ON GENERALIZED FRACTIONAL INTEGRALS IN THE ORLICZ SPACES ON SPACES OF HOMOGENEOUS TYPE

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Dedicated to Professor Marie Choda on her sixtieth birthday

Abstract. It is known that the fractional integral I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $0 < \alpha < n$, $1 < p < n/\alpha$ and $n/q = n/p - \alpha$ as the Hardy-Littlewood-Sobolev theorem. In [10] the author introduced generalized fractional integrals and extended this theorem to the Orlicz spaces. The purpose of this paper is twofold. First, we extend this to spaces of homogeneous type. Secondly, we give several examples and compare with known results. For example, we show the boundedness from $\exp L^p$ to $\exp L^q$, from $\exp\exp L^p$ to $\exp\exp L^q$, from $L(\log L)^\alpha$ to $L(\log L)^\beta_{weak}$, from $L(\log L)^\alpha$ to $\exp L^q_{weak}$, etc.

1. Introduction

It is known that the fractional integral I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $0 < \alpha < n$, $1 < p < n/\alpha$ and $n/q = n/p - \alpha$ as the Hardy-Littlewood-Sobolev theorem. The fractional integral was studied by many authors (see, for example, Rubin [16] or Chapter 5 in Stein [17]). The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory.

In [10] the author introduced generalized fractional integrals I_ρ and extended the above boundedness to the Orlicz spaces on the n-dimensional Euclidean space $\mathbb{R}^n$. If $\rho(r) = r^\alpha$, then I_ρ is the usual fractional integral I_α. The purpose of this paper is twofold. First, we extend this to spaces of homogeneous type. Secondly, we give several examples and compare with known results. For example, we have the following; the generalized fractional integral I_ρ is bounded from $\exp L^p$ to $\exp L^q$ (Remark 5.1), where

$$\rho(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \quad \alpha > 0,$$

$0 < p < 1/\alpha$, $1/q = 1/p - \alpha$ and $\exp L^p$ is the Orlicz space L^Φ with

$$\Phi(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r, \\ \exp(r^p) & \text{for large } r. \end{cases}$$

Gatto and Vági [4], and, Gatto, Segovia and Vági [5] studied the fractional integral of functions defind on the space of homogeneous type. We state our results on the space of homogeneous type which contains $\mathbb{R}^n$ case.

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The fractional integral in the Orlicz spaces was studied in [19], [7] [6], [1], etc. Torchinsky [19] treated sublinear operators with weak type (p_i, q_i) $(i = 1, 2)$ and used interpolation. Kokilashvili and Krbec [7] considered the boundedness of I_α with weights, and gave a necessary and sufficient condition on the weights so that weighted inequalities hold. Recently, Genebashvili, Gogatishvili, Kokilashvili and Krbec [6] gave the weighted theory for integral transforms on spaces of homogeneous type. Cianchi [1] gave a necessary and sufficient condition on Φ and Ψ so that the fractional integral I_α is bounded from L^Φ to L^Ψ or from L^Φ to L^Ψ_{weak}. The result in [1] can cover Trudinger's inequality [20] and is better than ours in the case that $\rho(r) = r^\alpha$ and $L^\Psi = \exp L^q$. However, fractional integral I_α is not well-defind on $\exp L^p(\mathbb{R}^n)$.

O'Neil [13] gave a sufficient condition on Φ, Ψ and g so that the convolution operator T_g, $T_g f = g * f$, is bounded from L^Φ to L^Ψ on $\mathbb{R}^n$. Our results are better in the case that $L^\Phi = \exp L^p$, $L^\Psi = \exp L^q$ and $g(x) = \rho(|x|)/|x|^n$.

It was proved by Pustylnik [14] that one of our conditions (2.10) is necessary for the boundedness from L^Φ to L^Ψ. In [14] the generalized fractional integrals with $\rho(r) = r^n/\varphi(r)$ are treated. However, the sufficient condition in [14] does not valid for $\exp L^p$.

Definitions and results are stated in the next section. Section 3 is for the preliminaries. We give a proof of the theorem in Section 4. In Section 5, as examples of our results, we investigate the cases

$$\rho(r) = (\log(1/r))^{-\alpha},\ (\log(1/r))^{-1}(\log\log(1/r))^{-\alpha},\ r^Q(\log(1/r))^\alpha,\ \text{etc.},$$

and consider the boundedness of I_ρ from $\exp L^p$ to $\exp L^q$, from $\exp\exp L^p$ to $\exp\exp L^q$, from $L(\log L)^\alpha$ to $L(\log L)^\beta_{weak}$, from $L(\log L)^\alpha$ to $\exp L^q_{weak}$, etc.

The author stated Theorem 2.1 in $\mathbb{R}^n$ case in [10]. The proof method is essentially the same. The author also reported some of his results in [9] without proofs.

The letter C shall always denote a constant, not necessarily the same one.

2. Definitions and results

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x,y) \ge 0 \quad \text{and} \quad d(x,y) = 0 \text{ if and only if } x = y,$$
$$d(x,y) = d(y,x),$$
$$d(x,y) \le K_1\,(d(x,z) + d(z,y)),$$

the balls $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) \le K_2\,\mu(B(x,r)) < +\infty,$$

where $K_i \ge 1$ $(i = 1, 2)$ are constants independent of $x, y, z \in X$ and $r > 0$.

We assume that $\mu(\{x\}) = 0$ for all $x \in X$ and that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$.

If $\mu(X) < +\infty$, then there exists a constant $R_0 > 0$ such that

$$X = B(x, R_0) \quad \text{for all } x \in X \tag{2.1}$$

(see Lemma 5.1 in [12]).

X is called Q-homogeneous $(Q > 0)$, if there exists constant $K_3 \ge 1$ such that

$$K_3{}^{-1} r^Q \le \mu(B(x,r)) \le K_3 r^Q \quad \text{for } \begin{cases} 0 < r < +\infty & \text{when } \mu(X) = +\infty, \\ 0 < r < R_0 & \text{when } \mu(X) < +\infty, \end{cases} \tag{2.2}$$

where R_0 is the constant in (2.1). The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous. If $Q = 1$, then (X, d, μ) is said to be normal. Macías and Segovia [8] showed that for any space of homogeneous type (X, d, μ) there exists a quasi-distance δ such that (X, δ, μ) is normal and that the topologies induced on X by d and δ coincide.

Let a function $\rho : (0, +\infty) \to (0, +\infty)$ satisfy the following:

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le A_1 \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.3}$$

$$\frac{\rho(r)}{r^Q} \le A_2 \frac{\rho(s)}{s^Q} \quad \text{for} \quad s \le r, \tag{2.4}$$

$$\int_0^1 \frac{\rho(t)}{t}\, dt < +\infty, \tag{2.5}$$

where $A_i > 0$ $(i = 1, 2)$ are independent of $r, s > 0$. For a Q-homogeneous space (X, d, μ), let

$$I_\rho f(x) = \int_X f(y) \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y).$$

If $\rho(r) = r^\alpha$, $0 < \alpha < Q$, then I_ρ is the fractional integral or the Riesz potential denoted by I_α.

Without the assumption Q-homogeneous, we define

$$\bar{\bar{I}}_\rho f(x) = \int_X f(y) \frac{\rho(\mu(B(x, d(x,y))))}{\mu(B(x, d(x,y)))}\, d\mu(y),$$

where ρ satisfies (2.3)–(2.5) with $Q = 1$ (see also [6, p. 121]).

A function $\Phi : [0, +\infty] \to [0, +\infty]$ is called a Young function if Φ is convex, $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \to +\infty} \Phi(r) = \Phi(+\infty) = +\infty$. Any Young function is increasing.

For a Young function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

Then $\widetilde{\Phi}$ is also a Young function. For example, if $\Phi(r) = r^p/p$, $1 < p < \infty$, then $\widetilde{\Phi}(r) = r^{p'}/p'$, $1/p + 1/p' = 1$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0 (0 \le r \le 1), = +\infty (r > 1)$.

For a Young function Φ, let

$$L^\Phi(X) = \left\{ f \in L^1_{\mathrm{loc}}(X) : \int_X \Phi(\epsilon|f(x)|)\, d\mu(x) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_\Phi = \inf\left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\},$$

$$L^\Phi_{weak}(X) = \left\{ f \in L^1_{\mathrm{loc}}(X) : \sup_{r>0} \Phi(r)\, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi, weak} = \inf\left\{ \lambda > 0 : \sup_{r>0} \Phi(r)\, m\left(r, \frac{f}{\lambda}\right) \le 1 \right\},$$

where $m(r, f) = \mu(\{x \in X : |f(x)| > r\})$.

If a Young function Φ satisfies

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty, \tag{2.6}$$

then Φ is continuous and bijective from $[0, +\infty)$ to itself. The inverse function Φ^{-1} is also increasing and continuous.

A function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

Let $Mf(x)$ be the maximal function, i.e.

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum be taken over all balls B containing x.

We assume that Φ satisfies (2.6). Then M is bounded from $L^\Phi(X)$ to $L^\Phi_{weak}(X)$ and

$$\|Mf\|_{\Phi,weak} \le C_0 \|f\|_\Phi. \tag{2.7}$$

If $\Phi \in \nabla_2$, then M is bounded on $L^\Phi(X)$ and

$$\|Mf\|_\Phi \le C_0 \|f\|_\Phi. \tag{2.8}$$

Let

$$\omega = \begin{cases} +\infty & \text{when } \mu(X) = +\infty, \\ R_0 & \text{when } \mu(X) < +\infty, \end{cases}$$

where R_0 is the constant in (2.1). Then our results are as follows.

Theorem 2.1. *Let (X, d, μ) be Q-homogeneous and ρ satisfy (2.3)–(2.5). Let Φ and Ψ be Young functions with (2.6). Assume that there exist constants $A, A', A'' > 0$ such that*

$$\int_r^\omega \widetilde{\Phi}\left(\frac{\rho(t)}{A\int_0^r (\rho(s)/s)\, ds \ \Phi^{-1}(1/r^Q) t^Q}\right) t^{Q-1} \, dt \le A' \quad \text{for } 0 < r < \omega, \tag{2.9}$$

$$\int_0^{\min(r,\omega)} \frac{\rho(t)}{t} \, dt \ \Phi^{-1}\left(\frac{1}{r^Q}\right) \le A'' \ \Psi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < r < +\infty, \tag{2.10}$$

where $\widetilde{\Phi}$ is the complementary function with respect to Φ. Then, for any $C_0 > 0$, there exists a constant $C_1 > 0$ such that, for $f \in L^\Phi(X)$,

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1 \|f\|_\Phi}\right) \le \Phi\left(\frac{Mf(x)}{C_0 \|f\|_\Phi}\right). \tag{2.11}$$

Therefore I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi_{weak}(X)$. Moreover, if $\Phi \in \nabla_2$, then I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

Remark 2.1. Let

$$Tf(x) = \sup_{t>0} \left| \int_X f(y) K(t, x, y) \, d\mu(y) \right|,$$

where $K : (0, +\infty) \times X \times X \to \mathbb{C}$ is a kernel such that

$$|K(t, x, y)| \le C \frac{\rho(d(x, y))}{d(x, y)^Q}, \tag{2.12}$$

for some $C > 0$ independently of t, x, y. Then the theorem also holds for the operator T.

Remark 2.2. We define a generalized fractional maximal function M_ρ by

$$M_\rho f(x) = \sup_{B \ni x} \frac{\rho(\mu(B)^{1/Q})}{\mu(B)} \int_B |f(y)| \, d\mu(y).$$

Let

$$K(t,x,y)=\begin{cases}\rho(t)/t^Q & y\in B(x,t),\\ 0 & y\notin B(x,t).\end{cases}$$

Then K satisfies (2.12) and $T|f|\sim M_\rho f$. Hence the theorem also holds for the operator M_ρ.

The next corollary is for the operator $\bar{\bar{I}}_\rho$. For any space of homogeneous type (X,d,μ) there exists a quasi-distance δ such that (X,δ,μ) is normal and that

$$\delta(x,y)\le C\mu(B(x,d(x,y)))$$

(see Macías and Segovia [8]). From (2.3) and (2.4) with $Q=1$ it follows that

$$\frac{\rho(\mu(B(x,d(x,y))))}{\mu(B(x,d(x,y)))}\le C\frac{\rho(\delta(x,y))}{\delta(x,y)}.$$

By Remark 2.1 we have the following:

Corollary 2.2. *Let ρ satisfy (2.3)–(2.5) with $Q=1$. Let Φ and Ψ be Young functions with (2.6). Assume that (2.9) and (2.10) hold with $Q=1$. Then, for any $C_0>0$, there exists a constant $C_1>0$ such that, for $f\in L^\Phi(X)$,*

$$\Psi\left(\frac{\left|\bar{\bar{I}}_\rho f(x)\right|}{C_1\|f\|_\Phi}\right)\le\Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right). \tag{2.13}$$

Therefore $\bar{\bar{I}}_\rho$ is bounded from $L^\Phi(X)$ to $L^\Psi_{weak}(X)$. Moreover, if $\Phi\in\nabla_2$, then $\bar{\bar{I}}_\rho$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

For functions $\theta,\kappa:(0,+\infty)\to(0,+\infty)$, we denote $\theta(r)\sim\kappa(r)$ if there exists a constant $C>0$ such that

$$C^{-1}\theta(r)\le\kappa(r)\le C\theta(r),\quad r>0.$$

A function $\theta:(0,+\infty)\to(0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C>0$ such that $\theta(r)\le C\theta(s)$ $(\theta(r)\ge C\theta(s))$ for $r\le s$.

Remark 2.3. From (2.3) it follows that

$$\rho(r)\le C\int_0^r\frac{\rho(t)}{t}\,dt. \tag{2.14}$$

If $\rho(r)/r^\varepsilon$ is almost increasing for some $\varepsilon>0$ and $\rho(t)/t^Q$ is almost decreasing, then ρ satisfies (2.3)–(2.5) and $\int_0^r(\rho(t)/t)\,dt\sim\rho(r)$. Let ρ satisfy (2.4) and

$$\rho(r)=\begin{cases}1/(\log(1/r))^{\alpha+1} & \text{for small } r,\\ (\log r)^{\alpha-1} & \text{for large } r,\end{cases}\quad \alpha>0.$$

Then

$$\int_0^r\frac{\rho(t)}{t}\,dt\sim\begin{cases}1/(\log(1/r))^{\alpha} & \text{for small } r,\\ (\log r)^{\alpha} & \text{for large } r.\end{cases}$$

Remark 2.4. In the case $\Phi(r)=r$, (2.9) is equivalent to

$$\frac{\rho(t)}{t^Q}\le\frac{A\int_0^r(\rho(s)/s)\,ds}{r^Q},\quad 0<r\le t.$$

This inequality follows from (2.4) and (2.14).

Remark 2.5. If $\mu(X) < +\infty$, then (2.10) for large r is equivalent to $\Psi(r) \le \Phi(Cr)$ for small r.

We will apply Theorem 2.1 to prove Propositions 5.3 and 5.4. The following corollaries are stated without the complementary function. We will apply Corollary 2.3 to prove Propositions 5.1 and 5.2. We cannot use, however, the corollaries to prove Propositions 5.3 and 5.4. The proof of the next corollary is the same as [10, Proof of Cor. 3.2].

Corollary 2.3. *Let (X, d, μ) be Q-homogeneous and ρ satisfy (2.3)–(2.5). Let Φ and Ψ be Young functions with (2.6). Assume that*

$$\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}\left(\frac{1}{r^Q}\right)$$

is almost decreasing for $0 < r < \omega$ and that there exist constants $A, A' > 0$ such that

$$\int_r^\omega \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right)\,dt \le A\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < r < \omega, \tag{2.15}$$

$$\int_0^{\min(r,\omega)} \frac{\rho(t)}{t}\,dt\ \Phi^{-1}\left(\frac{1}{r^Q}\right) \le A'\ \Psi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < r < +\infty. \tag{2.16}$$

Then (2.11) holds. Therefore I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi_{weak}(X)$. Moreover, if $\Phi \in \nabla_2$, then I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

Remark 2.6. If $r^\varepsilon\rho(r)\Phi^{-1}(1/r^Q)$ is almost decreasing for some $\varepsilon > 0$, then

$$\int_r^\omega \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right)\,dt \le C\rho(r)\ \Phi^{-1}\left(\frac{1}{r^Q}\right).$$

This inequality and (2.14) yield (2.15).

Corollary 2.4. *Let (X, d, μ) be Q-homogeneous and $\rho(r) = r^\alpha$ with $0 < \alpha < Q$. Let Φ and Ψ be Young functions with (2.6). Assume that there exist constants $A, A' > 0$ such that*

$$\int_r^\omega t^{\alpha-1}\Phi^{-1}\left(\frac{1}{t^Q}\right)\,dt \le Ar^\alpha\ \Phi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < r < \omega, \tag{2.17}$$

$$\min(r,\omega)^\alpha\ \Phi^{-1}\left(\frac{1}{r^Q}\right) \le A'\ \Psi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < r < +\infty. \tag{2.18}$$

Then (2.11) holds. Therefore I_α is bounded from $L^\Phi(X)$ to $L^\Psi_{weak}(X)$. Moreover, if $\Phi \in \nabla_2$, then I_α is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.

To prove this corollary by Corollary 2.3, we need the almost decreasingness of $r^\alpha\Phi^{-1}(1/r^Q)$. Since the function $r^\alpha\Phi^{-1}(1/r^Q)$ satisfies (2.3), it follows from (2.17) that

$$\int_r^\omega t^{\alpha-1}\Phi^{-1}\left(\frac{1}{t^Q}\right)\,dt \sim r^\alpha\ \Phi^{-1}\left(\frac{1}{r^Q}\right) \quad \text{for } 0 < 2r < \omega.$$

Hence $r^\alpha\Phi^{-1}(1/r^Q)$ is almost decreasing.

The Hardy-Littlewood-Sobolev theorem follows immediately from Corollary 2.4.

Corollary 2.5 (Hardy-Littlewood-Sobolev)**.** *Let (X, d, μ) be Q-homogeneous and $\rho(r) = r^\alpha$, $\Phi(r) = r^p$ and $\Psi(r) = r^q$ with $0 < \alpha < Q$, $1 \le p < Q/\alpha$ and $Q/q = Q/p - \alpha$. Then (2.11) holds. Therefore I_α is bounded from $L^1(X)$ to $L^q_{weak}(X)$ for $p = 1$ and from $L^p(X)$ to $L^q(X)$ for $1 < p < Q/\alpha$.*

Similarly to Corollaries 2.3–2.5, we can state the results for the operator $\bar{\bar{I}}_\rho$.

3. Preliminalies

Let Φ be a Young function. By the convexity and $\Phi(0) = 0$, we have

$$\Phi(r) \le \frac{r}{s}\Phi(s) \quad \text{for } r \le s. \tag{3.1}$$

Let $\widetilde{\Phi}$ be the complementary function with respect to Φ. Then

$$\widetilde{\Phi}\left(\frac{\Phi(r)}{r}\right) \le \Phi(r), \quad r > 0. \tag{3.2}$$

Actually,

$$\frac{\Phi(r)}{r}s - \Phi(s) \le \Phi(r) \quad \text{for } s < r$$

and

$$\frac{\Phi(r)}{r}s - \Phi(s) \le 0 \quad \text{for } s \ge r.$$

We note that

$$\int_X |f(x)g(x)|\, d\mu(x) \le 2\|f\|_\Phi \|g\|_{\widetilde{\Phi}} \tag{3.3}$$

(see for example [15]).

A function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r), \quad r \ge 0,$$

for some $k > 0$.

A Young function Φ with (2.6) is called an N-function if $\Phi(r)/r \to 0$ as $r \to +0$ and $\Phi(r)/r \to +\infty$ as $r \to +\infty$. If Φ is an N-function, then the complementary function $\widetilde{\Phi}$ is also an N-function, and

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \quad r \ge 0. \tag{3.4}$$

$\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$.

Let K_3 be the constant in (2.2) and let ${K_4}^Q = K_3(1 + K_3)$. Then

$$r^Q = \left(K_3^{-1}{K_4}^Q - K_3\right) r^Q \le \mu\left(B(x, K_4 r) \setminus B(x, r)\right) \le K_3 {K_4}^Q r^Q \quad \text{for } K_4 r < \omega. \tag{3.5}$$

4. Proof of Theorem 2.1

Let

$$\begin{aligned} J_1 &= \int_{d(x,y)<r} f(y)\frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y) \quad \text{and} \\ J_2 &= \int_{d(x,y)\ge r} f(y)\frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y). \end{aligned}$$

Let

$$h(r) = \inf\left\{\frac{\rho(s)}{s^Q} : s \le r\right\}, \quad r > 0.$$

Then h is nonincreasing. It follows that

$$\int_{d(x,y)<r} |f(y)| h(d(x,y))\, d\mu(y) \le Mf(x)\int_{d(x,y)<r} h(d(x,y))\, d\mu(y)$$

(see Stein[18, p.57]). Since $h(r) \sim \rho(r)/r^Q$,

$$|J_1| \le CMf(x)\int_{d(x,y)<r} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y) = CMf(x)\int_{d(x,y)<\min(r,\omega)} \frac{\rho(d(x,y))}{d(x,y)^Q}\, d\mu(y).$$

By (2.3) and (3.5), we have

$$\int_{s\le d(x,y)<K_4 s} \frac{\rho(d(x,y))}{d(x,y)^Q}\,d\mu(y) \sim \int_s^{K_4 s} \frac{\rho(t)}{t}\,dt \quad \text{for } K_4 s<\omega.$$

Hence

$$|J_1| \le CMf(x)\int_0^{\min(r,\omega)} \frac{\rho(t)}{t}\,dt. \tag{4.1}$$

Next we estimate J_2. If $r\ge\omega$, then $J_2=0$. So we assume that $r<\omega$. By (3.3) we have

$$|J_2| \le 2\left\|\frac{\rho(d(x,\cdot))}{d(x,\cdot)^Q}\chi_{B(x,r)^{\mathrm{C}}}(\cdot)\right\|_{\widetilde{\Phi}}\|f\|_{\Phi}. \tag{4.2}$$

where $\chi_{B(x,r)^{\mathrm{C}}}$ is the characteristic function of the complement of $B(x,r)$. Let

$$F(r)=\int_0^r \frac{\rho(s)}{s}\,ds\;\Phi^{-1}\left(\frac{1}{r^Q}\right). \tag{4.3}$$

We show

$$\left\|\frac{\rho(d(x,\cdot))}{d(x,\cdot)^Q}\chi_{B(x,r)^{\mathrm{C}}}(\cdot)\right\|_{\widetilde{\Phi}} \le CF(r). \tag{4.4}$$

From (2.3), (3.5) and the increasingness of $\widetilde{\Phi}$ it follows that

$$\int_{s\le d(x,y)<K_4 s} \widetilde{\Phi}\left(\frac{\rho(d(x,y))}{\lambda d(x,y)^Q}\right)d\mu(y) \le C_3\int_s^{K_4 s}\widetilde{\Phi}\left(\frac{C_2\rho(t)}{\lambda t^Q}\right)t^{Q-1}\,dt, \tag{4.5}$$

where C_2 and C_3 are independent of $\lambda>0$, $s>0$ and $x\in X$. We may assume that $C_3A'\ge 1$. By (3.1) and (2.9) we have

$$\int_r^{\omega}\widetilde{\Phi}\left(\frac{\rho(t)}{C_3AA'F(r)t^Q}\right)t^{Q-1}\,dt \le \frac{1}{C_3A'}\int_r^{\omega}\widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^Q}\right)t^{Q-1}\,dt \le \frac{1}{C_3}. \tag{4.6}$$

Let $\lambda=C_2C_3AA'F(r)$. Then, by (4.5) and (4.6) we have

$$\int_{d(x,y)\ge r}\widetilde{\Phi}\left(\frac{\rho(d(x,y))}{\lambda d(x,y)^Q}\right)d\mu(y)\le 1, \tag{4.7}$$

and so (4.4). By (4.1), (4.2) and (4.4) we have

$$|I_\rho f(x)|=|J_1+J_2|\le C\left(Mf(x)+\|f\|_{\Phi}\Phi^{-1}\left(\frac{1}{r^Q}\right)\right)\int_0^{\min(r,\omega)}\frac{\rho(t)}{t}\,dt. \tag{4.8}$$

Choose $r>0$ so that

$$\Phi^{-1}\left(\frac{1}{r^Q}\right)=\frac{Mf(x)}{C_0\|f\|_{\Phi}}. \tag{4.9}$$

Then

$$\int_0^{\min(r,\omega)}\frac{\rho(t)}{t}\,dt \le A''\frac{\Psi^{-1}\left(\frac{1}{r^Q}\right)}{\Phi^{-1}\left(\frac{1}{r^Q}\right)} = A''\frac{\Psi^{-1}\circ\Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Phi}}\right)}{\frac{Mf(x)}{C_0\|f\|_{\Phi}}}. \tag{4.10}$$

By (4.8), (4.9) and (4.10) we have

$$|I_\rho f(x)|\le C_1\|f\|_{\Phi}\Psi^{-1}\circ\Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Phi}}\right).$$

Therefore we have (2.11).

Let C_0 be as in (2.7). Then

$$\sup_{r>0} \Psi(r)\, m\left(r, \frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right) = \sup_{r>0} r\, m\left(r, \Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right)\right)$$
$$\le \sup_{r>0} r\, m\left(r, \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right)\right) = \sup_{r>0} \Phi(r)\, m\left(r, \frac{Mf(x)}{C_0\|f\|_\Phi}\right) \le 1,$$

i.e.

$$\|I_\rho f\|_{\Psi,weak} \le C_1\|f\|_\Phi.$$

Let C_0 be as in (2.8). Then

$$\int_X \Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right) d\mu(x) \le \int_X \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right) d\mu(x) \le 1,$$

i.e.

$$\|I_\rho f\|_\Psi \le C_1\|f\|_\Phi.$$

5. Propositions

In this section we investigate the cases

$$\rho(r) = (\log(1/r))^{-\alpha},\ (\log(1/r))^{-1}(\log\log(1/r))^{-\alpha},\ r^Q(\log(1/r))^{\alpha},\ \text{etc.}$$

We assume that $\mu(X) = +\infty$. (For the case $\mu(X) < +\infty$, see Remark 2.5.)

For large r, let

$$l_1(r) = \log r, \quad l_{i+1}(r) = \log l_i(r)\ (i = 1, 2, \dots),$$
$$e_1(r) = \exp r, \quad e_{i+1}(r) = \exp e_i(r)\ (i = 1, 2, \dots).$$

Let $-\infty < \alpha < +\infty$. For small r, let

$$L_{[n,\alpha]}(r) = \begin{cases} r^\alpha & n = 0, \\ (\log(1/r))^{-\alpha} & n = 1, \\ \left(\prod_{i=1}^{n-1} l_i(1/r)\right)^{-1} (l_n(1/r))^{-\alpha} & n \ge 2. \end{cases}$$

For large r, let

$$L^{[n,\alpha]}(r) = \begin{cases} r^\alpha & n = 0, \\ (\log r)^{\alpha} & n = 1, \\ \left(\prod_{i=1}^{n-1} l_i(r)\right)^{-1} (l_n(r))^{\alpha} & n \ge 2. \end{cases}$$

Let $0 < p < \infty$. For small ξ, let

$$E_{[n,p]}(\xi) = \begin{cases} \xi^p & n = 0, \\ 1/e_n(1/\xi^p) & n \ge 1. \end{cases}$$

For large ξ, let

$$E^{[n,p]}(\xi) = \begin{cases} \xi^p & n = 0, \\ e_n(\xi^p) & n \ge 1. \end{cases}$$

We define $G_i \subset (\{0\} \cup \mathbb{N}) \times (-\infty, +\infty) \times (0, +\infty) \times (0, +\infty)$ $(i = 1, 2)$ as follows:

$(n, \alpha, p, q) \in G_1 \iff$

$$\begin{cases} 0 < \alpha < Q,\ 1 < p < Q/\alpha,\ q > 1,\ Q/q \ge Q/p - \alpha, & \text{when } n = 0, \\ \alpha > 1,\ 0 < p < 1/(\alpha - 1),\ 1/q \ge 1/p - (\alpha - 1), & \text{when } n \ge 1. \end{cases}$$

$(n, \alpha, p, q) \in G_2 \iff$

$$\begin{cases} 1 < p < Q/\alpha,\ Q/q \le Q/p - \alpha, & \text{when } n = 0 \text{ and } 0 < \alpha < Q, \\ 1 < p < q < \infty, & \text{when } n = 0 \text{ and } \alpha = 0, \\ 1 < p \le q < \infty, & \text{when } n = 0 \text{ and } \alpha < 0, \\ 0 < p < 1/(\alpha+1),\ 1/q \le 1/p - (\alpha+1), & \text{when } n \ge 1 \text{ and } \alpha > -1, \\ 0 < p < q < \infty, & \text{when } n \ge 1 \text{ and } \alpha = -1, \\ 0 < p \le q < \infty, & \text{when } n \ge 1 \text{ and } \alpha < -1. \end{cases}$$

Proposition 5.1. *Let* $(n_i, \alpha_i, p_i, q_i) \in G_i$ $(i = 1, 2)$. *Let* ρ *satisfy* (2.3) *and*

$$\rho(r) = \begin{cases} L_{[n_1,\alpha_1]}(r) & \text{for small } r, \\ L^{[n_2,\alpha_2]}(r) & \text{for large } r. \end{cases}$$

Let Φ *and* Ψ *be N-functions such that*

$$\Phi(\xi) = \begin{cases} E_{[n_2,p_2]}(\xi) & \text{for small } \xi, \\ E^{[n_1,p_1]}(\xi) & \text{for large } \xi, \end{cases} \qquad \Psi(\xi) = \begin{cases} E_{[n_2,q_2]}(\xi) & \text{for small } \xi, \\ E^{[n_1,q_1]}(\xi) & \text{for large } \xi. \end{cases}$$

Then (2.11) *holds and* I_ρ *is bounded from* $L^\Phi(X)$ *to* $L^\Psi(X)$.

Remark 5.1. The case

$$(n_1, \alpha_1, p_1, q_1) = (n_2, \alpha_2, p_2, q_2) = (0, \alpha, p, q)$$

is the Hardy-Littlewood-Sobolev theorem. Let

$$(n_1, \alpha_1, p_1, q_1) = (1, \alpha+1, p, q) \quad \text{and} \quad (n_2, \alpha_2, p_2, q_2) = (1, \alpha-1, p, q).$$

Then we have that I_ρ is bounded from $\exp L^p$ to $\exp L^q$.

Proof. There exist small constant r_1 and large constant r_2 such that, for $0 < r \le r_1$,

$$\begin{cases} \rho(r) = r^{\alpha_1}, \\ \Phi^{-1}(1/r^Q) = r^{-Q/p_1}, \\ \Psi^{-1}(1/r^Q) = r^{-Q/q_1}, \\ \rho(r)\Phi^{-1}(1/r^Q) = r^{\alpha_1 - Q/p_1}, \end{cases} \qquad \text{when } n_1 = 0.$$

$$\begin{cases} \rho(r) = L_{[n_1,\alpha_1]}(r), \\ \Phi^{-1}(1/r^Q) \sim (l_{n_1}(1/r))^{1/p_1}, \\ \Psi^{-1}(1/r^Q) \sim (l_{n_1}(1/r))^{1/q_1}, \\ \rho(r)\Phi^{-1}(1/r^Q) \sim L_{[n_1,\alpha_1-1/p_1]}(r), \end{cases} \qquad \text{when } n_1 \ge 1,$$

and, for $r \ge r_2$,

$$\begin{cases} \rho(r) = r^{\alpha_2}, \\ \Phi^{-1}(1/r^Q) = r^{-Q/p_2}, \\ \Psi^{-1}(1/r^Q) = r^{-Q/q_2}, \\ \rho(r)\Phi^{-1}(1/r^Q) = r^{\alpha_2 - Q/p_2}, \end{cases} \qquad \text{when } n_2 = 0.$$

$$\begin{cases} \rho(r) = L^{[n_2,\alpha_2]}(r), \\ \Phi^{-1}(1/r^Q) \sim (l_{n_2}(r))^{-1/p_2}, \\ \Psi^{-1}(1/r^Q) \sim (l_{n_2}(r))^{-1/q_2}, \\ \rho(r)\Phi^{-1}(1/r^Q) \sim L^{[n_2,\alpha_2-1/p_2]}(r), \end{cases} \qquad \text{when } n_2 \ge 1.$$

Hence, for $0 < r \le r_1$,

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim \begin{cases} r^{\alpha_1}, & \text{when } n_1 = 0, \\ (l_{n_1}(1/r))^{-\alpha_1+1}, & \text{when } n_1 \ge 1, \end{cases}$$

$$\int_r^{r_1} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right) dt \le \begin{cases} Cr^{\alpha_1 - Q/p_1}, & \text{when } n_1 = 0, \\ C(l_{n_1}(1/r))^{-\alpha_1+1+1/p_1}, & \text{when } n_1 \ge 1, \end{cases}$$

and, for $r \ge r_2$,

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim \begin{cases} r^{\alpha_2}, & \text{when } n_2 = 0 \text{ and } 0 < \alpha_2 < Q, \\ \log r, & \text{when } n_2 = 0 \text{ and } \alpha_2 = 0, \\ 1, & \text{when } n_2 = 0 \text{ and } \alpha_2 < 0, \\ (l_{n_2}(r))^{\alpha_2+1}, & \text{when } n_2 \ge 1 \text{ and } \alpha_2 > -1, \\ (l_{n_2+1}(r)), & \text{when } n_2 \ge 1 \text{ and } \alpha_2 = -1, \\ 1, & \text{when } n_2 \ge 1 \text{ and } \alpha_2 < -1, \end{cases}$$

$$\int_r^{+\infty} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right) dt \sim \begin{cases} r^{\alpha_2 - Q/p_2}, & \text{when } n_2 = 0, \\ (l_{n_2}(r))^{\alpha_2+1-1/p_2}, & \text{when } n_2 \ge 1. \end{cases}$$

Let $F(r)$ be as (4.3). Then

$$\int_r^{r_1} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right) dt \le CF(r) \le C'\Psi^{-1}\left(\frac{1}{r^Q}\right), \quad 0 < r \le r_1,$$

and

$$\int_r^{+\infty} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right) dt \le CF(r) \le C'\Psi^{-1}\left(\frac{1}{r^Q}\right), \quad r \ge r_2.$$

Since $F(r)$ and $\Psi^{-1}(1/r^Q)$ are continuous,

$$\int_{r_1}^{r_2} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^Q}\right) dt \le CF(r) \le C'\Psi^{-1}\left(\frac{1}{r^Q}\right), \quad r_1 \le r \le r_2.$$

Using the almost decreasingness of $F(r)$, we have (2.15) and (2.16). Applying Corollary 2.3, we obtain the desired result. □

We define $H_i \subset [1, Q) \times (-\infty, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty)$ $(i = 1, 2)$ as follows:

$$(p, \alpha, \beta, \gamma) \in H_1 \iff \begin{cases} \alpha > 1,\ 0 \le \beta < +\infty,\ 0 \le \gamma \le \alpha + \beta - 1 & \text{when } p = 1, \\ \alpha > 1,\ -\infty < \beta < +\infty,\ \gamma \le \alpha + \beta - 1 & \text{when } 1 < p < Q. \end{cases}$$

$$(p, \alpha, \beta, \gamma) \in H_2 \iff$$

$$\begin{cases} 0 \le \beta < +\infty,\ \gamma \ge \alpha + \beta + 1 & \text{when } p = 1 \text{ and } \alpha > -1, \\ 0 \le \beta < \gamma < +\infty & \text{when } p = 1 \text{ and } \alpha = -1, \\ 0 \le \beta \le \gamma < +\infty & \text{when } p = 1 \text{ and } \alpha < -1, \\ -\infty < \beta < +\infty,\ \gamma \ge \alpha + \beta + 1 & \text{when } 1 < p < Q \text{ and } \alpha > -1, \\ -\infty < \beta < \gamma < +\infty & \text{when } 1 < p < Q \text{ and } \alpha = -1, \\ -\infty < \beta \le \gamma < +\infty & \text{when } 1 < p < Q \text{ and } \alpha < -1. \end{cases}$$

Proposition 5.2. *Let $n_i \ge 1$ and $(p_i, \alpha_i, \beta_i, \gamma_i) \in H_i$ $(i = 1, 2)$. Let ρ satisfy (2.3) and*

$$\rho(r) = \begin{cases} L_{[n_1, \alpha_1]}(r) & \textit{for small } r, \\ L^{[n_2, \alpha_2]}(r) & \textit{for large } r. \end{cases}$$

Let Φ and Ψ be Young functions such that

$$\Phi(\xi)=\begin{cases}\xi^{p_2}(l_{n_2}(1/\xi))^{-p_2\beta_2} & \textit{for small } \xi,\\ \xi^{p_1}(l_{n_1}(\xi))^{p_1\beta_1} & \textit{for large } \xi,\end{cases}$$

$$\Psi(\xi)=\begin{cases}\xi^{p_2}(l_{n_2}(1/\xi))^{-p_2\gamma_2} & \textit{for small } \xi,\\ \xi^{p_1}(l_{n_1}(\xi))^{p_1\gamma_1} & \textit{for large } \xi.\end{cases}$$

Then (2.11) *holds. Therefore I_ρ is bounded from $L^\Phi(X)$ to $L^\Psi_{weak}(X)$ for $p_1=1$ or $p_2=1$, and, from $L^\Phi(X)$ to $L^\Psi(X)$ for $1<p_i<Q$ $(i=1,2)$.*

Proof. First we note that

$$\Phi^{-1}(\xi)\sim\begin{cases}\xi^{1/p_2}(l_{n_2}(1/\xi))^{\beta_2} & \text{for small } \xi,\\ \xi^{1/p_1}(l_{n_1}(\xi))^{-\beta_1} & \text{for large } \xi\end{cases}$$

follows from

$$\begin{cases}\Phi(\xi^{1/p_2}(l_{n_2}(1/\xi))^{\beta_2})\sim\xi & \text{for small } \xi,\\ \Phi(\xi^{1/p_1}(l_{n_1}(\xi))^{-\beta_1})\sim\xi & \text{for large } \xi,\end{cases}$$

and $\Phi^{-1}\in\Delta_2$. Similarly,

$$\Psi^{-1}(\xi)\sim\begin{cases}\xi^{1/p_2}(l_{n_2}(1/\xi))^{\gamma_2} & \text{for small } \xi,\\ \xi^{1/p_1}(l_{n_1}(\xi))^{-\gamma_1} & \text{for large } \xi.\end{cases}$$

There exist small constant r_1 and large constant r_2 such that, for $0<r\le r_1$,

$$\int_0^r\frac{\rho(t)}{t}\,dt\sim(l_{n_1}(1/r))^{-\alpha_1+1},$$
$$\Phi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_1}(l_{n_1}(1/r))^{-\beta_1},$$
$$\Psi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_1}(l_{n_1}(1/r))^{-\gamma_1},$$
$$\rho(r)\Phi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_1}L_{[n_1,\alpha_1+\beta_1]}(r),$$

and, for $r\ge r_2$,

$$\int_0^r\frac{\rho(t)}{t}\,dt\sim\begin{cases}(l_{n_2}(r))^{\alpha_2+1}, & \text{when } \alpha_2>-1,\\ (l_{n_2+1}(r)), & \text{when } \alpha_2=-1,\\ 1, & \text{when } \alpha_2<-1,\end{cases}$$
$$\Phi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_2}(l_{n_2}(r))^{\beta_2},$$
$$\Psi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_2}(l_{n_2}(r))^{\gamma_2},$$
$$\rho(r)\Phi^{-1}(1/r^Q)\sim(1/r^Q)^{1/p_2}L^{[n_2,\alpha_2+\beta_2]}(r).$$

Then we have (2.16). Since $r^\varepsilon\rho(r)\Phi^{-1}(1/r^Q)$ is almost decreasing for some $\varepsilon>0$, by Remark 2.6 we have (2.15). Applying Corollary 2.3, we obtain the desired result. ☐

Proposition 5.3. *Let $n_i\ge 1,\alpha_i>0$ $(i=1,2)$. Let ρ satisfy* (2.3) *and*

$$\rho(r)=\begin{cases}r^Q(l_{n_1}(1/r))^{\alpha_1} & \textit{for small } r,\\ r^Q(l_{n_2}(r))^{-\alpha_2} & \textit{for large } r.\end{cases}$$

Let $\Phi(\xi)=\xi$, and Ψ be N-function such that

$$\Psi(\xi)=\begin{cases}1/e_{n_2}((1/\xi)^{1/\alpha_2}) & \textit{for small } \xi,\\ e_{n_1}(\xi^{1/\alpha_1}) & \textit{for large } \xi.\end{cases}$$

Then (2.11) *holds and* I_ρ *is bounded from* $L^1(X)$ *to* $L^{\Psi}_{weak}(X)$.

Proof. By Remark 2.4 we have (2.9). There exist small constant r_1 and large constant r_2 such that, for $0 < r \le r_1$,

$$\begin{aligned}&\int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r) = r^Q(l_{n_1}(1/r))^{\alpha_1},\\ &\Phi^{-1}(1/r^Q) = 1/r^Q,\\ &\Psi^{-1}(1/r^Q) \sim (l_{n_1}(1/r))^{\alpha_1},\end{aligned}$$

and, for $r \ge r_2$,

$$\begin{aligned}&\int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r) = r^Q(l_{n_2}(r))^{-\alpha_2}\\ &\Phi^{-1}(1/r^Q) = 1/r^Q,\\ &\Psi^{-1}(1/r^Q) \sim (l_{n_2}(r))^{-\alpha_2}.\end{aligned}$$

Then we have (2.10). Applying Theorem 2.1, we obtain the desired result. □

Proposition 5.4. *Let* $n_i \ge 1, \alpha_i > \beta_i > 0$ $(i = 1, 2)$. *Let* ρ *satisfy* (2.3) *and*

$$\rho(r) = \begin{cases} r^Q(l_{n_1}(1/r))^{\alpha_1} & \text{for small } r,\\ r^Q(l_{n_2}(r))^{-\alpha_2} & \text{for large } r.\end{cases}$$

Let Φ *and* Ψ *be N-functions such that*

$$\Phi(\xi) = \begin{cases} \xi(l_{n_2}(1/\xi))^{-\beta_2} & \text{for small } \xi,\\ \xi(l_{n_1}(\xi))^{\beta_1} & \text{for large } \xi,\end{cases}$$

$$\Psi(\xi) = \begin{cases} 1/e_{n_2}((1/\xi)^{1/(\alpha_2-\beta_2)}) & \text{for small } \xi,\\ e_{n_1}(\xi^{1/(\alpha_1-\beta_1)}) & \text{for large } \xi.\end{cases}$$

Then (2.11) *holds and* I_ρ *is bounded from* $L^\Phi(X)$ *to* $L^{\Psi}_{weak}(X)$.

Proof. First we note that

$$\Phi^{-1}(\xi) \sim \begin{cases} \xi(l_{n_2}(1/\xi))^{\beta_2} & \text{for small } \xi,\\ \xi(l_{n_1}(\xi))^{-\beta_1} & \text{for large } \xi.\end{cases}$$

Let $\widetilde{\Phi}$ be the complementary function with respect to Φ. From (3.4) it follows that

$$\widetilde{\Phi}^{-1}(\xi) \sim \begin{cases} (l_{n_2}(1/\xi))^{-\beta_2} & \text{for small } \xi,\\ (l_{n_1}(\xi))^{\beta_1} & \text{for large } \xi.\end{cases}$$

Then there exist constants ξ_1, ξ_2 $(0 < \xi_2 < \xi_1)$ such that

$$1/e_{n_2}((\xi/C)^{-1/\beta_2}) \le \widetilde{\Phi}(\xi) \le 1/e_{n_2}((C\xi)^{-1/\beta_2}), \qquad 0 \le \xi \le \xi_2, \tag{5.1}$$

$$e_{n_1}((\xi/C)^{1/\beta_1}) \le \widetilde{\Phi}(\xi) \le e_{n_1}((C\xi)^{1/\beta_1}), \qquad \xi \ge \xi_1. \tag{5.2}$$

There exist small constant r_1 and large constant r_2 such that, for $0 < r \le r_1$,

$$\begin{aligned}&\int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r) = r^Q(l_{n_1}(1/r))^{\alpha_1},\\ &\Phi^{-1}(1/r^Q) \sim (1/r^Q)(l_{n_1}(1/r))^{-\beta_1},\\ &\Psi^{-1}(1/r^Q) \sim (l_{n_1}(1/r))^{\alpha_1-\beta_1},\end{aligned}$$

and, for $r \geq r_2$,

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r) = r^Q (l_{n_2}(r))^{-\alpha_2}$$
$$\Phi^{-1}(1/r^Q) \sim (1/r^Q)(l_{n_2}(r))^{\beta_2},$$
$$\Psi^{-1}(1/r^Q) \sim (l_{n_2}(r))^{-\alpha_2+\beta_2}.$$

Then we have (2.10).

There exist constants $\delta, M, r_3, r_4 > 0$ ($r_3 \leq r_1$, $r_2 \leq r_4$) such that

$$e_{n_1}\left(\delta\, l_{n_1}\left(\frac{1}{t}\right)\right) \leq \left(\frac{1}{t}\right)^{Q/2}, \qquad 0 < t \leq r_3, \tag{5.3}$$

$$e_{n_2}(M\, l_{n_2}(t)) \geq t^{2Q}, \qquad t \geq r_4. \tag{5.4}$$

Let $A > 0$ be sufficiently large. Let $F(r)$ be as (4.3).

Let $0 < r \leq t \leq r_3$. If $\rho(t)/(AF(r)t^Q) > \xi_1$, then (5.2) and (5.3) show

$$\begin{aligned}\widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^Q}\right) &\leq e_{n_1}\left(\left(\frac{C\rho(t)}{AF(r)t^Q}\right)^{1/\beta_1}\right) \leq e_{n_1}\left(\left(\frac{C_4(l_{n_1}(1/t))^{\alpha_1}}{A(l_{n_1}(1/r))^{\alpha_1-\beta_1}}\right)^{1/\beta_1}\right)\\ &= e_{n_1}\left(\left(\frac{C_4(l_{n_1}(1/t))^{\alpha_1-\beta_1}}{A(l_{n_1}(1/r))^{\alpha_1-\beta_1}}\right)^{1/\beta_1} l_{n_1}\left(\frac{1}{t}\right)\right) \leq e_{n_1}\left(\left(\frac{C_4}{A}\right)^{1/\beta_1} l_{n_1}\left(\frac{1}{t}\right)\right) \leq \left(\frac{1}{t}\right)^{Q/2}.\end{aligned}$$

If $\rho(t)/(AF(r)t^Q) \leq \xi_1$, then $\widetilde{\Phi}(\rho(t)/(AF(r)t^Q)) \leq \widetilde{\Phi}(\xi_1)$. Hence

$$\int_r^{r_3} \widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^Q}\right) t^{Q-1}\,dt \leq C, \quad 0 < r \leq r_3. \tag{5.5}$$

Let $r_4 \leq r \leq t$. Since

$$\frac{\rho(t)}{F(r)t^Q} \sim \frac{(l_{n_2}(t))^{-\alpha_2}}{(l_{n_2}(r))^{-\alpha_2+\beta_2}} \leq \frac{1}{(l_{n_2}(r_4))^{\beta_2}},$$

we may assume $\rho(t)/(AF(r)t^Q) < \xi_2$. Then (5.1) and (5.4) show

$$\begin{aligned}\widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^Q}\right) &\leq 1/e_{n_2}\left(\left(\frac{C\rho(t)}{AF(r)t^Q}\right)^{-1/\beta_2}\right) \leq 1/e_{n_2}\left(\left(\frac{C_5(l_{n_2}(t))^{-\alpha_2}}{A(l_{n_2}(r))^{-\alpha_2+\beta_2}}\right)^{-1/\beta_2}\right)\\ &= 1/e_{n_2}\left(\left(\frac{C_5(l_{n_2}(t))^{-\alpha_2+\beta_2}}{A(l_{n_2}(r))^{-\alpha_2+\beta_2}}\right)^{-1/\beta_2} l_{n_2}(t)\right) \leq 1/e_{n_2}\left(\left(\frac{C_5}{A}\right)^{-1/\beta_2} l_{n_2}(t)\right) \leq t^{-2Q}.\end{aligned}$$

Hence

$$\int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^Q}\right) t^{Q-1}\,dt \leq C, \quad r \geq r_4. \tag{5.6}$$

Using (5.5), (5.6) and the almost decreasingness of $F(r)$, we have (2.9).

Applying Theorem 2.1, we obtain the desired result. □

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