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Q–FUZZY SUBALGEBRAS OF BCK/BCI–ALGEBRAS

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Abstract. Given a set Q, we introduce the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provide some appropriate examples. Using fuzzy subalgebras, we describe Q-fuzzy subalgebras. Conversely, we construct fuzzy subalgebras by using Q-fuzzy subalgebras. How the homomorphic images and inverse images of Q-fuzzy subalgebras become Q-fuzzy subalgebras is stated.

1. Introduction

The notion of BCK-algebras was proposed by Iami and Iséki in 1966. In the same year, Iséki [1] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [5]. In this paper, given a set Q, we introduce the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provide some appropriate examples. Using fuzzy subalgebras, we describe Q-fuzzy subalgebras. Conversely, we construct fuzzy subalgebras by using Q-fuzzy subalgebras. How the homomorphic images and inverse images of Q-fuzzy subalgebras become Q-fuzzy subalgebras is stated.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following axioms:

(I) $((x*y)*(x*z))*(z*y) = 0$,
(II) $(x*(x*y))*y = 0$,
(III) $x*x = 0$, and
(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$

for every $x, y, z \in X$. A BCI-algebra X satisfying the condition

(V) $0*x = 0$ for all $x \in X$

is called a *BCK-algebra*. A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. For further information on BCK/BCI-algebras, the reader refer to the textbook **BCK-algebras** (Meng and Jun [4]) and [1, 2, 3].

3. Q-fuzzy subalgebras

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Let X be a BCK/BCI-algebra. A fuzzy set A in X, i.e., a mapping $A : X \to [0,1]$, is called a *fuzzy subalgebra* of X if $A(x * y) \geq \min\{A(x), A(y)\}$ for all $x, y \in X$. Note that if A is a fuzzy subalgebra of a BCK/BCI-algebra X, then $A(0) \geq A(x)$ for all $x \in X$.

Proposition 3.1. *Let A be a fuzzy subalgebra of a BCK/BCI-algebra X. Define a fuzzy set B in X by $B(x) = \frac{A(x)}{A(0)}$ for all $x \in X$. Then B is a fuzzy subalgebra of X and $B(0) = 1$.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} B(x * y) &= \frac{1}{A(0)} A(x * y) \geq \frac{1}{A(0)} \min\{A(x), A(y)\} \\ &= \min\{\frac{A(x)}{A(0)}, \frac{A(y)}{A(0)}\} = \min\{B(x), B(y)\}. \end{aligned}$$

Hence B is a fuzzy subalgebra of X, and clearly $B(0) = 1$. □

According to Proposition 3.1, we may suppose that a fuzzy subalgebra A of a BCK/BCI-algebra X satisfies $A(0) = 1$.

In what follows, let Q and X denote a set and a BCK/BCI-algebra, respectively, unless otherwise specified. A mapping $H : X \times Q \to [0,1]$ is called a *Q-fuzzy set* in X.

Definition 3.2. A Q-fuzzy set $H : X \times Q \to [0,1]$ is called a *fuzzy subalgebra* of X over Q (briefly, *Q-fuzzy subalgebra of X*) if $H(x * y, q) \geq \min\{H(x, q), H(y, q)\}$ for all $x, y \in X$ and $q \in Q$.

Example 3.3. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a Q-fuzzy set H in X as follows: for every $q \in Q$, $H(0, q) = H(b, q) = 0.6$ and $H(a, q) = H(c, q) = 0.2$. It is easy to verify that H is a Q-fuzzy subalgebra of X.

Example 3.4. Consider a BCI-algebra $X = \{0, x\}$ with Cayley table as follows (Iséki [2]):

$*$	0	x
0	0	x
x	x	0

Let $Q = \{1, 2\}$ and let H be a Q-fuzzy set in X defined by $H(0, 1) = H(0, 2) = 1$, $H(x, 1) = 0.8$ and $H(x, 2) = 0.5$. It is easy to verify that H is a Q-fuzzy subalgebra of X.

Example 3.5. Let X be a BCK/BCI-algebra and let

$$Q = \{A \mid A \text{ is a fuzzy subalgebra of } X\}.$$

Let H be a mapping from $X \times Q$ into $[0,1]$ defined by $H(x, A) = A(x)$ for all $x \in X$ and $A \in Q$. Then H is a Q-fuzzy subalgebra of X.

Note that for $q \in Q$, if H is a q-fuzzy subalgebra of X, then

$$H(0, q) = H(x * x, q) \geq \min\{H(x, q), H(x, q)\} = H(x, q)$$

for all $x \in X$.

Proposition 3.6. *Let H be a Q-fuzzy subalgebra of X. Define a Q-fuzzy set G in X by $G(x,q) = \frac{H(x,q)}{H(0,q)}$ for all $x \in X$ and $q \in Q$. Then G is a Q-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned} G(x*y,q) &= \frac{H(x*y,q)}{H(0,q)} \geq \frac{1}{H(0,q)} \min\{H(x,q), H(y,q)\} \\ &= \min\{\frac{H(x,q)}{H(0,q)}, \frac{H(y,q)}{H(0,q)}\} = \min\{G(x,q), G(y,q)\}. \end{aligned}$$

Hence G is a Q-fuzzy subalgebra of X. □

Let X^Q denote the collection of all functions from Q into X, and define a binary operation $\circledast$ on X^Q by

$$(u \circledast v)(q) = u(q) * v(q)$$

for all $u, v \in X^Q$ and $q \in Q$. Then $(X^Q, \circledast, \theta)$ is a BCK/BCI-algebra, where θ is the zero map in X^Q, i.e., $\theta(q) = 0$ for all $q \in Q$.

Proposition 3.7. *Let A be a fuzzy subalgebra of X and let H be a mapping from $X^Q \times Q$ into $[0,1]$ defined by $H(u,q) = A(u(q))$ for all $u \in X^Q$ and $q \in Q$. Then H is a Q-fuzzy subalgebra of X^Q.*

Proof. For any $u, v \in X^Q$, we have

$$\begin{aligned} H(u \circledast v, q) &= A((u \circledast v)(q)) = A(u(q) * v(q)) \\ &\geq \min\{A(u(q)), A(v(q))\} \\ &= \min\{H(u,q), H(v,q)\}. \end{aligned}$$

Hence H is a Q-fuzzy subalgebra of X^Q.

Proposition 3.8. *Let H be a Q-fuzzy subalgebra of X. For any $q \in Q$, define $H_q : X \to [0,1]$ by $H_q(x) = H(x,q)$ for all $x \in X$. Then H_q is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$H_q(x*y) = H(x*y,q) \geq \min\{H(x,q), H(y,q)\} = \min\{H_q(x), H_q(y)\}.$$

Hence H_q is a fuzzy subalgebra of X. □

We now consider the converse of Proposition 3.8.

Proposition 3.9. *Let H_q, $q \in Q$, be a fuzzy subalgebra of X. Let H be a Q-fuzzy set in X defined by $H(x,q) = H_q(x)$ for all $x \in X$ and $q \in Q$. Then H is a Q-fuzzy subalgebra of X.*

Proof. For every $x, y \in X$ and $q \in Q$, we have

$$H(x*y,q) = H_q(x*y) \geq \min\{H_q(x), H_q(y)\} = \min\{H(x,q), H(y,q)\}.$$

Thus H is a Q-fuzzy subalgebra of X. □

Proposition 3.10. *Let Ω be a subalgebra of X^Q. Then for any $q \in Q$, the set $\Omega_q := \{u(q) \mid u \in \Omega\}$ is a subalgebra of X.*

Proof. For any $q \in Q$, let $u(q), v(q) \in \Omega_q$. Then $u(q) * v(q) = (u \circledast v)(q) \in \Omega_q$ since $u \circledast v \in \Omega$. Hence Ω_q, $q \in Q$, is a subalgebra of X. □

Theorem 3.11. *Let A be a fuzzy subalgebra of X^Q. Define a mapping*

$$H : X \times Q \to [0,1] \text{ by } H(x,q) := \sup\{A(u) \mid u \in X^Q, u(q) = x\}$$

for all $x \in X$ and $q \in Q$. Then H is a Q-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned} H(x * y, q) &= \sup\{A(u) \mid u \in X^Q, u(q) = x * y\} \\ &\geq \sup\{A(u \circledast v) \mid u, v \in X^Q, u(q) = x, v(q) = y\} \\ &\geq \sup\{\min\{A(u), A(v)\} \mid u, v \in X^Q, u(q) = x, v(q) = y\} \\ &= \min\{\sup\{A(u) \mid u \in X^Q, u(q) = x\}, \sup\{A(v) \mid v \in X^Q, v(q) = y\}\} \\ &= \min\{H(x,q), H(y,q)\}. \end{aligned}$$

Hence H is a Q-fuzzy subalgebra of X. □

Example 3.12. Let $X = \{0, x\}$ be a BCI-algebra in Example 3.4 and let $Q = \{1, 2\}$. Then $X^Q := \{\theta, u, v, w\}$, where $\theta(1) = \theta(2) = 0$, $u(1) = u(2) = x$, $v(1) = 0$, $v(2) = x$, $w(1) = x$ and $w(2) = 0$, is a BCI-algebra under the following Cayley table:

$\circledast$	θ	u	v	w
θ	θ	u	v	w
u	u	θ	w	v
v	v	w	θ	u
w	w	v	u	θ

Let A be a fuzzy subalgebra of X^Q defined by $A(\theta) = 0.8$, $A(u) = A(v) = 0.3$ and $A(w) = 0.7$. Then we can obtain a Q-fuzzy subalgebra of X as follows:

$$\begin{aligned} H(0,1) &= \sup\{A(u) \mid u \in X^Q, u(1) = 0\} \\ &= \sup\{A(\theta), A(v)\} = \sup\{0.8, 0.3\} = 0.8, \end{aligned}$$

and similarly we have $H(0,2) = 0.8$, $H(x,1) = 0.7$ and $H(x,2) = 0.3$.

Theorem 3.13. *Let H be a Q-fuzzy subalgebra of X and let A be a fuzzy set in X^Q defined by $A(u) := \inf\{H(u(q), q) \mid q \in Q\}$ for all $u \in X^Q$. Then A is a fuzzy subalgebra of X^Q.*

Proof. Let $u, v \in X^Q$. Then

$$\begin{aligned} A(u \circledast v) &= \inf\{H((u \circledast v)(q), q) \mid q \in Q\} \\ &= \inf\{H(u(q) * v(q), q) \mid q \in Q\} \\ &\geq \inf\{\min\{H(u(q), q), H(v(q), q)\} \mid q \in Q\} \\ &= \min\{\inf\{H(u(q), q) \mid q \in Q\}, \inf\{H(v(q), q) \mid q \in Q\}\} \\ &= \min\{A(u), A(v)\}. \end{aligned}$$

Therefore A is a fuzzy subalgebra of X^Q. □

Example 3.14. Let H be a Q-fuzzy subalgerba of X in Example 3.4. Then we can induce a fuzzy subalgebra A of X^Q as follows:

$$\begin{aligned} A(\theta) &= \inf\{H(\theta(q), q) \mid q \in Q\} \\ &= \inf\{H(\theta(1), 1), H(\theta(2), 2)\} = 1 \end{aligned}$$

and similarly we obtain $A(u) = A(v) = 0.5$ and $A(w) = 0.8$, where X^Q is a BCI-algebra in Example 3.12.

Definition 3.15. Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras and let H be a Q-fuzzy set in Y. Then the *inverse image* of H, denoted by $f^{-1}[H]$, is the Q-fuzzy set in X given by $f^{-1}[H](x, q) = H(f(x), q)$ for all $x \in X$ and $q \in Q$. Conversely, let G be a Q-fuzzy set in X. The *image* of G, written as $f[G]$, is a Q-fuzzy set in Y defined by

$$f[G](y, q) = \begin{cases} \sup_{z \in f^{-1}(y)} G(z, q) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$ and $q \in Q$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.16. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If H is a Q-fuzzy subalgebra of Y, then the inverse image $f^{-1}[H]$ of H is a Q-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned} f^{-1}[H](x * y, q) &= H(f(x * y), q) = H(f(x) * f(y), q) \\ &\geq \min\{H(f(x), q), H(f(y), q)\} \\ &= \min\{f^{-1}[H](x, q), f^{-1}[H](y, q)\}. \end{aligned}$$

Hence $f^{-1}[H]$ is a Q-fuzzy subalgebra of X. $\square$

Theorem 3.17. *Let $f : X \to Y$ be a homomorphism between BCK/BCI-algebras X and Y. If G is a Q-fuzzy subalgebra of X, then the image $f[G]$ of G is a Q-fuzzy subalgebra of Y.*

Proof. We first prove that

$$(*1) \qquad f^{-1}(y_1) * f^{-1}(y_2) \subseteq f^{-1}(y_1 * y_2)$$

for all $y_1, y_2 \in Y$. For, if $x \in f^{-1}(y_1) * f^{-1}(y_2)$, then $x = x_1 * x_2$ for some $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Since f is a homomorphism, it follows that $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$ so that $x \in f^{-1}(y_1 * y_2)$. Hence $(*1)$ holds. Now let $y_1, y_2 \in Y$ and $q \in Q$. Assume that $y_1 * y_2 \notin \mathrm{Im}(f)$. Then $f[G](y_1 * y_2, q) = 0$. But if $y_1 * y_2 \notin \mathrm{Im}(f)$, i.e., $f^{-1}(y_1 * y_2) = \emptyset$, then $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$ by $(*1)$. Thus $f[G](y_1, q) = 0$ or $f[G](y_2, q) = 0$, and so

$$f[G](y_1 * y_2, q) = 0 = \min\{f[G](y_1, q), f[G](y_2, q)\}.$$

Suppose that $f^{-1}(y_1 * y_2) \neq \emptyset$. Then we should consider two cases as follows:

(i) $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$,

(ii) $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$.

For the case (i), we have $f[G](y_1, q) = 0$ or $f[G](y_2, q) = 0$, and so

$$f[G](y_1 * y_2, q) \geq 0 = \min\{f[G](y_1, q), f[G](y_2, q)\}.$$

Case (ii) implies from (∗1) that

$$\begin{aligned}
f[G](y_1 * y_2, q) &= \sup_{z \in f^{-1}(y_1 * y_2)} G(z, q) \geq \sup_{z \in f^{-1}(y_1) * f^{-1}(y_2)} G(z, q) \\
&= \sup_{x_1 \in f^{-1}(y_1),\, x_2 \in f^{-1}(y_2)} G(x_1 * x_2, q) \\
&\geq \sup_{x_1 \in f^{-1}(y_1),\, x_2 \in f^{-1}(y_2)} \min\{G(x_1, q), G(x_2, q)\} \\
&= \min\{\sup_{x_1 \in f^{-1}(y_1)} G(x_1, q), \sup_{x_2 \in f^{-1}(y_2)} G(y_2, q)\} \\
&= \min\{f[G](y_1, q), f[G](y_2, q)\}.
\end{aligned}$$

Hence $f[G](y_1 * y_2, q) \geq \min\{f[G](y_1, q), f[G](y_2, q)\}$ for all $y_1, y_2 \in Y$ and $q \in Q$. This completes the proof. □

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