# A NOTE ON PRIMITIVE EQUIVALENCE 

NÁNDOR SIEBEN

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#### Abstract

Primitive equivalence of graphs and matrices was used by Enomoto, Fujii and Watatani to classify Cuntz-Krieger algebras of $3 \times 3$ irreducible matrices. In this paper it is shown that the definition of primitive equivalence can be simplified using primitive transfers of matrices that involve only two rows of the matrix.


## 1. Introduction

Primitive equivalence of graphs and matrices was used by Enomoto, Fujii and Watatani [3] to classify Cuntz-Krieger algebras [1] of $3 \times 3$ irreducible matrices. It was shown by Drinen and the author [2] that a graph and its primitive transfer have isomorphic groupoids and therefore isomorphic $C^{*}$-algebras. Franks [4, Corollary 2.2] used a similar operation to find a canonical form for the flow equivalence class of a matrix. His definition is more general but involves only two columns of a matrix. Parry and Sullivan [5] also used a similar equivalence relation of graphs called expansion.

The purpose of this paper is to show that the definition of primitive equivalence can be simplified using primitive transfers that involve only two rows of the matrix. This fact can be used to simplify proofs about primitive equivalence in [3] and in [2]. The author thanks Doug Drinen and John Quigg for their help.

## 2. Preliminaries

A digraph $E$ is a pair $\left(E^{0}, E^{1}\right)$ of possibly infinite sets where $E^{1} \subset E^{0} \times E^{0} . E^{0}$ is called the set of vertices and $E^{1}$ is called the set of edges. We say that the edge $e=(v, w)$ starts at vertex $v$ and ends at vertex $w$. If $v=w$ then $e$ is called a loop. Note that a digraph has no multiple edges but it can have loops. The vertex matrix $A$ of a graph $E$ is an $E^{0}$ by $E^{0}$ matrix such that

$$
A_{v, w}= \begin{cases}1 & \text { if }(v, w) \in E^{1} \\ 0 & \text { else }\end{cases}
$$

There is a bijective correspondence between digraphs and $0-1$ square matrices. A connected component of a digraph is a maximal subgraph in which every two vertices can be connected by an undirected path.

If $A$ is a matrix then we denote the $i$-th row of $A$ by $A_{i}$ and we use the notation $E_{j}=(0, \ldots, 0,1,0, \ldots, 0)$ for a row which has a 1 in the $j$-th column and 0 in all the other columns.

Let $A$ be the vertex matrix of a digraph $E$, that is, a 0-1 square matrix. If $A_{p}=A_{m_{1}}+$ $\cdots+A_{m_{s}}+E_{k_{1}}+\cdots+E_{k_{r}}$ for some distinct $k_{1}, \ldots, k_{r}, m_{1}, \ldots, m_{s}$ and $p \notin\left\{m_{1}, \ldots, m_{s}\right\}$

[^0]then the $0-1$ matrix $B$ defined by
\[

B_{i, j}= $$
\begin{cases}A_{i, j} & \text { if } i \neq p \\ 1 & \text { if } i=p \text { and } j \in\left\{m_{1}, \ldots, m_{s}, k_{1}, \ldots, k_{r}\right\} \\ 0 & \text { else }\end{cases}
$$
\]

is called by [3] a primitive transfer of $A$ at $p$ (see also [2]).
We call the number of elements in $M=\left\{m_{1}, \ldots, m_{s}\right\}$ the size of the primitive transfer. Alternatively, $B$ can be defined as

$$
B_{i}= \begin{cases}A_{i} & \text { if } i \neq p \\ A_{p}-\sum_{m \in M} A_{m}+\sum_{m \in M} E_{m} & \text { if } i=p\end{cases}
$$

The digraph $F$, whose vertex matrix is $B$, is also called a primitive transfer of $E$.
Example 1. If
then $B$ is a primitive transfer of $A$ since $A_{1}=A_{3}+A_{4}+A_{6}+A_{7}+A_{8}$ and $B_{1}=A_{1}-$ $A_{3}-A_{4}-A_{6}-A_{7}-A_{8}+E_{1}+E_{3}+E_{4}+E_{6}+E_{7}+E_{8}$. The corresponding graphs are:


Definition 2. If $A$ and $B$ are the vertex matrices of the digraphs $E$ and $F$ then $A$ and $B$ are called primitively equivalent if there exist matrices $A=C_{1}, \ldots, C_{i}, \ldots, C_{n}=B$ such that for all $1 \leq i \leq n-1$ one of the following holds:
(i) $C_{i}$ is a primitive transfer of $C_{i+1}$;
(ii) $C_{i+1}$ is a primitive transfer of $C_{i}$;
(iii) $C_{i}=P C_{i+1} P^{-1}$ for some permutation matrix $P$.

The digraphs $E$ and $F$ are also called primitively equivalent.

## 3. Main theorem

The purpose of this paper is to show that in the definition of primitive equivalence we could consider only size 1 primitive transfers. To see this we are going to show that a primitive transfer can be replaced by a sequence of size 1 primitive transfers. First we need a few tools.

Definition 3. Let $E$ be a digraph and let $B$ be a primitive transfer of the vertex matrix $A$ of $E$, corresponding to the equation $A_{p}=\sum_{m \in M} A_{m}+\sum_{k \in K} E_{k}$. The graph of the primitive transfer is the subgraph of $E$ induced by $M$.

Note that the graph of a primitive transfer has no vertex with more than one incoming edge since $A_{p}$ contains only 0's and 1's, hence no two $A_{m}$ 's can have a 1 at the same location.

Example 4. The graph of the primitive transfer in Example 1 has three connected components:


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Lemma 5. If $G$ is the graph of a primitive transfer with size $s$ from the matrix $A$ to $B$ and there is an edge $(l, n)$ in $G$ which is not a loop, then there is a matrix $C$ such that $A$ is a size 1 primitive transfer of $C$ and $B$ is a size $s-1$ primitive transfer of $C$. If $H$ is the graph of the latter primitive transfer then $(l, m) \in H^{1}$ whenever $(n, m) \in G^{1}$.

Proof. Let $B$ be the primitive transfer of $A$ at $p$ corresponding to the equation $A_{p}=$ $\sum_{m \in M} A_{m}+\sum_{k \in K} E_{k}$. Note that $M$ and $K$ are disjoint and $p \notin M$. Define

$$
C_{i}= \begin{cases}A_{i} & \text { if } i \neq l \\ A_{l}+A_{n}-E_{n} & \text { if } i=l\end{cases}
$$

$C$ is a $0-1$ matrix since $A_{l}+A_{n}$ is a $0-1$ row and $A_{l, n}=1$. If $J=\left\{j: A_{l, j}=1\right.$ and $\left.j \neq n\right\}$ then the equation $C_{l}=A_{n}+\sum_{j \in J} E_{j}$ determines a primitive transfer of $C$ at $l$. This primitive transfer is $A$ since

$$
A_{i}= \begin{cases}C_{i}= & \text { if } i \neq l \\ C_{l}-C_{n}+E_{n} & \text { if } i=l\end{cases}
$$

The equation

$$
\begin{aligned}
C_{p} & =A_{p}=\sum_{m \in M} A_{m}+\sum_{k \in K} E_{k} \\
& =\sum_{m \in M \backslash\{l\}} C_{m}+\left(C_{l}-C_{n}+E_{n}\right)+\sum_{k \in K} E_{k} \\
& =\sum_{m \in M \backslash\{n\}} C_{m}+\sum_{k \in K \cup\{n\}} E_{k}
\end{aligned}
$$

determines another primitive transfer of $C$ at $p$. This primitive transfer is $B$ since $B_{i}=$ $A_{i}=C_{i}$ for $i \neq p$ and

$$
\begin{aligned}
B_{p} & =A_{p}-\sum_{m \in M} A_{m}+\sum_{m \in M} E_{m} \\
& =C_{p}-\sum_{m \in M \backslash\{n, l\}} C_{m}+\sum_{m \in M \backslash\{n\}} E_{m}-\left(A_{l}+A_{n}-E_{n}\right) \\
& =C_{p}-\sum_{m \in M \backslash\{n\}} C_{m}+\sum_{m \in M \backslash\{n\}} E_{m} .
\end{aligned}
$$

The last part of the lemma follows from the construction of $C$.
Note that in the previous lemma, $H$ is connected if $G$ is. This follows from the last sentence of the lemma and the fact that $(l, n)$ is the only edge in $G$ that ends at $n$.

Lemma 6. If $G$ is the graph of the primitive transfer from the matrix $A$ to $B$ and $F$ is one of several connected components of $G$, then there is a matrix $D$ such that $D$ is the primitive transfer of $A$ with graph $F$, and $B$ is the primitive transfer of $D$ with graph $H:=G \backslash F$.

Proof. Let the original primitive transfer correspond to the equation $A_{p}=\sum_{m \in M} A_{m}+$ $\sum_{k \in K} E_{k}$. Note that $M=F^{0} \cup H^{0}$ and $F^{0} \cap H^{0}=\emptyset$. If $J=\left\{j: A_{h, j}=1\right.$ for some $\left.h \in H^{0}\right\}$ then the equation

$$
A_{p}=\sum_{m \in F^{0}} A_{m}+\sum_{k \in K \cup J} E_{k}
$$

determines a primitive transfer of $A$ at $p$. Note that $F^{0}$ and $K \cup J$ are disjoint since $M \cap K=\emptyset$ and there is no edge from $H^{0}$ to $F^{0}$ in $G$. Let $D$ be this primitive transfer of $A$ at $p$, that is,

$$
D_{i}= \begin{cases}A_{i} & \text { if } i \neq p \\ A_{p}-\sum_{m \in F^{0}} A_{m}+\sum_{m \in F^{0}} E_{m} & \text { if } i=p\end{cases}
$$

The equation

$$
\begin{aligned}
D_{p} & =A_{p}-\sum_{m \in F^{0}} A_{m}+\sum_{m \in F^{0}} E_{m} \\
& =\sum_{m \in M} A_{m}+\sum_{k \in K} E_{k}-\sum_{m \in F^{0}} A_{m}+\sum_{m \in F^{0}} E_{m} \\
& =\sum_{m \in H^{0}} D_{m}+\sum_{k \in K \cup F^{0}} E_{k}
\end{aligned}
$$

determines a primitive transfer of $D$ at $p$. This primitive transfer is $B$ since if $i \neq p$ then $B_{i}=A_{i}=D_{i}$ and

$$
\begin{aligned}
B_{p} & =A_{p}-\sum_{m \in M} A_{m}+\sum_{m \in M} E_{m} \\
& =A_{p}-\sum_{m \in F^{0}}^{m} A_{m}-\sum_{m \in H^{0}} A_{m}+\sum_{m \in F^{0}} E_{m}+\sum_{m \in H^{0}} E_{m} \\
& =D_{p}-\sum_{m \in H^{0}} D_{m}+\sum_{m \in H^{0}} E_{m}
\end{aligned}
$$

The statement about the graphs of the primitive transfers is obvious.
We are now in position to show our main result.
Theorem 7. If the matrix $B$ is a primitive transfer of $A$ then there is a sequence of matrices $A=C_{1}, \ldots, C_{i}, \ldots, C_{n}=B$ such that for all $1 \leq i \leq n-1$ either $C_{i+1}$ is a size 1 primitive transfer of $C_{i}$ or $C_{i}$ is a size 1 primitive transfer of $C_{i+1}$.
Proof. By Lemma 6 there is a sequence of matrices $A=D_{1}, \ldots, D_{i}, \ldots, D_{m}=B$ such that for all $1 \leq i \leq m-1, D_{i+1}$ is the primitive transfer of $D_{i}$ and the graph of the primitive transfer is connected. Inductively applying Lemma 5 we can transform $D_{i}$ to $D_{i+1}$ for all $i$, using size 1 primitive transfers. By the note after Lemma 5, we never run out of non-loop edges.

## References

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Arizona State University, Math. Dept., Tempe, AZ 85287-1804, USA
E-mail address: nandor.sieben@asu.edu


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