# BAYES SOLUTION TO DYNAMIC PERISHABLE INVENTORY PROBLEM WITH TWO TYPES OF STATES 

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#### Abstract

This paper discusses the model for a perishable product with two types of states, a good state and a bad one. The commodities sell well when the good state occurs at the beginning of a period, but do not when the bad state occurs. An indicator random variable is defined to express the states, and it is a Bernoulli random variable with unknown parameter which has a conjugate beta prior. We express the maximum expected profit for remaining periods by a dynamic program and obtain the optimal ordering policy in the first period. As a main result, we get a relationship between the numbers of times at which the good and the bad state occurred and the expected profit for remaining periods. Moreover, numerical examples are given to illustrate an optimal ordering policy for remaining $n>1$ periods.


1. Introduction Many perishable inventory models studied since Nahmias[1] and Fries[2] presented models for commodities with a fixed life time. Nahmias assumes that the cost of outdating is charged to the period in which the order arrives rather than the period in which the outdating actually occurs. For this reason, his model is extended to many one-period models[3-7]. Ishii et al.[3, 4] and Nose et al.[5] have determined an optimal inventory level under stochastic procurement leadtime is the case of 0 or 1 period. Nose et al.[5] have also considered two different selling prices. Ishii[6], and Ishii and Nose[7] have generalized these models by introducing two types of customers although they do not take stochastic leadtime into consideration. Ishii and Nose[7] have designed more generalized model which considered two storage facilities.

Fries[2] assumes that the outdating cost is incurred in the period the commodity perishes. Fries' approach is more efficient for actually computing an optimal policy[8].

For $n>1$ planning periods, the optimal ordering policy for both Nahmias' and Fries' approach cannot be characterized in a simple manner. In order to simplify calculation, good approximations have been developed[9, 10, 11]. Nahmias[9, 10] and Nandakumar[11] follow the methodology of Nahmias[1] and Fries[2], respectively. However, a demand in each period does not depend on a past history in these models mentioned above.

On the other hand, for non-perishable inventory model, a Bayesian dynamic inventory problem was modeled[12]. Azoury[12] presented the periodic review inventory model for which parameters of the demand distribution are unknown with a known prior distribution chosen from the natural conjugate family.

In this paper, we propose a Bayesian dynamic perishable inventory model based on Fries' approach with two types of states. We consider the following method as a way of taking a past history into consideration: The states at the beginning of the period are classified into two types, called the "good state" and the "bad state". The commodities sell well when the good state occurs at the beginning of a period, but do not when the bad state occurs. For instance, the commodities like beer sell well when temperature is high, but not

[^0]when temperature is not high. We define the indicator random variable $Y$ by $Y=1$ if the good state occurs and $Y=0$ otherwise. The probability mass function of a Bernoulli random variable having unknown parameter $\pi$ is given by $\psi(Y=i \mid \pi)=\pi^{i}(1-\pi)^{1-i}$ for $i=0,1$. The true value of $\pi$ is unknown and is given the beta prior distribution with parameters $(s, t)$. Since the beta distribution is a conjugate family for a sample from a Bernoulli distribution[13], we can state as follows: If the good state occurs, then the posterior distribution of $\pi$ becomes beta with parameter $(s+1, t)$, otherwise the posterior distribution of $\pi$ becomes beta with parameter $(s, t+1)$.

Using the above method, we first formulate the maximum expected profit for remaining periods and obtain the optimal ordering policy in the first period. As the main theorem, we next analyze the relationship between the numbers of times at which the good and the bad state occurred and the expected profit for remaining periods. Moreover, since it is difficult to derive an optimal ordering policy for $n>1$ planning periods, numerical examples are given to illustrate an optimal ordering quantity for remaining $n>1$ periods. Finally, we end this paper with further research problems.
2. The Model We consider the case of a single product and a finite planning horizon. We assume that two types of states, the "good state" and the "bad state", and demand occurs depending on the state at the start of the period. The length of period is arbitrary but fixed.

We will make the following assumptions:
(i) The planning horizon is divided into finite periods that are numbered backwards.
(ii) An order is placed at the beginning of a period and the new products arrive instantly.
(iii) Any units that have reached age $m$ are removed from the inventory.
(iv) All demand is fulfilled with available units or by following emergency procurement[2]: Given a situation in the supermarket, is there not the brand, the consumer selects the alternative brand, or an employee who have time to spare goes to the neighbor store belonging to the same chain, in order to procure commodities.
(v) Inventory is depleted according to the FIFO (First-In, First-Out) issuing policy.
(vi) The positive integers $s$ and $t$ are the numbers of times at which the good state and the bad state have occurred respectively.
(vii) The type 1 demand in the good state $D^{1}$ and type 2 demand in the bad state $D^{2}$ are independent nonnegative random variables with known distributions $F^{1}(\cdot), F^{2}(\cdot)$, densities $f^{1}(\cdot), f^{2}(\cdot)$ and means $\mu_{1}, \mu_{2}\left(\mu_{1}>\mu_{2}\right)$, respectively.
(viii) All items are sold by price $p$.

The following costs for any given period are specified:
$c=$ purchasing cost per unit.
$K=$ set-up cost per order $(K>0)$.
$h=$ holding cost per unit.
$r=$ emergency procurement cost per unit.
$\theta=$ disposal cost per unit.

The inventory vector at the beginning of a period is $\boldsymbol{x}=\left(x_{m-1}, x_{m-2}, \cdots, x_{1}\right)$, where $x_{i}$ is the amount of product on hand that will outdate exactly $i$ periods into the future. The decision variable $y$ is the quantity of fresh stock ordered. Also, let $\mathbf{1}$ be the $(m-1)$ dimensional column vector all of whose elements are 1 , and let $\boldsymbol{x} \mathbf{1}$ be product of vectors $\boldsymbol{x}$ and 1. If $u$ is a realization of demand in a period, the inventory position at the end of the period is:
(1) $z_{i}(\boldsymbol{x}, y, u) \equiv \begin{cases}\max \left[0, \min \left(x_{i+1}, \sum_{j=1}^{i+1} x_{j}-u\right)\right], & \text { if } i=1,2, \cdots, m-2 \\ \max \left[0, \min \left(y, \sum_{j=1}^{m-1} x_{j}+y-u\right)\right], & \text { if } i=m-1\end{cases}$
and let $\boldsymbol{z}(\boldsymbol{x}, y, u)=\left(z_{m-1}(\boldsymbol{x}, y, u), z_{m-2}(\boldsymbol{x}, y, u), \cdots, z_{1}(\boldsymbol{x}, y, u)\right)$.
Moreover, The probability mass function of a Bernoulli random variable with an unknown value of the parameter $\pi$ is given by $\psi(Y=i \mid \pi)=\pi^{i}(1-\pi)^{1-i}$ for $i=0,1$. And let $g(\pi \mid s, t)$ represent the prior beta density function with parameters $(s, t)$.

Then the maximum expected profit for remaining $n$ periods, $A_{n}(\boldsymbol{x} \mid s, t)$ is given by

$$
\begin{aligned}
& A_{n}(\boldsymbol{x} \mid s, t) \\
& =\quad \sup _{y \geq 0}\{-K \delta(y)-c y \\
& \quad+\int_{0}^{1} \psi(Y=1 \mid \pi)\left[L^{1}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{n-1}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s+1, t) f^{1}(u) d u\right] g(\pi \mid s, t) d \pi \\
& \left.(2) \quad+\int_{0}^{1} \psi(Y=0 \mid \pi)\left[L^{2}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{n-1}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s, t+1) f^{2}(u) d u\right] g(\pi \mid s, t) d \pi\right\}
\end{aligned}
$$

where

$$
\delta(y)= \begin{cases}1, & \text { if } y>0  \tag{3}\\ 0, & \text { otherwise }\end{cases}
$$

and

$$
\begin{align*}
L^{i}(\boldsymbol{x}, y)= & p \int_{0}^{\infty} u f^{i}(u) d u-h \int_{0}^{\boldsymbol{x} \mathbf{1}+y}(\boldsymbol{x} \mathbf{1}+y-u) f^{i}(u) d u \\
& -r \int_{\boldsymbol{x} \mathbf{1}+y}^{\infty}(u-\boldsymbol{x} \mathbf{1}-y) f^{i}(u) d u-\theta \int_{0}^{x_{1}}\left(x_{1}-u\right) f^{i}(u) d u \quad \text { for } i=1,2 \tag{4}
\end{align*}
$$

In the real world, when there are goods left unsold at the end of planning horizon, the retailer will hold the clearance sale for the goods disposal. In this case, although the goods are usually sold bellow the purchasing cost, the profit of clearance sale will make up for the disposal cost. Thus, we let $A_{0}(\boldsymbol{x} \mid s, t)=0$ and

$$
\begin{align*}
& A_{1}(\boldsymbol{x} \mid s, t)= \sup _{y \geq 0}\{-K \delta(y)- \\
& c y+\int_{0}^{1} \psi(Y=1 \mid \pi) L^{1}(\boldsymbol{x}, y) g(\pi \mid s, t) d \pi  \tag{5}\\
&\left.+\int_{0}^{1} \psi(Y=0 \mid \pi) L^{2}(\boldsymbol{x}, y) g(\pi \mid s, t) d \pi\right\}
\end{align*}
$$

Using basic properties on the beta density function, $A_{n}(\boldsymbol{x} \mid s, t)$ in (2) and $A_{1}(\boldsymbol{x} \mid s, t)$ in (5) become, respectively,

$$
\begin{align*}
A_{n}(\boldsymbol{x} \mid s, t)= & \sup _{y \geq 0}\{-K \delta(y)-c y \\
& +\frac{s}{s+t}\left[L^{1}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{n-1}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s+1, t) f^{1}(u) d u\right] \\
& \left.+\frac{t}{s+t}\left[L^{2}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{n-1}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s, t+1) f^{2}(u) d u\right]\right\} \tag{6}
\end{align*}
$$

and

$$
\begin{equation*}
A_{1}(\boldsymbol{x} \mid s, t)=\sup _{y \geq 0}\{-K \delta(y)+B(\boldsymbol{x}, y \mid s, t)\} \tag{7}
\end{equation*}
$$

where

$$
\begin{equation*}
B(\boldsymbol{x}, y \mid s, t)=-c y+\frac{s}{s+t} L^{1}(\boldsymbol{x}, y)+\frac{t}{s+t} L^{2}(\boldsymbol{x}, y) \tag{8}
\end{equation*}
$$

## 3. Analysis of the Expected Profit

Lemma 1 There exists a unique $y=y^{*}(\boldsymbol{x})$ which maximizes $-K \delta(y)+B(\boldsymbol{x}, y \mid s, t)$.
Proof. By partial differentiating $B(\boldsymbol{x}, y \mid s, t)$ with respect to $y$, we have

$$
\begin{equation*}
\frac{\partial B(\boldsymbol{x}, y \mid s, t)}{\partial y}=r-c-(h+r)\left[\frac{s}{s+t} F^{1}(\boldsymbol{x} \mathbf{1}+y)+\frac{t}{s+t} F^{2}(\boldsymbol{x} \mathbf{1}+y)\right] \tag{9}
\end{equation*}
$$

Then $\frac{\partial B(\boldsymbol{x}, y \mid s, t)}{\partial y} \geq 0$ agrees with

$$
\begin{equation*}
\frac{s}{s+t} F^{1}(\boldsymbol{x} \mathbf{1}+y)+\frac{t}{s+t} F^{2}(\boldsymbol{x} \mathbf{1}+y) \leq \frac{r-c}{h+r}(<1) . \tag{10}
\end{equation*}
$$

Let $\varphi_{1}(\boldsymbol{x}, y)$ express the left-hand-side of Inequality (10), we have

$$
\begin{align*}
\varphi_{1}(\boldsymbol{x}, 0) & =\frac{s}{s+t} F^{1}(\boldsymbol{x} \mathbf{1})+\frac{t}{s+t} F^{2}(\boldsymbol{x} \mathbf{1})  \tag{11}\\
\lim _{y \rightarrow+\infty} \varphi_{1}(\boldsymbol{x}, y) & =1  \tag{12}\\
\frac{\partial \varphi_{1}(\boldsymbol{x}, y)}{\partial y} & =\frac{s}{s+t} f^{1}(\boldsymbol{x} \mathbf{1}+y)+\frac{t}{s+t} f^{2}(\boldsymbol{x} \mathbf{1}+y)>0 \tag{13}
\end{align*}
$$

The above equations reveal as follows:
(a) If $\varphi_{1}(\boldsymbol{x}, 0)<\frac{r-c}{h+r}$, then $B(\boldsymbol{x}, y \mid s, t)$ first increases and then decreases with increasing $y$. Therefore there exists a unique $y=S_{0}(\boldsymbol{x})>0$ which maximizes $B(\boldsymbol{x}, y \mid s, t)$.
(b) If $\varphi_{1}(\boldsymbol{x}, 0) \geq \frac{r-c}{h+r}$, then $y=S_{0}(\boldsymbol{x})=0$ makes $B(\boldsymbol{x}, y \mid s, t)$ maximize since $B(\boldsymbol{x}, y \mid s, t)$ is non-increase function with respect to $y$.

If an order is placed, then an expected profit is $-K+B\left(\boldsymbol{x}, S_{0}(\boldsymbol{x}) \mid s, t\right)$, otherwise $B(\boldsymbol{x}, 0 \mid s, t)$. Hence, there exists a unique optimal ordering quantity $y=y^{*}(\boldsymbol{x})$ in the first period ( $n=1$ ). This is given by:

$$
y^{*}(\boldsymbol{x})=\left\{\begin{array}{cl}
S_{0}(\boldsymbol{x}), & \text { if } B(\boldsymbol{x}, 0 \mid s, t) \leq B\left(\boldsymbol{x}, S_{0}(\boldsymbol{x}) \mid s, t\right)-K,  \tag{14}\\
0, & \text { otherwise }
\end{array}\right.
$$

Remark. When $F^{1}(x) \leq F^{2}(x)$ for all $x$ (such as both demands distribution are exponentially distributed), $y=y^{*}(\boldsymbol{x})$ tends to become large if the number of times at which the good state occurred in the past (i.e. s) becomes large, but do not if $t$ becomes large.
Proof. When $y=y^{*}(\boldsymbol{x})>0, y^{*}(\boldsymbol{x})$ is the unique solution of

$$
\begin{equation*}
\frac{s}{s+t} F^{1}(\boldsymbol{x} \mathbf{1}+y)+\frac{t}{s+t} F^{2}(\boldsymbol{x} \mathbf{1}+y)=\frac{r-c}{h+r} \tag{15}
\end{equation*}
$$

Since the left-hand-side of (15) decreases in $s$ for any fixed $\boldsymbol{x} \mathbf{1}+y, y^{*}(\boldsymbol{x})$ increases gradually in $s$. On the contrary, since left-hand-side of (15) increases in $t, y^{*}(\boldsymbol{x})$ decreases in $t$.

In this kind of perishable inventory model, it is known that the ordering region becomes much more complex when $K>0[14]$. Although we tried to derive an optimal ordering policy explicitly which maximizes the expected profit for remaining $n>1$ periods when the demands $D^{1}, D^{2}$ are well-known random variables such as a uniform, an exponential and a Weibull, it turned out that it is quite difficult. Thus, some properties of $A_{n}(\boldsymbol{x} \mid s, t)$ is analyzed by assuming that there exists a $y=y^{*}(\boldsymbol{x})$ for $n>1$ periods. We also use a shorthand notation as follows:

$$
\begin{equation*}
\rho_{n}^{j}(\boldsymbol{x}, y \mid s, t)=-\int_{0}^{\infty} \frac{\partial A_{n}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s, t)}{\partial u} F^{j}(u) d u \tag{16}
\end{equation*}
$$

Theorem 1 If $L^{2}(\boldsymbol{x}, y) \leq(\geq$ resp. $) L^{1}(\boldsymbol{x}, y)$, then we have following relations for positive integer $k$.
(i) Case $n=k=1$.

Then it holds $A_{1}(\boldsymbol{x} \mid s, t+1) \leq(\geq) A_{1}(\boldsymbol{x} \mid s, t) \leq(\geq) A_{1}(\boldsymbol{x} \mid s+1, t)$.
(ii) Case $n=k>1$.

When $\rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t+1) \leq(\geq) \rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t) \leq(\geq) \rho_{k-1}^{1}(\boldsymbol{x}, y \mid s, t) \leq(\geq) \rho_{k-1}^{1}(\boldsymbol{x}, y \mid s+$ $1, t)$, we obtain $A_{k}(\boldsymbol{x} \mid s, t+1) \leq(\geq) A_{k}(\boldsymbol{x} \mid s, t) \leq(\geq) A_{k}(\boldsymbol{x} \mid s+1, t)$.
Proof. First, we consider the case of $L^{2}(\boldsymbol{x}, y) \leq L^{1}(\boldsymbol{x}, y)$.
(i) From Lemma 1, there exists a unique $y=y^{*}(\boldsymbol{x})$ which attains the $\sup$ of $A_{1}(\boldsymbol{x} \mid s, t)$. Let $S_{1}(\boldsymbol{x})$ be this $y^{*}(\boldsymbol{x})$, then $A_{1}(\boldsymbol{x} \mid s, t)$ and $A_{1}(\boldsymbol{x} \mid s+1, t)$ are respectively

$$
\begin{align*}
& A_{1}(\boldsymbol{x} \mid s, t) \\
& \quad=-K \delta\left(S_{1}(\boldsymbol{x})\right)-c S_{1}(\boldsymbol{x})+\frac{s}{s+t} L^{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)+\frac{t}{s+t} L^{2}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)  \tag{17}\\
& A_{1}(\boldsymbol{x} \mid s+1, t) \\
& \quad \geq-K \delta\left(S_{1}(\boldsymbol{x})\right)-c S_{1}(\boldsymbol{x})+\frac{s+1}{s+t+1} L^{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)+\frac{t}{s+t+1} L^{2}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)
\end{align*}
$$

Let $J_{1}(\boldsymbol{x}, y \mid s, t)$ be the inside of braces, $\}$, in right-hand-side of (7).

$$
\begin{equation*}
J_{1}(\boldsymbol{x}, y \mid s, t)=-K \delta(y)-c y+\frac{s}{s+t} L^{1}(\boldsymbol{x}, y)+\frac{t}{s+t} L^{2}(\boldsymbol{x}, y) \tag{19}
\end{equation*}
$$

Subtract the right-hand-side of (17) from the right-side-hand of (18), we have

$$
\begin{align*}
& J_{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x}) \mid s+1, t\right)-J_{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x}) \mid s, t\right) \\
& \quad=\left(\frac{s+1}{s+t+1}-\frac{s}{s+t}\right) L^{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)+\left(\frac{t}{s+t+1}-\frac{t}{s+t}\right) L^{2}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right) \\
& \quad=\frac{t}{(s+t)(s+t+1)}\left[L^{1}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)-L^{2}\left(\boldsymbol{x}, S_{1}(\boldsymbol{x})\right)\right](\geq 0) \tag{20}
\end{align*}
$$

Therefore, we get $A_{1}(\boldsymbol{x} \mid s, t) \leq A_{1}(\boldsymbol{x} \mid s+1)$. Also, we use same way to show that $A_{1}(\boldsymbol{x} \mid s, t+$ 1) $\leq A_{1}(\boldsymbol{x} \mid s, t)$.

$$
\begin{align*}
& J_{1}\left(\boldsymbol{x}, S_{2}(\boldsymbol{x}) \mid s, t\right)-J_{1}\left(\boldsymbol{x}, S_{2}(\boldsymbol{x}) \mid s, t+1\right) \\
& \quad=\frac{s}{(s+t)(s+t+1)}\left[L^{1}\left(\boldsymbol{x}, S_{2}(\boldsymbol{x})\right)-L^{2}\left(\boldsymbol{x}, S_{2}(\boldsymbol{x})\right)\right](\geq 0) \tag{21}
\end{align*}
$$

where $S_{2}(\boldsymbol{x})$ is an optimal ordering quantity which attains the sup of $A_{1}(\boldsymbol{x} \mid s, t+1)$. From $(20),(21)$, we have

$$
\begin{equation*}
A_{1}(\boldsymbol{x} \mid s, t+1) \leq A_{1}(\boldsymbol{x} \mid s, t) \leq A_{1}(\boldsymbol{x} \mid s+1, t) \tag{22}
\end{equation*}
$$

(ii) The proof is by induction on $k$. We now consider the case of $L^{2}(\boldsymbol{x}, y) \leq L^{1}(\boldsymbol{x}, y)$ and $\rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t+1) \leq \rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t) \leq \rho_{k-1}^{1}(\boldsymbol{x}, y \mid s, t) \leq \rho_{k-1}^{1}(\boldsymbol{x}, y \mid s+1, t)$.

Suppose that Case (ii) of Theorem 1 has been proved for $n=k-1$. For $n=k$, we assume that $S_{3}(\boldsymbol{x})$ is an optimal ordering quantity which attains the sup of $A_{k}(\boldsymbol{x} \mid s, t)$. Then, we have

$$
\begin{align*}
A_{k}(\boldsymbol{x} \mid s, t)= & -K \delta\left(S_{3}(\boldsymbol{x})\right)-c S_{3}(\boldsymbol{x}) \\
& +\frac{s}{s+t}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+\int_{0}^{\infty} A_{k-1}(\boldsymbol{z}(\boldsymbol{x}, y, u), \mid s+1, t) f^{1}(u) d u\right] \\
& +\frac{t}{s+t}\left[L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+\int_{0}^{\infty} A_{k-1}(\boldsymbol{z}(\boldsymbol{x}, y, u), \mid s, t+1) f^{2}(u) d u\right] \\
= & -K \delta\left(S_{3}(\boldsymbol{x})\right)-c S_{3}(\boldsymbol{x}) \\
& +\frac{s}{s+t}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+1, t)+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+1, t\right)\right] \\
& +\frac{t}{s+t}\left[L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s, t+1)+\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s, t+1\right)\right] \tag{23}
\end{align*}
$$

from the integration by parts. Using similar way as (23), we have

$$
\begin{align*}
& A_{2}(\boldsymbol{x} \mid s+1, t) \\
& \geq \quad-K \delta\left(S_{3}(\boldsymbol{x})\right)-c S_{3}(\boldsymbol{x}) \\
& \quad+\frac{s+1}{s+t+1}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+2, t)+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+2, t\right)\right] \\
& \quad+\frac{t}{s+t+1}\left[L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+1, t+1)+\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+1, t+1\right)\right] \tag{24}
\end{align*}
$$

Also, let

$$
\begin{align*}
J_{n}(\boldsymbol{x}, y \mid s, t)= & -K \delta(y)-c y \\
& +\frac{s}{s+t}\left[L^{1}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{k-1}(\boldsymbol{z}(\boldsymbol{x}, y, u), \mid s+1, t) f^{1}(u) d u\right] \\
& +\frac{t}{s+t}\left[L^{2}(\boldsymbol{x}, y)+\int_{0}^{\infty} A_{k-1}(\boldsymbol{z}(\boldsymbol{x}, y, u) \mid s, t+1) f^{2}(u) d u\right] \tag{25}
\end{align*}
$$

then it is derived from the inductive hypothesis that

$$
\begin{aligned}
J_{k}(\boldsymbol{x}, & \left.S_{3}(\boldsymbol{x}) \mid s+1, t\right)-J_{k}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s, t\right) \\
= & \frac{s+1}{s+t+1}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+2, t)+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+2, t\right)\right] \\
& +\frac{t}{s+t+1}\left[L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+1, t+1)+\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+1, t+1\right)\right] \\
& -\frac{s}{s+t}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s+1, t)+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+1, t\right)\right] \\
& -\frac{t}{s+t}\left[L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)+A_{k-1}(\mathbf{0} \mid s, t+1)+\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s, t+1\right)\right] \\
\geq & \frac{t}{(s+t)(s+t+1)}\left[L^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)-L^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x})\right)\right. \\
& +A_{k-1}(\mathbf{0} \mid s+2, t)-A_{k-1}(\mathbf{0} \mid s+1, t+1) \\
& \left.+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+2, t\right)-\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{3}(\boldsymbol{x}) \mid s+1, t+1\right)\right](\geq 0),
\end{aligned}
$$

and

$$
\begin{align*}
& J_{k}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x}) \mid s, t\right)-J_{k}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x}) \mid s, t+1\right) \\
& \geq \quad \frac{s}{(s+t)(s+t+1)}\left[L^{1}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x})\right)-L^{2}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x})\right)\right. \\
& \\
& \quad+A_{k-1}(\mathbf{0} \mid s+1, t)-A_{k-1}(\mathbf{0} \mid s, t+1)  \tag{27}\\
& \left.\quad+\rho_{k-1}^{1}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x}) \mid s+1, t\right)-\rho_{k-1}^{2}\left(\boldsymbol{x}, S_{4}(\boldsymbol{x}) \mid s, t+1\right)\right](\geq 0)
\end{align*}
$$

where $S_{4}(\boldsymbol{x})$ is an optimal order quantity which maximizes $J_{k}(\boldsymbol{x}, y \mid s, t+1)$.
Hence, we have

$$
\begin{equation*}
A_{k}(\boldsymbol{x} \mid s, t+1) \leq A_{k}(\boldsymbol{x} \mid s, t) \leq A_{k}(\boldsymbol{x} \mid s+1, t) \tag{28}
\end{equation*}
$$

The following case proved in the same manner:
If $L^{2}(\boldsymbol{x}, y) \geq L^{1}(\boldsymbol{x}, y)$, then it holds

$$
\begin{equation*}
A_{1}(\boldsymbol{x} \mid s, t+1) \geq A_{1}(\boldsymbol{x} \mid s, t) \geq A_{1}(\boldsymbol{x} \mid s+1, t) \tag{29}
\end{equation*}
$$

Also, if $L^{2}(\boldsymbol{x}, y) \geq L^{1}(\boldsymbol{x}, y)$ and $\rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t+1) \geq \rho_{k-1}^{2}(\boldsymbol{x}, y \mid s, t) \geq \rho_{k-1}^{1}(\boldsymbol{x}, y \mid s, t) \geq$ $\rho_{k-1}^{1}(\boldsymbol{x}, y \mid s+1, t)$, then we have

$$
\begin{equation*}
A_{k}(\boldsymbol{x} \mid s, t+1) \geq A_{k}(\boldsymbol{x} \mid s, t) \geq A_{k}(\boldsymbol{x} \mid s+1, t) \tag{30}
\end{equation*}
$$

for positive integer $k>1$. Equation (22) and (28) mean that the expected profit tends to large if the number of times at which the good state occurred is large. On the contrary, Equation (29) and (30) represent that the expected profit becomes small when the number of times at which the good state occurred is large.

We now consider some of relationships between $L^{1}(\boldsymbol{x}, y)$ and $L^{2}(\boldsymbol{x}, y)$ in the following Remarks.

## Remarks

(I) In the case of $p \geq(<$ resp. $)$ r, if $F^{1}(x) \leq(>) F^{2}(x)$ for $x \geq 0$, then we have $L^{2}(\boldsymbol{x}, y) \leq$ $(>) L^{1}(\boldsymbol{x}, y)$.
(II) Let $D^{1}$ and $D^{2}$ be uniformly distributed on these interval $\left(0, \alpha_{1}\right)$ and $\left(0, \alpha_{2}\right)\left(0<\alpha_{2}<\right.$ $\left.\alpha_{1}\right)$, respectively. For $i=1,2$, the uniform density function $f^{i}(u)$ is defined by

$$
f^{i}(u)= \begin{cases}\frac{1}{\alpha_{i}}, & \text { if } 0<u<\alpha_{i}  \tag{31}\\ 0, & \text { otherwise }\end{cases}
$$

Then it holds $L^{2}(\boldsymbol{x}, y) \leq(>$ resp. $) L^{1}(\boldsymbol{x}, y)$ when $\left[\alpha_{1} \alpha_{2}(p-r)+(h+r)(\boldsymbol{x} \mathbf{1}+y)^{2}\right.$ $\left.+\theta x_{1}^{2}\right] \geq(<) 0$.
(III) Assume that $D^{1}$ and $D^{2}$ are exponential random variables having respective parameters $\alpha_{1}$ and $\alpha_{2}\left(\alpha_{1}<\alpha_{2}\right)$. For $i=1,2$, the density of $D^{i}$ is defined by

$$
f^{i}(u)= \begin{cases}\alpha_{i} e^{-\alpha_{i} u}, & \text { if } u \geq 0  \tag{32}\\ 0, & \text { if } u<0\end{cases}
$$

Also there exists a unique $\alpha_{i}=\alpha_{0}$ which is a solution $\frac{\partial}{\partial \alpha_{i}} L^{i}(\boldsymbol{x}, y)=0$. Then we obtain the following relationship between $L^{1}(\boldsymbol{x}, y)$ and $L^{2}(\boldsymbol{x}, y)$.
(a) Case $r \leq p$.

Then we have $L^{2}(\boldsymbol{x}, y) \leq L^{1}(\boldsymbol{x}, y)$.
(b) Case $r>p$.
i. If $\alpha_{0} \leq \alpha_{1}$, then we have $L^{2}(\boldsymbol{x}, y) \leq L^{1}(\boldsymbol{x}, y)$.
ii. If $\alpha_{0} \geq \alpha_{2}$, then we have $L^{2}(\boldsymbol{x}, y) \geq L^{1}(\boldsymbol{x}, y)$.

## Proof

(I) Assume that $p \geq(<$ resp. $) r$ and $F^{1}(x) \leq(>) F^{2}(x)$, then we have

$$
\begin{aligned}
L^{1}(\boldsymbol{x}, y)- & L^{2}(\boldsymbol{x}, y) \\
=\quad & (p-r)\left\{E\left[D^{1}\right]-E\left[D^{2}\right]\right\}+(h+r) \int_{0}^{\boldsymbol{x} \mathbf{1}+y}\left[F^{2}(u)-F^{1}(u)\right] d u \\
& +\theta \int_{0}^{x_{1}}\left[F^{2}(u)-F^{1}(u)\right] d u \\
\geq(<) & 0 .
\end{aligned}
$$

(II) The following equation is obtained by substituting (4) into $f^{i}(u)$ from (31).

$$
\begin{aligned}
L^{i}(\boldsymbol{x}, y)= & p \int_{0}^{\infty} u \frac{1}{\alpha_{i}} d u-h \int_{0}^{\boldsymbol{x} \mathbf{1}+y}(\boldsymbol{x} \mathbf{1}+y-u) \frac{1}{\alpha_{i}} d u \\
& -r \int_{\boldsymbol{x} \mathbf{1}+y}^{\infty}(u-\boldsymbol{x} \mathbf{1}-y) \frac{1}{\alpha_{i}} d u-\theta \int_{0}^{x_{1}}\left(x_{1}-u\right) \frac{1}{\alpha_{i}}, \quad \text { for } i=1,2
\end{aligned}
$$

Subtract $L^{2}(\boldsymbol{x}, y)$ from $L^{1}(\boldsymbol{x}, y)$, it holds

$$
\begin{align*}
L^{1}(\boldsymbol{x}, y)-L^{2}(\boldsymbol{x}, y)= & (p-r) \frac{\alpha_{1}}{2}-(h+r) \frac{(\boldsymbol{x} \mathbf{1}+y)^{2}}{2 \alpha_{1}}+r(\boldsymbol{x} \mathbf{1}+y)-\theta \frac{x_{1}^{2}}{2 \alpha_{1}} \\
& -(p-r) \frac{\alpha_{2}}{2}+(h+r) \frac{(\boldsymbol{x} \mathbf{1}+y)^{2}}{2 \alpha_{2}}-r(\boldsymbol{x} \mathbf{1}+y)+\theta \frac{x_{1}^{2}}{2 \alpha_{2}} \\
= & \frac{\alpha_{1}-\alpha_{2}}{2 \alpha_{1} \alpha_{2}}\left[\alpha_{1} \alpha_{2}(p-r)+(h+r)(\boldsymbol{x} \mathbf{1}+y)^{2}+\theta x_{1}^{2}\right] . \tag{35}
\end{align*}
$$

Therefore when $\left[\alpha_{1} \alpha_{2}(p-r)+(h+r)(\boldsymbol{x} \mathbf{1}+y)^{2}+\theta x_{1}^{2}\right] \geq(<$ resp. $) 0$, we have $L^{2}(\boldsymbol{x}$, $y) \leq(>) L^{1}(\boldsymbol{x}, y)$.
(III) An expected value of exponential random variables become small when the rate $\alpha_{i}$ becomes large. Substituting the function of $f^{i}(u)$ from (32) in (4), we obtain

$$
\begin{align*}
L^{i}(\boldsymbol{x}, y)= & p \int_{0}^{\infty} u \alpha_{i} e^{-\alpha_{i} u} d u-h \int_{0}^{\boldsymbol{x} \mathbf{1}+y}(\boldsymbol{x} \mathbf{1}+y-u) \alpha_{i} e^{-\alpha_{i} u} d u \\
(36) \quad & -r \int_{\boldsymbol{x} \mathbf{1}+y}^{\infty}(u-\boldsymbol{x} \mathbf{1}-y) \alpha_{i} e^{-\alpha_{i} u} d u-\theta \int_{0}^{x_{1}}\left(x_{1}-u\right) \alpha_{i} e^{-\alpha_{i} u}, \text { for } i=1,2 \tag{36}
\end{align*}
$$

By partial differentiating $L^{i}(\boldsymbol{x}, y)$ with respect to $\alpha_{i}$, we have

$$
\begin{align*}
& \frac{\partial L^{i}\left(\boldsymbol{x}, y, \alpha_{i}\right)}{\partial \alpha_{i}} \\
& =-\frac{(p+h+\theta)}{\alpha_{i}^{2}}+\frac{(h+r)\left[1+\alpha_{i}(x+y)\right]}{\alpha_{i}^{2}} e^{-\alpha_{i}(\boldsymbol{x} \cdot \mathbf{1}+y)}+\frac{\theta\left(1+\alpha_{i} x_{1}\right)}{\alpha_{i}^{2}} e^{-\alpha_{i} x_{1}} \tag{37}
\end{align*}
$$

Then $\frac{\partial L\left(\boldsymbol{x}, y, \alpha_{i}\right)}{\partial \alpha_{i}} \geq 0$ agrees with
$(38)-(p+h+\theta)+(h+r)[1+\alpha(x+y)] e^{-\alpha(\boldsymbol{x} \cdot \mathbf{1}+y)}+\theta\left(1+\alpha x_{1}\right) e^{-\alpha x_{1}} \geq 0$
Let $\varphi_{2}\left(\alpha_{i}\right)$ express the left-hand-side of Inequality (38), we have

$$
\begin{align*}
\lim _{\alpha_{i} \rightarrow 0+0} \varphi_{2}\left(\alpha_{i}\right) & =-(p-r)  \tag{39}\\
\lim _{\alpha_{i} \rightarrow+\infty} \varphi_{2}\left(\alpha_{i}\right) & =-(p+h+\theta)(<0)  \tag{40}\\
\frac{d \varphi_{2}\left(\alpha_{i}\right)}{d \alpha_{i}} & =-\alpha_{i}(h+r)(x+y)^{2} e^{-\alpha_{i}(\boldsymbol{x} \cdot \mathbf{1}+y)}-\alpha_{i} \theta x_{1}^{2} e^{-\alpha_{i} x_{1}} \quad(<0) \tag{41}
\end{align*}
$$

Equation(39), (40) and (41) reveal as follows.
(a) If $p-r \geq 0$, then $L^{i}(\boldsymbol{x}, y)$ is non-increasing function with respect to $\alpha_{i}$. It means $L^{i}(\boldsymbol{x}, y)$ becomes large when expected values of $D^{i}$ are large.
(b) When $p-r<0, L^{i}(\boldsymbol{x}, y)$ first increases and then decreases with increasing $\alpha_{i}$. Therefore there exists a unique $\alpha_{i}=\alpha_{0}$ which maximizes the $L^{i}(\boldsymbol{x}, y)$. In this case, it is complicated that the relation between $L^{1}(\boldsymbol{x}, y)$ and $L^{2}(\boldsymbol{x}, y)$ is analyzed, but it becomes clear that $L^{2}(\boldsymbol{x}, y) \leq L^{1}(\boldsymbol{x}, y)$ if $\alpha_{0} \leq \alpha_{1}$ and $L^{1}(\boldsymbol{x}, y) \leq L^{2}(\boldsymbol{x}, y)$ if $\alpha_{2} \leq \alpha_{0}$.


Figure 1: Sensitivity analysis with respect to $K$.


Figure 2: Sensitivity analysis with respect to $c$.


Figure 3: Sensitivity analysis with respect to $\theta$.
4. Numerical Examples In this section, we calculate an optimal ordering quantity which maximizes the expected profit for remaining $n>1$ periods. As an example, suppose $D^{1}$ and $D^{2}$ are Weibull random variables having respective parameters $\left(\alpha_{1}, \beta\right)$ and $\left(\alpha_{2}, \beta\right)$, $\alpha_{1}<\alpha_{2}$, and $m=2, n=3, h=1, p=300, \alpha_{1}=0.5, \alpha=0.7, \beta=0.95$. Figure 1,2 , and 3 reveal the $y^{*}(\mathbf{0})$ and $A_{3}(\mathbf{0} \mid 1,1)$ for $r=250$ and 350 , respectively.

Figure 1 shows $y^{*}(\mathbf{0})$ and $A_{3}(\mathbf{0} \mid 1,1)$ when the cost, K, for ordering a lot (set-up) varies. It is observed in Figure 1 that $y^{*}(\mathbf{0})$ increases with an increase in $K$, and $y^{*}(\mathbf{0})$ increases with an increase in $r$. The following description explains these tendencies:

For large values of $K$, it is considered that there is a tendency to increase the ordering quantity in order to reduce the number of times of ordering. Since, in the case of $r=$ $350(>p)$, the big penalty cost is carried out to run out, it avoids selling out. Therefore, the ordering quantity is larger than in the case of $r=250(<p)$. This tendency also can be seen in Figure 2 and 3.

Moreover, it is observed in Figure 1 that $A_{3}(\mathbf{0} \mid 1,1)$ reduces with an increase in $K$ and in $r$, respectively.

Figure 2 illustrates $y^{*}(\mathbf{0})$ and $A_{3}(\mathbf{0} \mid 1,1)$ when the cost, c , for purchasing per unit varies. It is remarked in Figure 2 that $y^{*}(\mathbf{0})$ reduces with an increase in $c$. Large values of $c$ make the ordering quantity small in order to reduce the ordering and holding cost. In Figure 2, it is also observed that $A_{3}(\mathbf{0} \mid 1,1)$ reduces with an increase in $c$ and in $r$.

Figure 3 shows $y^{*}(\mathbf{0})$ and $A_{3}(\mathbf{0} \mid 1,1)$ when the cost, $\theta$, for disposal per unit changes. It is observed in Figure 3 that $y^{*}(\mathbf{0})$ decreases with an increase $\theta$. This can be explained in the following manner:

As increase $\theta$, the present ordering quantity is made small in order to small the disposal cost in the future. Also, it is observed that $A_{3}(\mathbf{0} \mid 1,1)$ reduces with an increase in $\theta$ and in $r$.

In addition, many examples of cases with $m>2$ and $n>3$ yielded results that were consistent with above mentioned.
5. Concluding Remarks This study proposed a Bayesian dynamic perishable inventory model based on Fries' approach. We consider the case that there exist two types of state, and that the commodities sell well when the good state occurs at the beginning of the period, but do not when the bad state occurs. First, the maximum expected profit for remaining periods was formulated and the optimal ordering policy in the first period was obtained. In the main theorem, we gave the relationship between the numbers of times at which the good and the bad state occurred and the maximum expected profit for remaining periods. Moreover, numerical examples were given to illustrate an optimal ordering quantity for remaining $n>1$ periods. Furthermore, the research of models with more general types of states is left as one of further problems.

## References

[1] S.Nahmias, Optimal ordering policies for perishable inventory-II, Operations Research, 23 , (1975), 735-749.
[2] B.E.Fries, Optimal ordering policy for a perishable commodity with fixed lifetime, Operations Research, 23, (1975), 46-61.
[3] H.Ishii, T.Nose, S.Shiode and T.Nishida, Perishable inventory management subject to stochastic leadtime, European Journal of Operational Research,8, (1981), 76-85.
[4] H.Ishii, T.Nose and T.Nishida, Some properties of perishable inventory control subject to stochastic lead-time, Journal of the Operations Research Society of Japan, 24, (1981),110-134.
[5] T.Nose, H.Ishii and T.Nishida, Perishable inventory management with stochastic leadtime and different selling price. European Journal of Operational Research,18, (1984),322-338.
[6] H.Ishii, Perishable inventory problem with two types of customers and different selling prices, Journal of the Operations Research Society of Japan, 36, (1993),199-205.
[7] H.Ishii,T.Nose, Perishable inventory control with two types of customers and different selling prices under the warehouse capacity constraint, Int. J. Production Economics, 44, (1996), 167176.
[8] S.Nahmias, Perishable inventory theory: a review, Operations Research, 30, (1982), 680-708.
[9] S.Nahmias, Myopic approximations for the perishable inventory problem, Management Science, 9, (1976), 1002-1008.
[10] S.Nahmias, Higher-order approximations for the perishable inventory problem, Operations Reseach, 25, (1977), 630-640.
[11] P.Nandakumar,T.E.Morton, Near myopic heuristics for the fixed-life perishability problem, Management Science, 39, (1993), 1490-1498.
[12] K.S.Azoury, Bayes solution to dynamic inventory models under unknown demand distribution, Management Science, 31, (1985), 1150-1160.
[13] H.M.Degroot, Optimal Statistical Decisions, New York, (1970).
[14] S.Nahmias, The fixed-charge perishable inventory problem, Operations Research, 26, (1978), 464-481.

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