# OPTIMAL STOPPING GAMES UNDER WINNING PROBABILITY MAXIMIZATION AND PLAYER'S WEIGHTED PRIVILEGE 

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#### Abstract

Some constant-sum $n$-stage sequential game version of best-choice problems under Winning Probability (WP) maximization are investigated and the explicit solutions are obtained. The essential feature contained in these sequential games is the fact that the players have their own weights by which at each stage, player's desired decision may be taken away by an opponent as an outcome of drawing a lottery. The game is, essentially, a generalization of the "horse game" first investigated by Enns and Ferenstein [2]in 1985.


## 1. Introduction

In this paper some constant-sum $n$-stage sequential-game versions of best-choice problems under Winning Probability (WP) maximization are investigated.

A zero-sum game version of the discrete-time, full-information best-choice problem under WP-maximization is studied in Section 2, and the extensions of this simplest game to the three-player case and no information versions are studied in Sections 3 and 4.

We first state the two-person game version as follows:
$\left(1^{0}\right)$ There are two players I and II, a sequence of $n$ iid r.v.s $\left\{X_{i}\right\}_{i=1}^{n}$, each r.v. obeying uniform distribution on $0 \leq x \leq 1$. Both players observe $X_{i}$ s sequentially one by one.
$\left(2^{0}\right)$ Observing each $X_{t}$, both players select simultaneously and independently, either to accept (A) or to reject (R) the $X_{t}$. If I-II choice is A-A, then player I(II) accepts to receive $X_{t}$ with probability $w(\bar{w}=1-w), \frac{1}{2} \leq w \leq 1$, and drops out from the game thereafter. The player remained continues his one-person game. If I-II choice is A-R (R-A), then I (II) accepts $X_{t}$ and drops out and his opponent continues the remaining one-person game. If $\mathrm{I}-\mathrm{II}$ choice is $\mathrm{R}-\mathrm{R}$, then $X_{t}$ is rejected and the players face the next $X_{t+1}$.
$\left(3^{0}\right)$ A player wins if he accepts a r.v. that is greater than the opponent's one, or if his opponent fails to accept any r.v. The aim of player I (II) in the game is to find his strategy by following which he maximizes (minimizes) probability of player I's winning.

If $w=1$, the problem reduces essentially to the "horse game" first investigated by Enns and Ferentein [2], in 1985.

The explicit solution to this problem is given in Section 2. The game is extended to the three-person case in Section 3, and under a vital and reasonable assumption the equilibrium play is explicitly derived for the equal-weight (i.e. $w_{1}=w_{2}=w_{3}=\frac{1}{3}$ ) game. The problems we consider in this paper belong to a class of best-choice problems combined with sequential games. Two and three-person optimal stopping games where players have weighted privilege, under full-information and expected net value (ENV) maximization

[^0]are investigated in Sakaguchi [10]. And those problems under no-information and WPmaximization, but with the player's aim being selecting-best of r.v..s are treated in Ramsey and Szajowski [11]. Recent works related to these area of problems are [1]~[6], [8], and [9]. Also a recent look for the optimal stopping games in various phases can be found in Sakaguchi [7].

## 2. Selecting Better-than-opponents-Two-person Full-information case.

Consider the zero-sum two-person game described in the previous section. Define state $(x \mid n)$ to mean that (1) both player remain in the game, and (2) there remain $n$ r.v.s to be observed and the players currently face the first observation $X_{1}=x$. Let $V_{n}$ be the value of the game for player I, for the $n$-problem. Since the players shoud choose A-A in state $(x \mid 1)$, and so draw of the game cannot occur.

The Optimality Equation is evidently

$$
\begin{equation*}
V_{n}=E \operatorname{val} M_{n}(X) \quad\left(n=1,2, \ldots, V_{0} \equiv 0\right) \tag{2.1}
\end{equation*}
$$

where $E_{\varphi}(X) \equiv \int_{0}^{1} \varphi(x) d x$ and

$$
M_{n}(x)=\begin{gather*}
 \tag{2.2}\\
R \\
A
\end{gather*}\left(\begin{array}{cc}
R & A \\
E \operatorname{val} M_{n-1}(X) & 1-x^{n-1} \\
x^{n-1} & (w-\bar{w}) x^{n-1}+\bar{w}
\end{array}\right)
$$

Lemma 1.1 Let $\frac{1}{2} \leq a, w \leq 1$ and $Y$ is a r.v. with $\operatorname{cdf} G(y)$, on $0 \leq y \leq 1$. Then

$$
\left.\begin{array}{rl} 
& E_{G} v a l\left[\begin{array}{cc}
a & 1-Y \\
Y & (w-\bar{w}) Y+\bar{w}
\end{array}\right] \\
= & w+\left[\int_{\bar{a}}^{\frac{1}{2}}-(w-\bar{w}) \int_{\frac{1}{2}}^{1}\right. \tag{2.3}
\end{array}\right] G(y) d y .
$$

Proof. The matrix $M(y) \equiv \begin{gathered} \\ \mathrm{R} \\ \mathrm{A}\end{gathered}\left(\begin{array}{cc}\mathrm{a} & \mathrm{R} \\ y & (w-\bar{w}) y+\bar{w}\end{array}\right)$ has the saddle point at R-R, R-A and A-A, if $0<y<\bar{a}, \bar{a}<y<\frac{1}{2}$, and $\frac{1}{2}<y<1$ respectively. Hence it follows that

$$
\operatorname{valM}(y)=\left\{\begin{array}{llc}
a, & \text { if } & 0<y<\bar{a} \\
1-y, & \text { if } & \bar{a}<y<\frac{1}{2} \\
(w-\bar{w}) y+\bar{w}, & \text { if } & \frac{1}{2}<y<1
\end{array}\right.
$$

and

$$
\begin{equation*}
E[\operatorname{val} M(Y)]=a G(\bar{a})+\int_{\bar{a}}^{\frac{1}{2}}(1-y) d G(y)+\int_{\frac{1}{2}}^{1}\{(w-\bar{w}) y+\bar{w}\} d G(y) \tag{2.4}
\end{equation*}
$$

The r.h.s. of (2.4) becomes (2.3) after intergration by parts and simplification.

Theorem 1. The solution to our OSG described by (2.1)-(2.2), for $\frac{1}{2}<w<1$, is as follows: The optimal strategy-pair in state $(x \mid n)$ is $R-R, R-A$ and $A-A$, if $0 \leq x<$ $\left(\bar{V}_{n-1}\right)^{\frac{1}{n-1}},\left(\bar{V}_{n-1}\right)^{\frac{1}{n-1}}<x<2^{-1 /(n-1)}$, and $2^{-1 /(n-1)}<x \leq 1$, respectively. The sequence $\left\{V_{n}\right\}$ is determined by recursion

$$
\begin{equation*}
V_{n}=\frac{1}{n}(w-\bar{w})+\bar{w}+\left(\frac{n-1}{n}\right)\left[w\left(\frac{1}{2}\right)^{\frac{1}{n-1}}-\left(\bar{V}_{n-1}\right)^{\frac{n}{n-1}}\right] \quad\left(n \geq 2 ; V_{1}=w\right) \tag{2.5}
\end{equation*}
$$

Table 1: Optimal strategies for the games where $w=3 / 4$ and $w=1$

| $n$ | $\left(\bar{V}_{n-1}\right)^{\frac{1}{n-1}}$ |  | $\left(\frac{1}{2}\right)^{\frac{1}{n-1}}$ | $V_{n}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $w=3 / 4$ | $w=1$ |  | $w=3 / 4$ | $w=1$ |
| 1 |  |  | 0.0000 | 0.7500 | 1.0000 |
| 2 | 0.2500 | 0.0000 | 0.5000 | 0.6563 | 0.7500 |
| 3 | 0.5863 | 0.5000 | 0.7071 | 0.6359 | 0.7214 |
| 4 | 0.7141 | 0.6532 | 0.7937 | 0.6264 | 0.7088 |
| 5 | 0.7818 | 0.7346 | 0.8409 | 0.6209 | 0.7016 |
| 6 | 0.8237 | 0.7852 | 0.8706 | 0.6172 | 0.6969 |
| 7 | 0.8521 | 0.8196 | 0.8909 | 0.6146 | 0.6935 |
| 8 | 0.8727 | 0.8445 | 0.9057 | 0.6126 | 0.6910 |
| 9 | 0.8882 | 0.8635 | 0.9170 | 0.6110 | 0.6891 |
| 10 | 0.9004 | 0.8783 | 0.9259 | 0.6097 | 0.6875 |
| 11 | 0.9102 | 0.8902 | 0.9330 | 0.6087 | 0.6863 |
| 12 | 0.9182 | 0.9000 | 0.9389 | 0.6078 | 0.6852 |

Proof. In Lemma 1.1, we have $G(y)=\mathrm{P}_{\mathrm{r}}(Y \leq y)=\mathrm{P}_{\mathrm{r}}\left(X^{n-1} \leq y\right)=y^{1 /(n-1)}$. Substituting this into (2.3) and using $\int y^{1 /(n-1)} d y=\left(\frac{n-1}{n}\right) y^{n /(n-1)}$, we obtain (2.5) after some computation.

From (2.5), we have, when $w=1 / 2$,

$$
V_{n}=\frac{1}{2}+\left(\frac{n-1}{n}\right)\left[\left(\frac{1}{2}\right)^{\frac{n}{n-1}}-\left(\bar{V}_{n-1}\right)^{\frac{n}{n-1}}\right]
$$

which gives $V_{n}=1 / 2,(n \geq 1)$, and the optimal play in state $(x \mid n)$ is R-R (A-A), if $x<(>) 2^{-1 /(n-1)}$. When $w=1$,

$$
\begin{equation*}
V_{n}=\frac{1}{n}+\left(\frac{n-1}{n}\right)\left[\left(\frac{1}{2}\right)^{\frac{1}{n-1}}-\left(\bar{V}_{n-1}\right)^{\frac{n}{n-1}}\right] \quad\left(n \geq 2 ; V_{1}=1\right) \tag{2.6}
\end{equation*}
$$

which gives $V_{2}=3 / 4, V_{3}=(4 \sqrt{2}+3) / 12 \dot{=} 0.7214$, and so on. Table 1 shows the characteristics of the optimal strategies in the game where $w=3 / 4$ and $w=1$, respectively, for $n=1(1) 12$.

From the table we find that, when $w=3 / 4$, for example, the opimal play in state $(x \mid 12)$ is R-R, R-A and A-A if $0 \leq x<0.9182,0.9182<x<0.9389$ and $0.9389<x \leq 1$, respectively, and the value of the game is $V_{12}=0.6078$. We also observe that $V_{n}$ is strictly decreasing, because the resistance by the opponent player II becomes more efficient as $n$ increases.

Corollary 1.1. If the sequence $\left\{\bar{V}_{n}\right\}$ converges to a value $\alpha \in(0,1)$, then $\alpha$ is a unique root of the equation

$$
\begin{equation*}
-\alpha \log \alpha=-\alpha+(\log 2) w+\bar{w} \tag{2.7}
\end{equation*}
$$

Proof. Denoting $\bar{V}_{n}$, simply by $\alpha_{n}$, (2.4) becomes

$$
\alpha_{n}=w-\frac{1}{n}(w-\bar{w})-\left(1-n^{-1}\right)\left\{w\left(\frac{1}{2}\right)^{\frac{1}{n-1}}-\alpha_{n-1}^{\frac{n}{n-1}}\right\} .
$$

Subtracting $\alpha_{n-1}$ from the both sides and multiplying by $n$, we obtain

$$
n\left(\alpha_{n}-\alpha_{n-1}\right)=\bar{w}-\alpha_{n-1}+\frac{w\left\{1-\left(\frac{1}{2}\right)^{\frac{1}{n-1}}\right\}-\alpha_{n-1}\left\{1-\alpha_{n-1}^{\frac{1}{n-1}}\right\}}{1 /(n-1)}
$$

Since $\lim _{x \rightarrow 0+}\left(1-\alpha^{x}\right) / x=-\log \alpha$, for any $\alpha \in(0,1)$, it follows that, if $\alpha_{n} \rightarrow \alpha$, then, by letting $n \rightarrow \infty$, we have

$$
0=\bar{w}-\alpha+w \log 2+\alpha \log \alpha
$$

This is (2.7).
Equation (2.7) gives the value of $\alpha$, for $w=0.5(0.1) 1.0$, as follows:

$$
\begin{array}{cccccccc}
w & = & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0 \\
\alpha & = & 0.5000 & 0.4594 & 0.4186 & 0.3860 & 0.3561 & 0.3276 \\
\bar{\alpha} & = & 0.5000 & 0.5406 & 0.5814 & 0.6140 & 0.6439 & 0.6724
\end{array}
$$

When $w=1$ our result coinsides with that in Enns and Ferenstein [2].

## 3. Selecting Better-than-opponent-Three-Person Full-information Case

### 3.1. A three-person optimal stopping game

The analysis made in the previous section can be extended to three-person games. We state the problem in correspondence to $\left(1^{0}\right) \sim\left(3^{0}\right)$ in Section 1, as follows:
$\left(1^{+}\right)$There are three persons I, II, and III. These players have their weights $w_{1}, w_{2}$, and $w_{3}$, respectively. Let $1 \geq w_{1} \geq w_{2} \geq w_{3} \geq 0, w_{1}+w_{2}+w_{3}=1$, and $w_{(i, j)} \equiv w_{i} /\left(w_{i}+w_{j}\right), i \neq j$.
$\left(2^{+}\right)$If three-players choice is A-A-A, then player I (II, III) accepts $X_{t}$ with probability $w_{1}\left(w_{2}, w_{3}\right)$ and drops out from the play thereafter. The two players remained continue their two-person game with their " revised" new weights. If three players' choice is R-A-A, then II (III) accepts $X_{t}$ with probability $w_{(2,3)}\left(w_{(3,2)}\right)$ dropping out from the game, and the remaining players III (II) and I continue their two-person game with their revised new weights. If three-players choice is R-R-A, then III acceps $X_{t}$ and drops out and his opponents I and II continue the remaining two-person game. If players' choice-triple is R-RR , then $X_{t}$ is rejected and the players face the next $X_{t+1}$. In case of other four choice-triples A-R-A, A-A-R, R-A-R, and A-R-R, the game is played similarly as mentioned above.
$\left(3^{+}\right)$A player wins if he accepts a r.v. that is larger than those accepted by his opponents. The purpose of each player is to find the strategy that maximizes the probability of his winning.

Definition of state is $(x \mid n)$ is the same as in Section 2, with a single difference that there are three players. Let $W_{n}^{i}, i=1,2,3$, be the value of the game for player $i$, for the $n$-problem.

The statement of the problem in dynamic programming framework is as follows. Denote, by $V_{n}\left(w_{(i, j)}, x\right)$, the value for player $i$ in the two-person game against $j$, with weights $w_{(i, j)}$ for $i$, and $w_{(j, i)}$ for $j$, under the condition that player $k(\neq i, j)$, has already dropped out from the game by accepting a past observation $x$. In state $(x \mid n)$, players face a trimatrix game with the payoff matrix $M_{n}(x)$, which is

$$
M_{n}(x)= \begin{cases}M_{n, R}(x), & \text { if } \mathrm{R} \text { is chosen by III }  \tag{3.1}\\ M_{n, A}(x), & \text { if } \mathrm{A} \text { is chosen by III }\end{cases}
$$

where
and

In these two matrices the subscript $n-1$ of $W^{i}$ and $V$, and $x$ inside $V(\cdot)$ are omitted. Also $w_{(i, j)}$ are rewritten as $w_{i j}$. The Optimality Equation is

$$
\begin{equation*}
\left(W^{1}, W^{2}, W^{3}\right)=E\left[\text { eq.val. } M_{n}(X)\right] \quad\left(n \geq 1 ; W_{0}^{i}=V_{0}\left(w_{i j}\right)=0, \forall i, j\right) \tag{3.2}
\end{equation*}
$$

provided the eq. value of $M_{n}(x)$ exists uniquely.
In the next section we present the explicit solution to the problem (3.1)-(3.2) in a special case of $<w_{1}, w_{2}, w_{3}>=<\frac{1}{3}, \frac{1}{3}, \frac{1}{3}>$.

### 3.2. Three-person equal-weight game.

Denote by $\Gamma_{n}^{(3)}$ the three-person equal-weight $n$-stage game. Also denote by $\Gamma_{m, x}^{(2)}(1 \leq m<$ $n$ ), the two-person equal-weight $m$-stage game under the condition that the r.v.s smaller that $x$ should be rejected. Also, denote by $W_{n}$ and $V_{m, x}$, the common equilibrium values (c.e.v.) in the games $\Gamma_{n}^{(3)}$ and $\Gamma_{m, x}^{(2)}$, respectively. We first consider the game $\Gamma_{m, x}^{(2)}$. We shall make an important assumption.

Assumption A. In the game $\Gamma_{m, x}^{(2)}(1 \leq m<n)$, players should choose $A$ - $A$ at the earliest r.v. that is larger that $x$.

Under Assumption A, the event that both players lose the game $\Gamma_{m, x}^{(2)}$ cannot happen, since even if no candidate appear until the $(m-1)$-th, each player will get $1 / 2$, by the choice-pair A-A at the $m$-th. Therefore by considering symmetry we can deduce that $V_{m, x}=\frac{1}{2}\left(1-x^{m}\right)$. Or more precisely

$$
\begin{aligned}
& V_{m, x}=x V_{m-1, x}+\int_{x}^{1} \frac{1}{2}\left\{y^{m-1}+\left(1-y^{m-1}\right)\right\} d y \\
& =x V_{m-1, x}+\frac{1}{2}(1-x)=x^{2} V_{m-2, x}+\frac{1}{2}\left(1-x^{2}\right) \\
& =\ldots=x^{m-1} V_{1, x}+\frac{1}{2}\left(1-x^{m-1}\right)=\frac{1}{2}\left(1-x^{m}\right)
\end{aligned}
$$

since $V_{1, x}=\int_{x}^{1} \frac{1}{2} d y=\frac{1}{2}(1-x)$. For the game $\Gamma_{n}^{(3)}$, the fact that the event in which all three palyers lose the game cannot happen, and symmetry in the game lead to $W_{n}=1 / 3, n \geq 1$.

Now we are interested in finding the equilibrium play in the game $\Gamma_{n}^{(3)}$, and this is the purpose of our study in this subsection. Under Assumption A the payoff matrix $M_{n}(x)$ in state $(x \mid n)$ of the game $\Gamma_{n}^{(3)}$ become (3.1) with

$$
M_{n, R}(x)=(\mathrm{I})\left\{\begin{array}{c}
\mathrm{R} \\
\overbrace{\mathrm{R}}  \tag{3.3}\\
\mathrm{~A}
\end{array} \quad\left[\begin{array}{cc}
1 / 3,1 / 3,1 / 3 & \bar{u} / 2, u, \bar{u} / 2 \\
u, \bar{u} / 2, \bar{u} / 2 & (1+u) / 4,(1+u) / 4, \bar{u} / 2
\end{array}\right]\right.
$$

and

$$
M_{n, A}(x)=(\mathrm{I})\left\{\begin{array}{c}
\overbrace{\mathrm{R}} \\
\mathrm{R}  \tag{3.4}\\
\mathrm{~A}
\end{array} \quad\left[\begin{array}{cc}
\bar{u} / 2, \bar{u} / 2, u & \bar{u} / 2,(1+u) / 4,(1+u) / 4 \\
(1+u) / 4, \bar{u} / 2,(1+u) / 4 & 1 / 3,1 / 3,1 / 3
\end{array}\right]\right.
$$

where we have set $u=x^{n-1}$ for simplicity. Note that, in (3.3)-(3.4), the element A-A-R is

$$
\left(\frac{1}{2}\left(x^{n-1}+V_{n-1, x}\right), \frac{1}{2}\left(x^{n-1}+V_{n-1, x}\right), V_{n-1, x}\right)=\left(\frac{1}{4}(1+u), \frac{1}{4}(1+u), \frac{\bar{u}}{2}\right)
$$

and the element $\mathrm{A}-\mathrm{A}-\mathrm{A}$ is the triple of

$$
\frac{1}{3}\left(x^{n-1}+2 V_{n-1, x}\right)=\frac{1}{3}\left[x^{n-1}+2 \cdot \frac{1}{2}\left(1-x^{n-1}\right)\right]=\frac{1}{3} .
$$

Now we prove
Theorem 2. Under Assumption $A$ the equilibriun play in the game $\Gamma_{n}^{(3)}$ is as follows: Common equilibrium strategy, in state $(x \mid n)$, is

$$
\text { Choose } R(A), \text { if } x<(>)\left(\frac{1}{3}\right)^{\frac{1}{n-1}}
$$

If the process goes on like

$$
\begin{equation*}
X_{t} \leq\left(\frac{1}{3}\right)^{\frac{1}{n-t}}, \quad t=1,2, \ldots, n-m-1, \quad \text { and } \quad X_{n-m}=x>\left(\frac{1}{3}\right)^{\frac{1}{m}} \tag{3.5}
\end{equation*}
$$

for some $m, 2 \leq m<n$, adopt the following strategy in the game $\Gamma_{m, x}^{(2)}$, thereafter. That is; Accept the earliest candidate (if any), and if the lottery comes out lose at that time, accept the 2nd earliest candidate (if any).

Proof. First note that in $M_{n, R}(x)$ and $M_{n, A}(x)$, given by (3.3) - (3.4),

$$
\frac{1}{2}\left(1-x^{n-1}\right)>\frac{1}{4}\left(1+x^{n-1}\right) \Longleftrightarrow \frac{1}{2}\left(1-x^{n-1}\right)>\frac{1}{3} \Leftrightarrow x<\left(\frac{1}{3}\right)^{\frac{1}{n-1}}
$$

Then if $x\left\{\begin{array}{l}< \\ >\end{array}\right\}\left(\frac{1}{3}\right)^{\frac{1}{n-1}},\left\{\begin{array}{l}\mathrm{R}-\mathrm{R} \\ \mathrm{A}-\mathrm{A}\end{array}\right\}$ is in equilibrium for I-II in both of $M_{n, R}(x)$ and $M_{n . A}(x)$, and besides for III $\left\{\begin{array}{l}\mathrm{R} \\ \mathrm{A}\end{array}\right\}$ is better that $\left\{\begin{array}{l}\mathrm{A} \\ \mathrm{R}\end{array}\right\}$. Thus the equilibrium in state $(x \mid n)$, is: Choose R-R-R (A-A-A), if $x<(>)\left(\frac{1}{3}\right)^{\frac{1}{n-1}}$. So , c.e.v. of $M_{n}(x)=1 / 3, \forall x \in[0,1]$, and hence $W_{n}$ is equal to $\mathrm{E}\left[\right.$ c.e.v. of $\left.M_{n}(X)\right]=1 / 3$, as we expected. If the r.v.s go on like (3.5), then one player drops out from the game $\Gamma_{n}^{(3)}$ and there is left game $\Gamma_{m, x}^{(2)}$ for the other two players.

Corollary 2.1 For $0 \leq x \leq 1$ and $m \geq 3$,

$$
x>\left(\frac{1}{3}\right)^{\frac{1}{m}} \Longrightarrow\left[\frac{1}{2}\left(1-x^{m-1}\right)\right]^{\frac{1}{m-1}}<x
$$

That is, under Assumption A, the player who drops out first stands more advantageous than the two opponents.

Proof. We prove the contraposition

$$
\left[\frac{1}{2}\left(1-x^{m-1}\right)\right]^{\frac{1}{m-1}} \geq x \Longrightarrow x^{m-1} \leq \frac{1}{3} \Longrightarrow x^{m} \leq \frac{1}{3} \Longrightarrow x \leq\left(\frac{1}{3}\right)^{\frac{1}{m}}
$$

Remark 1. By Theorem 1 with $w=1 / 2$, or by following the same line of proof of Theorem 2, where the payoff matrix is

$$
\left.\begin{array}{c} 
\\
\mathrm{R} \\
\mathrm{~A}
\end{array} \begin{array}{cc}
\mathrm{R} & \mathrm{~A} \\
1 / 2,1 / 2 & 1-x^{n-1}, x^{n-1} \\
x^{n-1}, 1-x^{n-1} & 1 / 2,1 / 2
\end{array}\right]
$$

we find that the equilibrium play in state $(x \mid n)$ of game $\Gamma_{n}^{(2)}$ is to choose $\mathrm{R}-\mathrm{R}$ (A-A), if $x<(>)\left(\frac{1}{2}\right)^{\frac{1}{n-1}}$. After the moment at which $x>\left(\frac{1}{2}\right)^{\frac{1}{m}}$, for some $2 \leq m<n$, the remaining player accepts the earliest r.v. larger that $x$, which appears thereafter. Assumption A determines the equilibrium play in $\Gamma_{m, x}^{(2)}$. Table 2 gives the decision points $3^{-1 /(n-1)}$ of $\Gamma_{n}^{(3)}$, for some small $n$, in contrast with those of $\Gamma_{n}^{(2)}$.

Remark 2. At some time $n-m, 2 \leq m<n$, and some value $x \in(0,1)$, the game $\Gamma_{n}^{(3)}$ is transferred to the game $\Gamma_{m, x}^{(2)}$. Let $\mu_{n}$, and $\xi_{n}$ be the expected value of such $m$ and $x$, respectively. Then we have

$$
\begin{equation*}
\mu_{n}=\sum_{m=1}^{n-1} m 3^{-\left(\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\frac{1}{m+1}\right)}\left(1-3^{-\frac{1}{m}}\right) \tag{3.6}
\end{equation*}
$$

which satisfies the recursion

$$
\begin{equation*}
\mu_{n+1}=3^{-\frac{1}{n}} \mu_{n}+n\left(1-3^{-\frac{1}{n}}\right),\left(n \geq 1 ; \mu_{1}=0\right) \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi_{n}=\frac{1}{2} \sum_{m=1}^{n-1} 3^{-\left(\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{m+1}\right)}\left(1-3^{-\frac{2}{m}}\right) \tag{3.8}
\end{equation*}
$$

which satisfies the recursion

$$
\begin{equation*}
\xi_{n+1}=3^{-\frac{1}{n-1}} \xi_{n}+\frac{1}{2}\left(1-3^{-\frac{2}{n}}\right),\left(n \geq 1 ; \xi_{1}=0\right) \tag{3.9}
\end{equation*}
$$

In (3.6) and (3.8), the negative exponents of 3 are $\sum_{j=m+1}^{n-1} j^{-1}$, which is interpreted as zero if the sum is vacuous. Numerical values of $\mu_{n}$ and $\xi_{n}$ are shown for some small $n$ in Table 3.

Remark 3. The Assumption A brings our game problem into the easier from too much extent. In the "subgame" $\Gamma_{m, x}^{(2)}$ here, each player fixes his strategy. We find from(3.7) and (3.9) (by using the similar argument as used in Corollary 1.1.) that $\lim _{n \rightarrow \infty} n^{-1} \mu_{n}=$ $\frac{\log 3}{1+\log 3} \dot{=} 0.5235$ and $\lim _{n \rightarrow \infty} \xi_{n}=1$, unwelcoming result for our three-person game $\Gamma_{n}^{(3)}$. Theorem 2 states only that, under Assumption A, each player can expect the winning probability $1 / 3$ by employing the stated strategy. The equilibrium strategy in $\Gamma_{n}^{(3)}$ without assuming Assumption A is as yet unknown. We have to think about the fact that the optimal play in the two-person equal-weight game $\Gamma_{m, x}^{(2)}$ is different from one in $\Gamma_{m}^{(2)}$ (See Theorem 1 with $w=1 / 2$ ).

Remark 4. The three-person unequal-weight games seem to be more difficult than the equal-weight game to derive the explicit solution, even in the cases with weight $<1 / 2,1 / 2,0>$ or $\langle 1,0,0\rangle$.

We showed by Theorem 1 , that for the two-person $<1,0>$-weight game, the equilibrium strategy in state $(x \mid n)$ is to choose

R-R, R-A and A-A, if $0 \leq x<\left(\bar{V}_{n-1}\right)^{\frac{1}{n-1}},\left(\bar{V}_{n-1}\right)^{\frac{1}{n-1}}<x<2^{-\frac{1}{n-1}}$ and $2^{-\frac{1}{n-1}}<x \leq 1$, respectively,
and the equilibrium values are $\left(V_{n}, \bar{V}_{n}\right)$, where $\left\{V_{n}\right\}$ is determined by the recursion (2.6). To derive the solutions to three-person $<1 / 2,1 / 2,0>$-weight and $<1,0,0>$-weight games are open problems.

Table 2: Decision points in the equal-weight games $\Gamma_{n}^{(3)}$ and $\Gamma_{n}^{(2)}$

|  | $\Gamma_{n}^{(3)}$ | $\Gamma_{n}^{(2)}$ |
| ---: | :---: | :---: |
|  | $\left(\frac{1}{3}\right)^{\frac{1}{n-1}}$ | $\left(\frac{1}{2}\right)^{\frac{1}{n-1}}$ |
| $\mathrm{n}=2$ | $0.3333(=1 / 3)$ | 0.5000 |
| 3 | $0.5774(=1 / \sqrt{3})$ | 0.7071 |
| 4 | 0.6934 | 0.7937 |
| 5 | 0.7598 | 0.8409 |
| 6 | 0.8027 | 0.8706 |
| 7 | 0.8327 | 0.8909 |
| 8 | 0.8548 | 0.9057 |
| 9 | 0.8717 | 0.9170 |
| 10 | 0.8851 | 0.9259 |
| 11 | 0.8960 | 0.9330 |
| 12 | 0.9050 | 0.9389 |
| $\vdots$ | $\vdots$ | $\vdots$ |

Table 3: Expected values of $m$ and $x$ in $\Gamma_{n}^{(3)}$

| Based on | Eq. (3.6) | Eq. (3.8) |
| ---: | :---: | :---: |
| n | $\mu_{n}$ | $\xi_{n}$ |
| 1 | 0.0000 | 0.0000 |
| 2 | $0.6667(=2 / 3)$ | $0.4444(=4 / 9)$ |
| 3 | 1.2302 | 0.5899 |
| 4 | 1.7729 | 0.6687 |
| 5 | 2.3078 | 0.7194 |
| 6 | 2.8388 | 0.7553 |
| 7 | 3.3678 | 0.7823 |
| 8 | 3.8956 | 0.8033 |
| 9 | 4.4220 | 0.8203 |
| 10 | 4.9481 | 0.8344 |
| 11 | 5.4737 | 0.8462 |
| 12 | 5.9989 | 0.8563 |
| $\vdots$ | $\vdots$ | $\vdots$ |

## 4. Selecting Better-than-opponent-No-information Case

A zero-sum no-information version of OSG is presented in this section. The problem is stated, similarly as in the begining of the previous Section 1, and hence only the difference is to be noted.
$\left(1^{+}\right)$Players I and II observe a sequence of $n$ independent r.v.s $\left\{r_{i}\right\}_{i=1}^{m}$ obeying the discrete distribution

$$
\operatorname{Pr}\left(Y_{i}=1\right)=\cdots=\operatorname{Pr}\left(Y_{i}=i\right)=i^{-1}
$$

$\left(2^{+}\right)$Observing each $Y_{i}$, players choose either A or R. If either player choose A, he obtains the reward

$$
g(i, y) \equiv \prod_{j=i+1}^{n}(1-y / j), \text { if } Y_{i}=y
$$

and drops out from the game. If both players choose A, the rule is the same as in $\left(2^{0}\right)$, with $X_{i}$ replaced by $g\left(i, Y_{i}\right)$.
$\left(3^{+}\right)$A player wins if he accepts a $Y_{i}$ that is absolutely better than the opponent's one, or if his opponent fails to accept any r.v. (c.f.: $g(i, y)=\operatorname{Pr}\{i$-th has the absolute rank $\left.y \mid Y_{i}=y\right\}$ ).

Define state $(i, y)$ to mean that (1) both players remain in the game, and (2) players currently face the r.v. $Y_{i}=y$. Let $V(i, y)$ be the value of the game in state $(i, y)$, for the $n$-problem. Note that $n$ is fixed throughout, players should choose A-A in state ( $n, y$ ) and hence draw of the game cannot occur.

The Optimality Equation is

$$
\begin{gather*}
\left.V(i, y)=v a l\left\{\begin{array}{c}
\mathrm{R} \\
\mathrm{R} \\
\mathrm{~A}
\end{array} \begin{array}{cc}
\mathrm{A} \\
\mu_{i+1} & 1-g(i, y) \\
g(i, y) & (w-\bar{w}) g(i, y)+\bar{w}
\end{array}\right]\right\}  \tag{4.1}\\
(i=1,2, \ldots, n-1 ; V(n, y)=w, \forall y \in[1, n])
\end{gather*}
$$

where

$$
\begin{align*}
g(i, y) & \equiv\left(1-\frac{y}{i+1}\right)\left(1-\frac{y}{i+2}\right) \ldots\left(1-\frac{y}{n}\right)=\binom{n-y}{i-y} /\binom{n}{i} \\
& =\binom{i}{y} /\binom{n}{y} \tag{4.2}
\end{align*}
$$

and

$$
\mu_{i+1} \equiv E V\left(i+1, Y_{i+1}\right) \equiv \frac{1}{i+1} \sum_{y^{\prime}=1}^{i+1} V\left(i+1, y^{\prime}\right)
$$

We prove
Theorem 3. The solution to our OSG described by $(4,1)-(4,2)$ for $1 / 2<w<1$, is as follows: The optimal strategy-pair in state ( $i, y$ ) is $R-R, R-A$ and $A-A$, if $0<g(i, y)<$ $\bar{\mu}_{i+1}, \bar{\mu}_{i+1}<g(i, y)<1 / 2$ and $1 / 2<g(i, y)<1$, respectively. The sequence $\left\{\mu_{n}\right\}$ is determined by the recursion

$$
\begin{gather*}
\mu_{i}=\frac{1}{i} \sum_{y=1}^{i}\left[\mu_{i+1} I\left\{g(i, y) \leq \bar{\mu}_{i+1}\right\}+(1-g(i, y)) I\left\{\bar{\mu}_{i+1}<g(i, y) \leq \frac{1}{2}\right\}\right. \\
\left.+\{(w-\bar{w}) g(i, y)+\bar{w}\} I\left\{g(i, y)>\frac{1}{2}\right\}\right] \\
\left(i=1,2, \ldots, n-1 ; \mu_{n} \equiv \frac{1}{n} \sum_{y=1}^{n} V(n, y)=w\right) \tag{4.3}
\end{gather*}
$$

Proof. Our common sense tells that $\mu_{i}>\frac{1}{2}, \forall i$. Moreover $g(i, y)$ is decreasing in $1 \leq y \leq i$ and $1-g(i, y)>(w-\bar{w}) g(i, y)+\bar{w} \Leftrightarrow g(i, y)<1 / 2$. Hence applying Lemma 1.1 in Section 2 to (4.1) we obtain

$$
V(i, y)=\left\{\begin{array}{lll}
\mu_{i+1}, & \text { if } 0<g(i, y)<\bar{\mu}_{i+1}, & \text { (R-R is optimal) }  \tag{4.4}\\
1-g(i, y), & \text { if } \bar{\mu}_{i+1}<g(i, y)<1 / 2, & \text { (R-A is optimal) } \\
(w-\bar{w}) g(i, y)+\bar{w}, & \text { if } \frac{1}{2}<g(i, y)<1, & \text { (A-A is optimal) }
\end{array}\right.
$$

from which (4.3) follows.

Figure 1: Optimal strategy-pair when $n=10$ and $w=3 / 4$


We illustrate the solution for the case when $n=10$ and $w=3 / 4$. First the values of $g(i, y), 1 \leq y \leq i \leq 10$, are computed by using

$$
g(i, y)=\frac{(i)_{y}}{(n)_{y}}=\frac{(n-y)_{n-i}}{(n)_{n-i}}=g(n-y, n-i)
$$

and

$$
g(i-1, y)=\left(1-\frac{y}{i}\right) g(i, y),(1 \leq y \leq i \leq n=10)
$$

The winning probabilities are computed from (4.3) from $\mu_{10}=3 / 4$ downward in $i$ until reaching $\mu_{1}=V(1,1)=0.578$. Figure 1 shows the optimal strategy-pair.

It is interesting to note the following two remarks, in connection with our result.

Figure 2: Optimal strategy-pair when $n=10$ and $w=1$


Remark 5. Enns and Ferenstein [2] obtained the result $\operatorname{Pr}(\mathrm{I}$ 's win $)=0.619$, and the optimal strategy-pair as shown by Figure 2, when $n=10$ and $w=1$.

Remark 6. For the case where $n=10$ and $w=3 / 4$, we have $\operatorname{Pr}(\mathrm{I}$ 's win $)=0.6097$ in Full-information case (see Table 1 in section 2), and $\operatorname{Pr}(\mathrm{I}$ 'swin $)=0.587$ in No-information case (See the example in Section 4).

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