ON CLASSES OF OPERATORS RELATED TO PARANORMAL OPERATORS

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Abstract. Recently, Furuta-Ito-Yamazaki [16] and Fujii-Nakamoto [11] introduce new classes of operators which are derived from the Furuta inequality. We will give some properties and generalized results for these classes.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *positive*, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if

$$(TT^*)^p \leq (T^*T)^p \tag{1.1}$$

for $p > 0$([1]). It is clear that 1-hyponormal operator is a hyponormal operator, and also it has been studied by many authors([1],[4] and [7]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *paranormal* if

$$||T^2x||||x|| \geq ||Tx||^2 \tag{1.2}$$

for all $x \in \mathcal{H}$([2],[12] and [13]). It is introduced as an intermediate class between hyponormal operators and normaloid ones. In [13], it is proved that every paranormal operator is normaloid. For $p > 0$, the p-paranormality was introduced by Fujii-Izumino-Nakamoto [8] and the p-paranormality is based on the fact that $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal[7, Lemma 1]. Actually, $T = U|T|$ is p-paranormal if and only if $S = U|T|^p$ is paranormal.

We have to state the order-preserving operator inequality because it is a base of our discussion in the below([14]).

The Furuta inequality. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q} \tag{1.3}$$

holds for $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

We denote by $A > 0$ if a positive operator A is invertible. For $A, B > 0$, $A \gg B$ if $\log A \geq \log B$([10]).

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Theorem A. *The following statements are mutually equivalent for $A, B > 0$:*

(1) $A \gg B$, *i.e.*, $\log A \geq \log B$.
(2) $(B^p A^{2p} B^p)^{\frac{1}{2}} \geq B^{2p}$ *for all* $p > 0$.
(3) $(B^r A^{2p} B^r)^{\frac{r}{p+r}} \geq B^{2r}$ *for all* $p, r > 0$.

We note that (2) is due to Ando [3] and (3) in [5], and that (3) is regarded as "the Furuta inequality for chaotic order". Based on the above operator inequalities, Furuta-Ito-Yamazaki [16] introduced absolute p-paranormality as an extension of paranormality. In addition, they proved that every paranormal operator is absolute p-paranormal operator for $p \geq 1$ and every absolute p-paranormal for $p > 0$ is normaloid.

Recently, Fujii-Jung-Lee-Lee-Nakamoto [9] proved that the p-paranormality has the monotone increasing property and every p-paranormal operator is normaloid. Very recently, Fujii-Nakamoto [11] introduced the (p, r, q)-paranormality and absolute (p, r)-paranormality using Furuta inequality and they gave some relations of those classes of operators. In this note, we will give some generalization properties of (p, r, q)-paranormal and absolute (p, r)-paranormal operators.

2. (p, r, q)**-paranormal operators.** In this section, we will consider the (p, r, q)-paranormality and the p-paranormality.

First, we recall definitions due to Furuta-Ito-Yamazaki and Fujii-Nakamoto([16] and [11]).

Definition. Let $T = U|T|$ be the polar decomposition of T and let $p, r, q > 0$
(1) An operator T on $\mathcal{H}$ is *p-paranormal* if

$$|||T|^p U |T|^p x|| \geq |||T|^p x||^2 \tag{2.1}$$

for all unit vectors $x \in \mathcal{H}$.
(2) An operator T on $\mathcal{H}$ is *(p, r, q)-paranormal* if

$$|||T|^p U |T|^r x||^{\frac{1}{q}} \geq |||T|^{\frac{p+r}{q}} x|| \tag{2.2}$$

for all unit vectors $x \in \mathcal{H}$.
(3) An operator T on $\mathcal{H}$ is *absolute p-paranormal* if it satisfies

$$|||T|^p T x|| \geq |||T| x||^{p+1} \tag{2.3}$$

for all unit vectors $x \in \mathcal{H}$.
(4) An operator T on $\mathcal{H}$ is *absolute (p, r)-paranormal* if it is $(p, r, p + r)$-paranormal, i.e., it satisfies

$$|||T|^p U |T|^r x|| \geq |||T| x||^{p+r} \tag{2.4}$$

for all unit vectors $x \in \mathcal{H}$.

We know that every p-paranormal operator is paranormal for $1 \geq p > 0$. Clearly, every $(p, p, 2)$-paranormal operator is p-paranormal operator. That is, the (p, r, q)-paranormality is a generalization of the p-paranormality. And it is easily seen that the $(p, 1, p + 1)$-paranormality is absolute p-paranormality. In [11, Theorem 4.1], Fujii-Nakamoto proved that the (p, r, q)-paranormality is monotone increasing on $q \geq 1$ and the $(p, r, 1)$-paranormality is monotone increasing on $r > 0$.

For the sake of convenience, we cite the following Hölder-McCarthy inequality.

Hölder-McCarthy inequality. *For $A \geq 0$ on $\mathcal{H}$, the following inequalities hold for all $x \in \mathcal{H}$*

$$(Ax,x)^r \geq \|x\|^{2(r-1)}(A^r x,x) \quad \text{if } 0 \leq r \leq 1 \tag{2.5}$$

and

$$(Ax,x)^r \leq \|x\|^{2(r-1)}(A^r x,x) \quad \text{if } r \geq 1. \tag{2.6}$$

Consequently, if $0 < t \leq s$ and $\|x\| = 1$, then

$$\|A^t x\|^s \leq \|A^s x\|^t. \tag{2.7}$$

The following theorem gives a monotonicity for the (p,r,q)-paranormality as a generalization of [11, Theorem 4.1].

Theorem 2.1. *The (p,r,q)-paranormality has monotone increasing property on $r > 0$.*

Proof. Suppose that T is (p,r,q)-paranormal and $\epsilon > 0$, $q \geq 1$ with $p+r+\epsilon \geq \epsilon q$. It suffices to show that T is $(p,r+\epsilon,q)$-paranormal. For a given unit vector $x \in \mathcal{H}$, it follows from (2.5) that

$$\begin{aligned}
&\||T|^p U|T|^{r+\epsilon}x\| \\
&= \||T|^p U|T|^r \frac{|T|^\epsilon x}{\||T|^\epsilon x\|}\| \cdot \||T|^\epsilon x\| \\
&\geq \||T|^{\frac{p+r}{q}}|T|^\epsilon x\|^q \||T|^\epsilon x\|^{1-q} \\
&= \||T|^{\frac{p+r+\epsilon}{q}\frac{p+r+q\epsilon}{p+r+\epsilon}} x\|^q \||T|^\epsilon x\|^{1-q} \\
&\geq \||T|^{\frac{p+r+\epsilon}{q}} x\|^{\frac{q(p+r+q\epsilon)}{p+r+\epsilon}} \||T|^\epsilon x\|^{1-q} \\
&= \||T|^{\frac{p+r+\epsilon}{q}} x\|^q \||T|^{\frac{p+r+\epsilon}{q}} x\|^{\frac{(q-1)q\epsilon}{p+r+\epsilon}} \||T|^\epsilon x\|^{1-q} \\
&\geq \||T|^{\frac{p+r+\epsilon}{q}} x\|^q \||T|^\epsilon x\|^{\frac{p+r+\epsilon}{q\epsilon}\frac{(q-1)q\epsilon}{p+r+\epsilon}} \||T|^\epsilon x\|^{1-q} \\
&= \||T|^{\frac{p+r+\epsilon}{q}} x\|^q.
\end{aligned}$$

Thus T is $(p,r+\epsilon,q)$-paranormal.

Theorem 2.2. *If T is (p,r,q)-paranormal, then T is (p',r',q')-paranormal for $p' \geq p$ and $q' \geq q \geq 1$, $r' \geq r$ with*

$$\frac{q'}{q} \geq \frac{p'+r'}{p+r+q(r'-r)} \tag{2.8}$$

and

$$p'+r' \geq q'r'. \tag{2.9}$$

Proof. Suppose that T is (p,r,q)-paranormal and $p' \geq p$, $q' \geq q \geq 1$, $r' = r+\epsilon, \epsilon \geq 0$ with $\frac{q'}{q} \geq \frac{p'+r+\epsilon}{p+r+q\epsilon}$ and $p'+r+\epsilon \geq q'(r+\epsilon)$. For a given unit vector $x \in \mathcal{H}$, it follows from (2.5) that

$$\begin{aligned}
&\||T|^{p'}U|T|^{r+\epsilon}x\| \\
&= \|(|T|^p)^{\frac{p'}{p}}\frac{U|T|^{r+\epsilon}x}{\|U|T|^{r+\epsilon}x\|}\|\cdot\|U|T|^{r+\epsilon}x\| \\
&\geq \||T|^pU|T|^r\frac{|T|^\epsilon x}{\||T|^\epsilon x\|}\|^{\frac{p'}{p}}\||T|^{r+\epsilon}x\|^{1-\frac{p'}{p}}\||T|^\epsilon x\|^{\frac{p'}{p}} \\
&\geq \||T|^{\frac{p+r}{q}}|T|^\epsilon x\|^{\frac{p'q}{p}}\||T|^{r+\epsilon}x\|^{\frac{p-p'}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}} \qquad \text{since } T \text{ is } (p,r,q)\text{-paranormal} \\
&= \||T|^{\frac{p'+r+\epsilon}{q'}\frac{q'(p+r+q\epsilon)}{q(p'+r+\epsilon)}}x\|^{\frac{p'q}{p}}\||T|^{r+\epsilon}x\|^{\frac{p-p'}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}} \\
&\geq \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{\frac{p'q'(p+r+q\epsilon)}{p(p'+r+\epsilon)}}\||T|^{r+\epsilon}x\|^{\frac{p-p'}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}} \qquad \text{by } \frac{p+r+\epsilon q}{p'+r+\epsilon}\geq\frac{q}{q'} \\
&= \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'}\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'(\frac{p'(p+r+q\epsilon)}{p(p'+r+\epsilon)}-1)}\||T|^{r+\epsilon}x\|^{\frac{p-p'}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}}.
\end{aligned}$$

Furthermore it follows from (2.9) that

$$\||T|^{\frac{p'+r+\epsilon}{q'}}x\| \geq \||T|^{r+\epsilon}x\|^{\frac{p'+r+\epsilon}{q'(r+\epsilon)}},$$

so that by $\frac{p-p'}{p}\leq 0$

$$\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{\frac{q'(r+\epsilon)}{p'+r+\epsilon}\cdot\frac{p-p'}{p}} \leq \||T|^{r+\epsilon}x\|^{\frac{p-p'}{p}}.$$

Hence it implies that

$$\begin{aligned}
&\||T|^{p'}U|T|^{r+\epsilon}x\| \\
&\geq \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'}\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'(\frac{p'(p+r+q\epsilon)}{p(p'+r+\epsilon)}-1)}\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{\frac{q'(r+\epsilon)}{p'+r+\epsilon}\cdot\frac{p-p'}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}} \\
&= \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'}\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{\frac{q'p'\epsilon(q-1)}{p(p'+r+\epsilon)}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}}.
\end{aligned}$$

Since $p'+r+\epsilon\geq q'(r+\epsilon)\geq q'\epsilon$, we have

$$\||T|^{\frac{p'+r+\epsilon}{q'}}x\| \geq \||T|^\epsilon x\|^{\frac{p'+r+\epsilon}{q'\epsilon}},$$

so that

$$\||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{\frac{q'p'\epsilon(q-1)}{p(p'+r+\epsilon)}} \geq \||T|^\epsilon x\|^{\frac{p'+r+\epsilon}{q'\epsilon}\cdot\frac{q'p'\epsilon(q-1)}{p(p'+r+\epsilon)}} = \||T|^\epsilon x\|^{\frac{p'(q-1)}{p}}$$

by $q\geq 1$. Consequently,

$$\begin{aligned}
\||T|^{p'}U|T|^{r+\epsilon}x\| &\geq \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'}\||T|^\epsilon x\|^{\frac{p'(q-1)}{p}}\||T|^\epsilon x\|^{\frac{p'(1-q)}{p}} \\
&= \||T|^{\frac{p'+r+\epsilon}{q'}}x\|^{q'},
\end{aligned}$$

which implies that T is (p',r',q')-paranormal.

Theorem 2.3. *If T is (p,r,q)-paranormal, then T is (p',r,q')-paranormal for $p'\geq p$ and $q'\geq q$ with $\frac{p+r}{q}\geq\frac{p'+r}{q'}\geq r$.*

Proof. Suppose that T is (p,r,q)-paranormal and $p'\geq p$, $q'\geq q$ with $\frac{p+r}{q}\geq\frac{p'+r}{q'}\geq r$. For a given unit vector $x\in\mathcal{H}$,

$$\begin{aligned}
|||T|^{p'}U|T|^r x|| &= ||(|T|^p)^{\frac{p'}{p}}\frac{U|T|^r x}{||U|T|^r x||}||\cdot||U|T|^r x||\\
&\geq |||T|^p U|T|^r x||^{\frac{p'}{p}}\frac{||U|T|^r x||}{||U|T|^r x||^{\frac{p'}{p}}}\\
&\geq |||T|^{\frac{p+r}{q}}x||^{\frac{p'q}{p}}|||T|^r x||^{1-\frac{p'}{p}} \text{ since } T \text{ is } (p,r,q)\text{-paranormal}\\
&= |||T|^{\frac{p'+r}{q'}\frac{(p+r)q'}{(p'+r)q}}x||^{\frac{p'q}{p}}||T|^r x||^{1-\frac{p'}{p}}\\
&\geq |||T|^{\frac{p'+r}{q'}}x||^{\frac{p'q}{p}\frac{(p+r)q'}{(p'+r)q}}|||T|^r x||^{1-\frac{p'}{p}}\\
&= |||T|^{\frac{p'+r}{q'}}x||^{q'}|||T|^{\frac{p'+r}{q'}}x||^{q'(\frac{p'(p+r)}{p(p'+r)}-1)}|||T|^r x||^{1-\frac{p'}{p}}\\
&\geq |||T|^{\frac{p'+r}{q'}}x||^{q'}|||T|^r x||^{\frac{p'+r}{rq'}\cdot\frac{q'r(p'-p)}{p(p'+r)}}|||T|^r x||^{\frac{p-p'}{p}} \text{ by } \frac{p'+r}{rq'}\geq 1\\
&= |||T|^{\frac{p'+r}{q'}}x||^{q'}.
\end{aligned}$$

Thus T is (p',r,q')-paranormal for $p'\geq p$ and $q'\geq q$ with $\frac{p+r}{q}\geq\frac{p'+r}{q'}\geq r$.

Theorem 2.4. *If T is (p,r,q)-paranormal for $q\geq\max\{p,r\}$ and $p+r\geq qr$, then T is q-paranormal.*

Proof. If $q\leq 1$, by [11, Theorem 4.2], T is $\max\{p,r\}$-paranormal. Since $q\geq\max\{p,r\}$, T is q-paranormal. Suppose that T is (p,r,q)-paranormal for $q\geq\max\{1,p,r\}$ and $p+r\geq qr$. Let $x\in\mathcal{H}$ be a unit vector. Then we have

$$\begin{aligned}
&|||T|^q U|T|^q x||\\
&= ||(|T|^p)^{\frac{q}{p}}\frac{U|T|^q x}{||U|T|^q x||}||\cdot||U|T|^q x||\\
&\geq |||T|^p U|T|^q x||^{\frac{q}{p}}|||T|^q x||^{1-\frac{q}{p}}\\
&= |||T|^p U|T|^r\frac{|T|^{q-r}x}{|||T|^{q-r}x||}||^{\frac{q}{p}}|||T|^q x||^{\frac{p-q}{p}}|||T|^{q-r}x||^{\frac{q}{p}}\\
&\geq |||T|^{\frac{p+r}{q}}|T|^{q-r}x||^{\frac{q^2}{p}}|||T|^{q-r}x||^{\frac{q}{p}(1-q)}|||T|^q x||^{\frac{p-q}{p}} \qquad \text{since } T \text{ is } (p,r,q)\text{-paranormal}\\
&= |||T|^{\frac{q^2+p+r-qr}{q}}x||^{\frac{q^2}{p}}|||T|^{q-r}x||^{\frac{q(1-q)}{p}}|||T|^q x||^{\frac{p-q}{p}}\\
&\geq |||T|^q x||^{\frac{q^2+p+r-qr}{p}}|||T|^{q-r}x||^{\frac{q(1-q)}{p}}|||T|^q x||^{\frac{p-q}{p}}\\
&= |||T|^q x||^2|||T|^q x||^{\frac{q^2-q+r-qr}{p}}|||T|^{q-r}x||^{\frac{q(1-q)}{p}}.
\end{aligned}$$

Moreover we have

$$|||T|^{q-r}x||\leq|||T|^q x||^{\frac{q-r}{q}}$$

and so $q\geq 1$ implies

$$|||T|^{q-r}x||^{\frac{q(1-q)}{p}}\geq|||T|^q x||^{\frac{q-r}{q}\cdot\frac{q(1-q)}{p}}=|||T|^q x||^{\frac{(q-r)(1-q)}{p}}.$$

Therefore it follows that

$$|||T|^qU|T|^qx|| \geq |||T|^qx||^2|||T|^qx||^{\frac{q^2-q+r-qr}{p}+\frac{(q-r)(1-q)}{p}} = |||T|^qx||^2.$$

Remark 2.5. It is not known that the monotonicity of (p,r,q)-paranormality on p by Theorem 2.2. Indeed, if $p' > p > 0, r = r'$ and $q = q'$, then such p, p', r, and q do not satisfy the conditions (2.8) and (2.9) of Theorem 2.2.

3. Absolute (p,r)-paranormal operators. In this section, we will give some relations between absolute p-paranormality and absolute (p,r)-paranormality. The following is a generalization of [11].

Theorem 3.1.

(1) *For $r > 1$, if T is absolute p-paranormal, then T is absolute (p,r)-paranormal.*
(2) *For $0 < r < 1$ and $1 \leq p+r$, if T is absolute (p,r)-paranormal, then T is absolute p-paranormal.*

Proof. (1) Suppose that T is absolute p-paranormal and $r > 1$. For a given unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
|||T|^pU|T|^rx|| &= |||T|^pU|T|\frac{|T|^{r-1}x}{|||T|^{r-1}x||}|| \cdot |||T|^{r-1}x|| \\
&\geq |||T||T|^{r-1}x||^{p+1}|||T|^{r-1}x||^{-p} \quad \text{since } T \text{ is absolute } p\text{-paranormal} \\
&= |||T|^rx||^{\frac{p+r}{r}}|||T|^rx||^{(p+r)(1-\frac{1}{r})}|||T|^rx||^{1-r}|||T|^{r-1}x||^{-p} \\
&= |||T|^rx||^{\frac{p+r}{r}}|||T|^rx||^{\frac{(r-1)p}{r}}|||T|^{r-1}x||^{-p} \\
&\geq |||T|x||^{p+r}|||T|^{r-1}x||^{\frac{r}{r-1}\frac{(r-1)p}{r}}|||T|^{r-1}x||^{-p} \quad \text{by } r > 1 \text{ and } \frac{r}{r-1} \geq 1 \\
&= |||T|x||^{p+r}.
\end{aligned}$$

(2) Suppose that T is absolute (p,r)-paranormal and $0 < r < 1$. For a given unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
&|||T|^pTx|| \\
&= |||T|^pU|T|x|| \\
&= |||T|^pU|T|^r\frac{|T|^{1-r}x}{|||T|^{1-r}x||}|| \cdot |||T|^{1-r}x|| \\
&\geq |||T||T|^{1-r}x||^{p+r}|||T|^{1-r}x||^{1-p-r} \quad \text{since } T \text{ is absolute } (p,r)\text{-paranormal} \\
&= |||T|^{2-r}x||^{\frac{p+1}{2-r}}|||T|^{2-r}x||^{(p+1)(1-\frac{1}{2-r})}|||T|^{2-r}x||^{r-1}|||T|^{1-r}x||^{1-p-r} \\
&= |||T|^{2-r}x||^{\frac{p+1}{2-r}}|||T|^{\frac{2-r}{1-r}(1-r)}x||^{\frac{(1-r)(p+r-1)}{2-r}}|||T|^{1-r}x||^{1-p-r} \\
&\geq |||T|x||^{p+1}|||T|^{1-r}x||^{\frac{2-r}{1-r}\frac{(1-r)(p+r-1)}{2-r}}|||T|^{1-r}x||^{1-p-r} \quad \text{by } 2-r > 1 \text{ and } \frac{2-r}{1-r} \geq 1 \\
&= ||Tx||^{p+1}.
\end{aligned}$$

In [11, Theorem 4.3], it is proved that every $(p,r,1)$-paranormal for $p+r \geq 1$ is absolute (p,r)-paranormal. Next we prove that every (p,r,q)-paranormal operator for $p+r \geq q$ is absolute (p,r)-paranormal.

Theorem 3.2.

(1) *For $p+r \geq q$, if T is (p,r,q)-paranormal, then T is absolute (p,r)-paranormal.*
(2) *For $p+r \leq q$, if T is absolute (p,r)-paranormal, then T is (p,r,q)-paranormal.*

Proof. For a given unit vector $x \in \mathcal{H}$,

$$\||T|^p U|T|^r x\| \geq \||T|^{\frac{p+r}{q}} x\|^q \geq \||T|x\|^{p+r},$$

and

$$\||T|^p U|T|^r x\| \geq \||T|x\|^{p+r} = \||T|x\|^{q\frac{p+r}{q}} \geq \||T|^{\frac{p+r}{q}} x\|^q.$$

Thus the proofs are completed.

Theorem 3.3.

(1) *If T is (p,r,q)-paranormal, then T is absolute (p,s)-paranormal for $r \leq s \leq r+1$ and $q \geq 1$ with $p+r \geq q(1-s+r)$. Consequently, T is absolute (p,s')-paranormal for $s' \geq s$.*
(2) *If T is (p,r,q)-paranormal, then T is absolute (p,s)-paranormal for $s \geq r+1$ and $q \leq 1$.*

Proof. (1) Suppose that T is (p,r,q)-paranormal and $r \leq s \leq r+1$. Then T is absolute (p,s)-paranormal for $q \geq 1$ with $p+r \geq q(1-s+r)$ by [11, Theorem 4.5]. Since $p+s \geq q(1-s+r)-r+s = (q-1)(1+r-s)+1 \geq 1$, T is absolute (p,s')-paranormal for $s' \geq s$.

(2) Suppose that T is (p,r,q)-paranormal and $s \geq r+1$ and $q \leq 1$. Then, it follows from the Hölder-McCarthy inequality that for a given unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
\||T|^p U|T|^s x\| &= \||T|^p U|T|^r |T|^{s-r} x\| \\
&\geq \||T|^{\frac{p+r}{q}} |T|^{s-r} x\|^q \||T|^{s-r} x\|^{1-q} \\
&\geq \||T|x\|^{q(\frac{p+r}{q}+s-r)} x\|^q \||T|^{s-r} x\|^{1-q} \\
&= \||T|x\|^{p+s}.
\end{aligned}$$

Question 3.4. *Is every (p,r,q)-paranormal operator normaloid?*

4. Examples. In this section, we will give a characterization of (p,r,q)-paranormal operators. Using this characterization, we will give some examples showing that the monotonicity of section 2 and 3 are all strict. In [2], Ando proved that an operator T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$$

for all $\lambda > 0$. As a generalization of Ando's result, we will give the characterization of (p,r,q)-paranormal and absolute (p,r)-paranormal.

The following lemma is clear by the simple calculation.

Lemma 4.1. *If $a,b \geq 0$, then $a^\mu b^\lambda \leq \lambda a + \mu b$ for $\lambda, \mu > 0, \lambda+\mu = 1$.*

Theorem 4.2. *Let $T = U|T|$ be the polar decomposition of T. An operator T is (p,r,q)-paranormal if and only if*

$$|T|^r U^*|T|^{2p}U|T|^r - q\lambda^{q-1}|T|^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \geq 0 \tag{4.1}$$

for all $\lambda > 0$.

Proof. Suppose that $T = U|T|$ is (p,r,q)-paranormal. Then

$$|||T|^{\frac{p+r}{q}}x|| \leq |||T|^p U|T|^r x||^{\frac{1}{q}}$$

for every unit vector x in $\mathcal{H}$. It is equivalent to

$$(|T|^{\frac{2(p+r)}{q}}x, x) \leq (|T|^r U^*|T|^{2p}U|T|^r x, x)^{\frac{1}{q}}(x,x)^{\frac{q-1}{q}} \tag{4.2}$$

for every vector x in $\mathcal{H}$. By Lemma 4.1,

$$\begin{aligned}(|T|^r U^*|T|^{2p}U|T|^r x, x)^{\frac{1}{q}}(x,x)^{\frac{q-1}{q}} &= \{\lambda^{1-q}(|T|^r U^*|T|^{2p}U|T|^r x, x)\}^{\frac{1}{q}}\{\lambda(x,x)\}^{\frac{q-1}{q}} \\ &\leq \frac{1}{q}\lambda^{1-q}(|T|^r U^*|T|^{2p}U|T|^r x, x) + \frac{q-1}{q}\lambda(x,x)\end{aligned} \tag{4.3}$$

for all $\lambda > 0$. This implies that

$$\frac{1}{q}\lambda^{1-q}|T|^r U^*|T|^{2p}U|T|^r - |T|^{\frac{2(p+r)}{q}} + \frac{q-1}{q}\lambda \geq 0$$

for every $\lambda > 0$. Thus it implies (4.1).

Conversely, let

$$\lambda = \{\frac{(|T|^r U^*|T|^{2p}U|T|^r x, x)}{(x,x)}\}^{\frac{1}{q}} \tag{4.4}$$

(If $x \in \ker |T|^p U|T|^r$, put λ is sufficiently small positive). Then it is satisfies (4.2) and hence T is (p,r,q)-paranormal.

Corollary 4.3. *An operator $T = U|T|$ is absolute (p,r)-paranormal if and only if*

$$|T|^r U^*|T|^{2p}U|T|^r - (p+r)\lambda^{p+r-1}|T|^2 + (p+r-1)\lambda^{p+r} \geq 0$$

for all $\lambda > 0$.

The following proposition is obtained from Theorem 4.2.

Proposition 4.4. *Let $\mathcal{K} = \oplus_{-\infty}^{\infty}\mathcal{H}$. For a given two positive operators A, B on $\mathcal{H}$, define the operator $T = T_{A,B}$ on $\mathcal{K}$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & & \\ \ddots & 0 & 0 & 0 & & & & \\ & B & 0 & 0 & & & & \\ & 0 & B & \boxed{0} & & & & \\ & 0 & 0 & B & 0 & & & \\ & & & 0 & A & 0 & & \\ & & & & 0 & A & 0 & \\ & & & & & & \ddots & \ddots \end{pmatrix} \tag{4.5}$$

where $\boxed{\cdot}$ *is the* (0,0)*-element. The following statements are hold:*

(1) T *is absolute* p*-paranormal if and only if*

$$BA^{2p}B - (p+1)\lambda^p B^2 + p\lambda^{p+1} \geq 0 \tag{4.6}$$

for all $\lambda > 0$.

(2) T *is absolute* (p,r)*-paranormal if and only if*

$$B^r A^{2p} B^r - (p+r)\lambda^{p+r-1} B^2 + (p+r-1)\lambda^{p+r} \geq 0 \tag{4.7}$$

for all $\lambda > 0$.

(3) T *is* (p,r,q)*-paranormal if and only if*

$$B^r A^{2p} B^r - q\lambda^{q-1} B^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \geq 0 \tag{4.8}$$

for all $\lambda > 0$.

Proof. It is enough to show that the operator T is (p,r,q)-paranormal if and only if

$$B^r A^{2p} B^r - q\lambda^{q-1} B^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \geq 0$$

for all $\lambda > 0$. Let U be the unilateral shift and $P = Diag(\cdots, B, \boxed{B}, A, A, A, \cdots)$. Then $T = UP$ is the polar decomposition of T. By the simple calculation, the inequality (4.2)

$$|T|^r U^* |T|^{2p} U |T|^r - q\lambda^{q-1} |T|^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \geq 0$$

is equivalent to

$$B^r A^{2p} B^r - q\lambda^{q-1} B^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \geq 0$$

for all $\lambda > 0$. Hence the proof is completed.

From Proposition 4.4, we give an example as follows:

Example 4.5. (An example for Theorem 2.2)

Put $A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ on $\mathbf{C}^2$. Define $T_{A,B}$ on $\mathcal{K}$ as (4.5) and denote

$$X_{p,r,q}(\lambda) = B^r A^{2p} B^r - q\lambda^{q-1} B^{\frac{2(p+r)}{q}} + (q-1)\lambda^q \tag{4.9}$$

in Proposition 4.4. Then, we have $T_{A,B}$ is (6,3,3)-paranormal, non (2,2,1)-paranormal. Since

$$\begin{aligned} X_{6,3,3}(\lambda) &= B^3 A^{12} B^3 - 3\lambda^2 B^6 + 2\lambda^3 \\ &= \begin{pmatrix} 3854061109710848 - 3\lambda^2 + 2\lambda^3 & 113447507052199936 \\ 113447507052199936 & 3339422103077126144 - 196608\lambda^2 + 2\lambda^3 \end{pmatrix} \end{aligned}$$

and

$$X_{2,2,1}(\lambda) = B^2 A^4 B^2 - B^8 = \begin{pmatrix} 137959 & 1015168 \\ 1015168 & 7407612 \end{pmatrix},$$

we have $X_{2,2,1}(\lambda)$ is not positive. It is easy that $3854061109710848 - 3\lambda^2 + 2\lambda^3$ is positive for all $\lambda > 0$. Since the determinant of $X_{6,3,3}(\lambda)$ has no positive real solution and since $3339422103077126144 - 196608\lambda^2 + 2\lambda^3 > 0$ for all $\lambda > 0$, we have $X_{6,3,3}(\lambda)$ is positive for all $\lambda > 0$.

The following example shows that the (p,r,q)-paranormality has no monotonicity on p.

Example 4.6. Put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 0 \end{pmatrix}$$

on $\mathbf{C}^2$. Then $T_{A,B}$ is (2,2,2)-paranormal, non (1,2,2)-paranormal. And, we put

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}^{\frac{1}{2}} \quad \text{and } D = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^{\frac{1}{2}}$$

on $\mathbf{C}^2$. Then $T_{C,D}$ is (1,1,1)-paranormal, non (2,1,1)-paranormal.

Proof. since

$$B^2A^4B^2 - 2\lambda B^4 + \lambda^2 = \begin{pmatrix} 1476 - 72\lambda + \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

and

$$B^2A^2B^2 - 2\lambda B^3 + \lambda^2 = \begin{pmatrix} 180 - 12\sqrt{6}\lambda + \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix},$$

we have $T_{A,B}$ is (2,2,2)-paranormal, non (1,2,2)-paranormal. And since D is invertible, $DC^2D - D^4 \geq 0$ is equivalent to $C^2 - D^2 \geq 0$ and $DC^4D - D^6 \geq 0$ is equivalent to $C^4 - D^4 \geq 0$. Thus

$$C^2 - D^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$C^4 - D^4 = \begin{pmatrix} 0 & -4 \\ -4 & 8 \end{pmatrix}.$$

Hence $T_{C,D}$ is (1,1,1)-paranormal, non (2,1,1)-paranormal.

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