and $(\mathcal{C}(\tau))_{s-c-\Omega}$, we denote the subclass of all elements $(\{f_\lambda, \lambda \in \Lambda\}, f)$ of $\mathcal{C}^*_w$, $\mathcal{C}^*_{w-\vartheta}$, $\mathcal{C}^*_{s-c}$, $(\mathcal{C}(\tau))_w$, $(\mathcal{C}(\tau))_{w-\vartheta}$ and $(\mathcal{C}(\tau))_{s-c}$, respectively for which $\Lambda \in \Omega$.

21. THEOREM. the following propositions are true:

(1) A topology τ on $W\Theta(Y,Z)$ is weakly $\Omega-\vartheta-$splitting if and only if

$$\mathcal{C}^*_{w-\vartheta-\Omega} \subseteq (\mathcal{C}(\tau))_{w-\vartheta-\Omega}.$$

(2) A topology τ on $W\Theta(Y,Z)$ is weakly $\Omega-\vartheta-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\vartheta-\Omega} \subseteq \mathcal{C}^*_{w-\vartheta-\Omega}.$$

(3) A topology τ on $W\Theta(Y,Z)$ is simultaneously, weakly $\Omega-\vartheta-$splitting and weakly $\Omega-\vartheta-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\vartheta-\Omega} = \mathcal{C}^*_{w-\vartheta-\Omega}.$$

The proof of this theorem is similar of the proof of Theorems 8 and 11 and Corollary 14.

Similarly, we have the following two theorems:

22. THEOREM. the following propositions are true:

(1) A topology τ on $WC(Y,Z)$ is weakly $\Omega-$splitting if and only if

$$\mathcal{C}^*_{w-\Omega} \subseteq (\mathcal{C}(\tau))_{w-\Omega}.$$

(2) A topology τ on $WC(Y,Z)$ is weakly $\Omega-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\Omega} \subseteq \mathcal{C}^*_{w-\Omega}.$$

(3) A topology τ on $WC(Y,Z)$ is simultaneously, weakly $\Omega-$splitting and weakly $\Omega-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{w-\Omega} = \mathcal{C}^*_{w-\Omega}.$$

23. THEOREM. the following propositions are true:

(1) A topology τ on $SUC(Y,Z)$ is super $\Omega-$splitting if and only if

$$\mathcal{C}^*_{s-c-\Omega} \subseteq (\mathcal{C}(\tau))_{s-c-\Omega}.$$

(2) A topology τ on $SUC(Y,Z)$ is super $\Omega-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{s-c-\Omega} \subseteq \mathcal{C}^*_{s-c-\Omega}.$$

(3) A topology τ on $SUC(Y,Z)$ is simultaneously, super $\Omega-$splitting and super $\Omega-$jointly continuous if and only if

$$(\mathcal{C}(\tau))_{s-c-\Omega} = \mathcal{C}^*_{s-c-\Omega}.$$

24. THEOREM. The following propositions are true:

(1) A topology τ on $W\Theta(Y,Z)$ is weakly $\vartheta-$splitting if and only if is weakly $\mathcal{A}-\vartheta-$splitting, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

(2) A topology τ on $W\Theta(Y,Z)$ is weakly $\vartheta-$jointly continuous if and only if is weakly $\mathcal{A}-\vartheta-$jointly continuous, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

PROOF. It is sufficient to prove that if τ is weakly $\mathcal{A}-\vartheta-$splitting, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point, then the topology τ is weakly $\vartheta-$splitting.

Let $(\{f_\lambda, \lambda \in \Lambda\}, f) \in \mathcal{C}^*_{w-\vartheta}$. We must prove that the net $\{f_\lambda, \lambda \in \Lambda\}$ $\vartheta-$converges to the element f in the topology τ.

Let $X = \Lambda \cup \{\infty\}$, where ∞ is a symbol such that $\infty \geq \lambda$, for every $\lambda \in \Lambda$. Then we topologize $X = \Lambda \cup \{\infty\}$ defining any singleton $\{\lambda\}$, $\lambda \in \Lambda$ to be open and neighbourhoods of ∞ the sets $\{\lambda \in X : \lambda \geq \lambda_0$ for some $\lambda_0 \in \Lambda\}$ (See [1] and [2]). Clearly the element ∞ is the unique non-isolated point of the space X and $X \in \mathcal{A}$.

We consider the map $F : X \times Y \to Z$ setting $F(\lambda, y) = f_\lambda(y)$, $\lambda \neq \infty$ and $F(\infty, y) = f(y)$. Obviously the map F is weakly $\vartheta-$continuous. Now we prove that $\{f_\lambda, \lambda \in \Lambda\}$ $\vartheta-$converges to f in the topology τ.

Indeed, let $U \in \tau$ be an open neighbourhood of f. By assumption the topology τ is weakly $\mathcal{A}-\vartheta-$splitting. Hence, the map $\widehat{F} : X \to W\Theta_\tau(Y, Z)$ is weakly $\vartheta-$continuous. Also, we have $\widehat{F}(\infty) = f$ and $\widehat{F}(\lambda) = f_\lambda$, $\lambda \neq \infty$. Thus, there exists an open neighbourhood V of ∞ such that

$$\widehat{F}(Int(Cl(V))) \subseteq Cl(U).$$

Since the set V is an open neighbourhood of ∞ in X, there exists an element $\lambda_0 \in \Lambda$ such that $\lambda \in V \subseteq Int(Cl(V))$, for every $\lambda \geq \lambda_0$, $\lambda \in \Lambda$.

Hence, $\widehat{F}(\lambda) = f_\lambda \in Cl(U)$, for every $\lambda \geq \lambda_0$, $\lambda \in \Lambda$. Thus, the net $\{\widehat{F}(\lambda) = f_\lambda, \lambda \in \Lambda\}$ $\vartheta-$converges to f in the topology τ and the topology τ is weakly $\vartheta-$splitting.

Similarly, we can prove the proposition (2).

Also, similarly, we can prove the following two theorems:

25. THEOREM. The following propositions are true:

(1) A topology τ on $WC(Y, Z)$ is weakly splitting if and only if is weakly $\mathcal{A}-$splitting, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

(2) A topology τ on $WC(Y, Z)$ is weakly jointly continuous if and only if is weakly $\mathcal{A}-$jointly continuous, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

26. THEOREM. The following propositions are true:

(1) A topology τ on $SUC(Y, Z)$ is super splitting if and only if is super $\mathcal{A}-$splitting, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

(2) A topology τ on $SUC(Y, Z)$ is super jointly continuous if and only if is super $\mathcal{A}-$jointly continuous, where $\mathcal{A}$ is the family of all spaces having exactly one non-isolated point.

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