NOTES ON GENERALIZATIONS OF HUREWICZ SPACES

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Received March 26, 2001

Abstract. Guido and Kočinac defined the properties (Π_1) and (Π_2) (respectively, properties (M^ω) and (sM^ω)). We shall show that they are equivalent(respectively).

Quite recently, C. Guido and Lj.D.R. Kočinac discussed with mappings between function spaces in [2]. In its paper, they introduced some properties of continous mappings and of covering properties of spaces, and they showed some interesting results. The purposes of this paper are to show their more strict relations and by use of the results, to give some characterizations with function spaces between covering properties.

Throughout this paper, all the spaces are completely regular T_1 topological spaces, and the mappings are continuous. Let N be a set of all natural numbers.

For a subset A of a space X, $\overline{A}$ denotes the closure of A in X.
For a space X, $C_p(X)$ denotes the space of all of the real-valued continuous maps on X with the pointwise convergence topology.
Furthermore the open cover $\mathbf{U}$ of a space X is said to be $\omega - cover$ of X if, each finite subset F of X is contained in some $U \in \mathbf{U}$. At first we shall define the following covering properties which were introduced by Guido and Kočinac.

Definition 1 (Guido and Kočinac[2]). Let Y be a subspace of a space X. Then:

1. Y has *property* (M^0) *in* X, if for each sequence $\{\mathbf{U}_n | n \in N\}$ of open covers of X, there is a sequence $\{\mathbf{V}_n | n \in N\}$ such that $\mathbf{V}_n$ is a finite subset of $\mathbf{U}_n$ for each $n \in N$, and for each $y \in Y$ there is some $k \in N$ such that $\mathbf{V}_n$ is nonempty and $y \in V$ for each $V \in \mathbf{V}_n$.

2. Y has *property* (M^ω) *in* X, if for each sequence $\{\mathbf{U}_n | n \in N\}$ of $\omega - covers$ of X, there is a sequence $\{\mathbf{V}_n | n \in N\}$ such that $\mathbf{V}_n$ is a finite subset of $\mathbf{U}_n$ for each $n \in N$, and for each finite subset F of Y, there is some $k \in N$ such that $\mathbf{V}_n$ is nonempty and $F \subset V$ for each $V \in \mathbf{V}_n$.

3. Y has *property* (sM^ω) *in* X, if for each sequence $\{\mathbf{U}_n | n \in N\}$ of $\omega - covers$ of X, then there is a sequence $\{\mathbf{V}_n | n \in N\}$ such that $\mathbf{V}_n$ is a nonempty finite subset of $\mathbf{U}_n$ for each $n \in N$, and each finite subset F of Y is contained in V for each $V \in \mathbf{V}$ for infinitely many $n \in N$.

2000 *Mathematics Subject Classification.* 54C35, 54D20.
Key words and phrases. $\omega-$cover,property (M^ω), property (sM^ω), property (Π_1), property (Π_2).

Secondly we shall give the definitions of maps as follows([1] and [2]):

Definitiont 2. Let f be a map from a space X to a space Y:

1. f has *property* (Π_1) if for each sequence $\{A_n|n \in N\}$ of subsets of X and each $x \in \bigcap\{\overline{A_n}|n \in N\}$, there is a sequence $\{B_n|n \in N\}$ such that B_n is a nonempty finite subset of A_n for each $n \in N$ and each neighborhood V of $f(x)$ contains $f(B_n)$ for some $n \in N$.

2. f has *property* (Π_2) if for each sequence $\{A_n|n \in N\}$ of subsets of X and each $x \in \bigcap\{\overline{A_n}|n \in N\}$, there is a sequence $\{B_n|n \in N\}$ such that B_n is a nonempty finite subset of A_n for each $n \in N$ and each neighborhood V of $f(x)$ contains $f(B_n)$ for infinitely many $n \in N$.

If we fix a point $x \in X$ in the above definitions, we say that a map has *property* (Π_i) *at* x (where $i = 1, 2$).
Furthermore, if $X = Y$ and f is an identity map in the above definitions, it is called that a space X *has property* (Π_1) (respectively *property* (Π_2)).

By the above definitons, it is clear that for a subspace Y of a space X, if Y has property (sM^ω), then Y has property (M^ω) and for each map f from a space X to a space Y (, where a space Y is not necessarily a subspace of X), if f has property (Π_2), then it has property (Π_1).

For a subspace Y of a space X, let π denotes the *projection* map from $C_p(X)$ to $C_p(Y)$. This means that for each $f \in C_p(X)$, $\pi(f)$ is the restriction map of f to Y.

In Guido and Kocinac[2], the following theorem between the above properties of covering porperties and the properties of mappings.

Theorem 3 (Guido and Kočinac[2]). Let Y be a subspace of a space X and π a projection map from $C_p(X)$ to $C_p(Y)$. Then

1. If π has property (Π_1), then Y has property (M^ω) in X.
2. If Y has property (sM^ω) in X, then π has property (Π_1).

Guido and Kočinac said that property (sM^ω) introduced in order to obtain a sort of converse of Theorem 3 (1).
The main purpose of this paper is to show that the converse of Theorem 3(1) is true by showing that properties Π_1 and Π_2 are equivalent and properties (M^ω) and $s(M^\omega)$ are equivalent.

Theorem 4. Let f be a map from a space X to a space Y, and x a point of X. Then:
f has a property (Π_1) at x, if and only if it has a property (Π_2) at x.

proof. Since "if part" is trivial by definitons, it is enough to show the "only if" part. Let $\{A_n|n \in N\}$ be a sequence of subsets of X with $x \in \bigcap\{\overline{A_n}|n \in N\}$. Let $\{N_i|i \in N\}$ be a mutually disjoint family of inifinite subsets of N with $N = \bigcup\{N_i|n \in N\}$. For each i, we

denote by $N_i = \{i_j | j = 1, 2, 3, ...\}$.
Let any $i \in N$ be fixed. Clearly x is in $\bigcap\{\overline{A_{i_j}} | j \in N\}$. So there is a finite subset $B_{i_j} \subset A_{i_j}$ for each $j \in N$ such that for each nbd (=neighborhood) V of $f(x)$, we have some $j_{j_i,V}$ with $f(B_{i_{j_i,V}}) \subset V$. Then the sequence $\{B_n | n \in N\}$ of finite sets will be desired with $B_n \subset A_n$ for each $n \in N$.
To show this, let V be any nbd of $f(x)$. For each $i \in N$, we have some $j_{i,V} \in N$ with $f(B_{i_{j_i,V}}) \subset V$. Since $i_{j_{i,V}} \in N_i$ and $N_i \cap N_k =$ for $i \neq k$, $\{i_{j_{i,V}} | i \in N\}$ is infinite, and hence $\{k \in N | f(B_k) \subset V\}$ is infinite. This means that f has a property (Π_2) at x. It completes the proof.

So we shall have the following theorem by Theorem 4:

Theorem 5. Let f be a map from a space X to a space Y. Then:
f has a property (Π_1) if and only if it has property (Π_2).

Guido and Kočinac defined the property (Π_1) at x. Really a space X is said to have *property* (Π_1) *at* x for $x \in X$, if for any sequence $\{A_n | n \in N\}$ of subsets of X with $x \in \overline{A_n - \{x\}}$ for $n \in N$, there is a finite subset B_n of A_n for each $n \in N$ such that every nbd of x contains B_k for some $k \in N$.

So by the similar method in the above proof of Theorem 4, we have the characterization of (Π_1) at x .

Theorem 6. Let X be a space and $x \in X$. Then:
X has property (Π_1) at x if and only if for any sequence $\{A_n | n \in N\}$ of subsets of X with $x \in \overline{A_n - \{x\}}$ for $n \in N$, there is a finite subset B_n of A_n for each $n \in N$ such that every nbd of x contains B_k for infinitely many $k \in N$.

Now we shall discuss among covering properties.

Theorem 7. Let Y a subspace of a space X. Then:
Y has the property (M^ω) in X if and only if Y has the property (sM^ω) in X.

proof. The "if part" is clear, and the proof of "only if part" follows by the similar method in the proof of Theorem 5.

In [1], Kočinac proved that a space X has property (sM^ω) if and only if its function space $C_p(X)$ has property Π_2. So we have the following theorems:

Theorem 8 (ref. [2: the last part of p.113]). Let Y be a subspace of a space X and π a projection from $C_p(X)$ to $C_p(Y)$. Then the following are equivalent:

1. Y has property (M^ω) in X.
2. Y has property (sM^ω) in X.
3. π has property (Π_1).
4. π has property (Π_2).

Theorem 9. The following are equivalent for a space X:

1. X has property (M^ω).
2. X has property (sM^ω).
3. $C_p(X)$ has property (Π_1).
4. $C_p(X)$ has property (Π_2).

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