ON GENERALIZED RIESZ POTENTIALS AND SPACES OF SOME SMOOTH FUNCTIONS

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Abstract. Let (X, δ, μ) be a normal space of homogeneous type of order γ. Gatto and Vági [7] showed that, if f and $I_\alpha f$ are in $L^p(X)$ $(0 < \alpha < \min(\gamma, 1/p))$, then $I_\alpha f$ is in $C^{p,\alpha}(X)$, where I_α is the Riesz potential of order α and $C^{p,\alpha}$ is the space of smooth functions of Calderón-Scott [1]. In this paper, we introduce a generalized Riesz potential I_ϕ and extend the result above. With this aim, we extend the Hardy-Littlewood-Sobolev inequality to the Orlicz space.

1. Introduction

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$
$$d(x, y) = d(y, x),$$
$$d(x, y) \leq K_1 \, (d(x, z) + d(z, y)),$$

the balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x, 2r)) \leq K_2 \, \mu(B(x, r)) < \infty,$$

where $K_i \geq 1$ $(i = 1, 2)$ are constants independent of $x, y, z \in X$ and $r > 0$. Following [5], we assume that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$.

We assume that $X = (X, d, \mu)$ is of order γ $(0 < \gamma \leq 1)$ and Q-homogeneous $(Q > 0)$, i.e.

$$|d(x, z) - d(y, z)| \leq K_3 d(x, y)^\gamma (d(x, z) + d(y, z))^{1-\gamma}, \tag{1.1}$$

$$K_4{}^{-1} r^Q \leq \mu(B(x, r)) \leq K_4 r^Q, \tag{1.2}$$

where $K_i \geq 1$ $(i = 3, 4)$ are constants independent of $x, y, z \in X$ and $r > 0$. From (1.2) it follows that $\mu(\{x\}) = 0$ for all $x \in X$.

The n-dimensional Euclidean space $\mathbb{R}^n$ is of order 1 and n-homogeneous.

For an increasing function $\phi : (0, \infty) \to (0, \infty)$, let

$$I_\phi f(x) = \int_X f(y) \frac{\phi(d(x, y))}{d(x, y)^Q} \, d\mu(y).$$

If $\phi(r) = r^\alpha$, $0 < \alpha < Q$, then I_ϕ is the Riesz potential of order α.

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For $f \in L^p(X)$, $1 < p < \infty$, we consider the sharp functions

$$f^\sharp_\phi(x) = \sup_{x\in B(a,r)} \frac{1}{\phi(r)\mu(B(a,r))} \int_{B(a,r)} |f(y) - f_{B(a,r)}|\, d\mu(y)$$

where $f_{B(a,r)} = \mu(B(a,r))^{-1} \int_{B(a,r)} f(y)\, d\mu(y)$ and the supremum is taken over all balls $B(a,r)$ containing x. The space $C^{p,\phi}(X)$ is the set of all functions $f \in L^p(X)$ with $f^\sharp_\phi \in L^p(X)$ equipped with the norm $\|f\|_{C^{p,\phi}} = \|f^\sharp_\phi\|_p + \|f\|_p$, where $\|\ \|_p$ denotes the L^p-norm.

Our main results are as follows:

Theorem 1.1. *Let $1 < p < \infty$. Assume that ϕ is increasing, $\phi(r)/r^{(Q/p-\varepsilon)}$ is decreasing for some $\varepsilon > 0$, and $\int_0^1 (\phi(t)/t)\, dt + \int_1^\infty (\phi(t)/t^{1+\gamma})\, dt < \infty$. Let*

$$\psi(r) = \int_0^r \frac{\phi(t)}{t}\, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}}\, dt, \quad 0 < r < \infty. \tag{1.3}$$

If f and $F = I_\phi f$ are in $L^p(X)$, then F is in $C^{p,\psi}(X)$ and $\|F\|_{C^{p,\psi}} \le C(\|F\|_p + \|f\|_p)$ with a constant C independent of F and f.

Remark 1.1. If ϕ is increasing and $\phi(r)/r^Q$ is decreasing, then ϕ is continuous and

$$\phi(r) \le \phi(2r) \le 2^Q \phi(r), \tag{1.4}$$

$$\phi(r) \le \left(\int_0^r \frac{\phi(t)}{t}\, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}}\, dt \right).$$

Corollary 1.2. *Let $1 < p < \infty$. Assume that ϕ is increasing, $\phi(r)/r^{(Q/p-\varepsilon)}$ is decreasing for some $\varepsilon > 0$, and there exists a constant $C_0 > 0$ such that*

$$\int_0^r \frac{\phi(t)}{t}\, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}}\, dt \le C_0 \phi(r), \quad 0 < r < \infty. \tag{1.5}$$

If f and $F = I_\phi f$ are in $L^p(X)$, then F is in $C^{p,\phi}(X)$ and $\|F\|_{C^{p,\phi}} \le C(\|F\|_p + \|f\|_p)$ with a constant C independent of F and f.

Remark 1.2. If $\phi(r) = r^\alpha$, $0 < \alpha < \min(\gamma, Q/p)$, then ϕ satisfies (1.5). Therefore the result of [7, Theorem 2.1] is contained in this corollary.

To prove the results above, we extend the Hardy-Littlewood-Sobolev inequality to the Orlicz space L^Φ. The definitions of the N-function Φ and the Orlicz space L^Φ are in next section.

Theorem 1.3. *Let $1 < s < \infty$. Assume that ϕ is increasing, $\phi(r)/r^{(Q/s-\varepsilon)}$ is decreasing for some $\varepsilon > 0$, and $\int_0^1 (\phi(t)/t)\, dt < \infty$. Then there exists an N-function Φ such that*

$$C^{-1}\Phi^{-1}\left(\frac{1}{r^Q}\right) \le \frac{1}{r^{Q/s}} \int_0^r \frac{\phi(t)}{t}\, dt \le C\Phi^{-1}\left(\frac{1}{r^Q}\right), \quad 0 < r < \infty, \tag{1.6}$$

and I_ϕ is bounded from $L^s(X)$ to $L^\Phi(X)$.

Section 3 is for preliminalies. In Section 4 we give proofs of the theorems. In Section 5 we give examples.

The letter C will denote a constant, not neccessarily the same indifferent occurrences.

2. Orlicz spaces

In this section, we recall the definition of Orlicz spaces.

A function $\Phi : [0, \infty) \to [0, \infty)$ is called an N-function if it can be represented as

$$\Phi(r) = \int_0^r a(t)\, dt,$$

where $a : [0, \infty) \to [0, \infty)$ is a right continuous nondecreasing function such that $a(0) = 0$, $a(t) > 0$ if $t > 0$, and, $a(t) \to \infty$ as $t \to \infty$. Let

$$b(r) = \sup\{s : a(s) \le r\}.$$

Then

$$\Psi(r) = \int_0^r b(t)\, dt$$

is also an N-function, and (Φ, Ψ) is called a complementary pair.

Let (X, μ) be a measure space. For an N-function Φ, let

$$L^{\Phi}(X) = \left\{ f : \int_X \Phi(\varepsilon|f(x)|)\, d\mu(x) < \infty \ \text{ for some } \ \varepsilon > 0 \right\},$$

$$\|f\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda} \right) d\mu(x) \le 1 \right\}.$$

Let (Φ, Ψ) be a complementary pair of N-functions. We note that

$$\int_X |f(x)g(x)|\, d\mu(x) \le 2\|f\|_{\Phi}\|g\|_{\Psi}, \tag{2.1}$$

and that

$$r \le \Phi^{-1}(r)\Psi^{-1}(r), \quad r \ge 0, \tag{2.2}$$

where Φ^{-1} and Ψ^{-1} are inverse functions of Φ and Ψ, respectively. Let (X, d, μ) be a space of homogeneous type, and $\chi_{B(a,r)}$ be the characteristic function of a ball $B(a, r)$. Then

$$\begin{aligned} \|\chi_{B(a,r)}\|_{\Psi} &= \inf \left\{ \lambda > 0 : \int_X \Psi\left(\frac{\chi_{B(a,r)}(x)}{\lambda} \right) d\mu(x) \le 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \Psi\left(\frac{1}{\lambda} \right) \mu(B(a,r)) \le 1 \right\} \\ &= \frac{1}{\Psi^{-1}(1/\mu(B(a,r)))} \le \mu(B(a,r))\Phi^{-1}\left(\frac{1}{\mu(B(a,r))} \right). \end{aligned} \tag{2.3}$$

3. Preliminalies

In this section, we show lemmas to prove theorems.

Lemma 3.1. *Let $\alpha > 0$, $\beta > 0$, $\delta > 0$, $\phi : (0, \infty) \to (0, \infty)$ be increasing and $\phi(r)/r^{\alpha}$ be decreasing. Then, for $0 < r < \infty$,*

$$\left(\frac{1}{(\alpha + \beta)\delta} \right)^{1/\delta} \frac{\phi(r)}{r^{\alpha+\beta}} \le \left(\int_r^{\infty} \left(\frac{\phi(t)}{t^{\alpha+\beta}} \right)^{\delta} t^{-1}\, dt \right)^{1/\delta} \le \left(\frac{1}{\beta\delta} \right)^{1/\delta} \frac{\phi(r)}{r^{\alpha+\beta}}.$$

Proof. By the increasingness of ϕ we have

$$\int_r^\infty \left(\frac{\phi(t)}{t^{\alpha+\beta}}\right)^\delta t^{-1}\,dt = \int_r^\infty \phi(t)^\delta t^{-1-(\alpha+\beta)\delta}\,dt \ge \phi(r)^\delta \int_r^\infty t^{-1-(\alpha+\beta)\delta}\,dt = \frac{1}{(\alpha+\beta)\delta}\left(\frac{\phi(r)}{r^{\alpha+\beta}}\right)^\delta.$$

By the decreasingness of $\phi(r)/r^\alpha$ we have

$$\int_r^\infty \left(\frac{\phi(t)}{t^{\alpha+\beta}}\right)^\delta t^{-1}\,dt = \int_r^\infty \left(\frac{\phi(t)}{t^{\alpha}}\right)^\delta t^{-1-\beta\delta}\,dt \le \left(\frac{\phi(r)}{r^{\alpha}}\right)^\delta \int_r^\infty t^{-1-\beta\delta}\,dt = \frac{1}{\beta\delta}\left(\frac{\phi(r)}{r^{\alpha+\beta}}\right)^\delta. \quad \square$$

Lemma 3.2. *Let $\alpha > 0$, $\beta > 0$, $\alpha+\beta < Q$, $h : (0,\infty) \to (0,\infty)$ be increasing and differentiable, and $h(r)/r^\alpha$ be decreasing. Then there exists an N-function Φ such that*

$$C^{-1}\Phi^{-1}\left(\frac{1}{r^Q}\right) \le \frac{h(r)}{r^{\alpha+\beta}} \le C\Phi^{-1}\left(\frac{1}{r^Q}\right), \quad 0 < r < \infty, \tag{3.1}$$

where $C > 0$ is independent of r.

Proof. Let

$$H(r) = \int_r^\infty \frac{h(t)}{t^{\alpha+\beta}} t^{-1}\,dt.$$

Then H is decreasing, differentiable and $H'(r) < 0$ for all $r > 0$. Applying Lemma 3.1 with $\delta = 1$, we have that $H(r)$ is comparable to $h(r)/r^{\alpha+\beta}$, and so

$$\lim_{r\to+0} H(r) = \infty \quad \text{and} \quad \lim_{r\to\infty} H(r) = 0.$$

Hence H is bijective from $(0,\infty)$ to itself. Let

$$\Phi(u) = \begin{cases} 0, & u = 0 \\ 1/(H^{-1}(u))^Q & u > 0. \end{cases}$$

Then

$$\Phi^{-1}\left(\frac{1}{r^Q}\right) = \int_r^\infty \frac{h(t)}{t^{\alpha+\beta}} t^{-1}\,dt,$$

and we have (3.1). Next we show that Φ is an N-function, i.e., $\lim_{u\to+0}\Phi'(u) = 0$, $\lim_{u\to\infty}\Phi'(u) = \infty$ and $\Phi''(u) \ge 0$. Let

$$u = H(r) = \Phi^{-1}\left(\frac{1}{r^Q}\right), \quad v = \frac{1}{r^Q}.$$

Then $v = \Phi(u)$ and

$$\Phi'(u) = \frac{dv}{du} = \frac{dv}{dr}\Big/\frac{du}{dr} = \left(-\frac{Q}{r^{Q+1}}\right)\Big/\left(-\frac{h(r)}{r^{\alpha+\beta+1}}\right) = \frac{Q}{r^{Q-\alpha-\beta}h(r)}.$$

If $u \to +0$, then $r \to \infty$ and $\Phi'(u) \to 0$. If $u \to \infty$, then $r \to +0$ and $\Phi'(u) \to \infty$. Since du/dv is decreasing with respect to r, we have $d(du/dv)/dr \le 0$. Hence

$$\frac{d^2 v}{du^2} = \left(\frac{d}{dr}\frac{dv}{du}\right)\Big/\frac{du}{dr} \ge 0. \quad \square$$

Remark 3.1. If ϕ is increasing, $\phi(r)/r^\alpha$ is decreasing, and $\int_0^1 (\phi(t)/t)\,dt < \infty$, then $h(r) = \int_0^r (\phi(t)/t)\,dt$ is increasing and differentiable, and $h(r)/r^\alpha$ is decreasing. Actually,

$$\begin{aligned}\frac{d}{dr}\left(\frac{h(r)}{r^\alpha}\right) = \frac{rh'(r)-\alpha h(r)}{r^{\alpha+1}} = \frac{1}{r^{\alpha+1}}\left(\phi(r) - \alpha\int_0^r \frac{\phi(t)}{t^\alpha}t^{\alpha-1}\,dt\right)\\ \le \frac{1}{r^{\alpha+1}}\left(\phi(r) - \alpha\frac{\phi(r)}{r^\alpha}\int_0^r t^{\alpha-1}\,dt\right) = 0.\end{aligned}$$

Lemma 3.3. *Let ϕ be increasing and $\phi(r)/r^Q$ be decreasing. If $2K_1 d(x,x') \le d(x,y)$, then*

$$\left|\frac{\phi(d(x,y))}{d(x,y)^Q} - \frac{\phi(d(x',y))}{d(x',y)^Q}\right| \le C d(x,x')^\gamma \frac{\phi(d(x,y))}{d(x,y)^{Q+\gamma}}, \tag{3.2}$$

where $C>0$ is independent of $x, x', y \in X$.

Proof. By mean value theorem we have that, for $u < v$, there exists r_0 such that

$$\frac{1}{u^Q} - \frac{1}{v^Q} = \frac{v-u}{{r_0}^{Q+1}}, \quad u < r_0 < v.$$

Hence

$$0 \le \frac{\phi(u)}{u^Q} - \frac{\phi(v)}{v^Q} \le \phi(u)\left(\frac{1}{u^Q} - \frac{1}{v^Q}\right) = Q\phi(u)\frac{v-u}{{r_0}^{Q+1}} \le Q(v-u)\frac{\phi(u)}{u^{Q+1}}.$$

Let $u = \min(d(x,y), d(x',y))$ and $v = \max(d(x,y), d(x',y))$. Then

$$\begin{aligned} v-u &\le K_3 d(x,x')^\gamma (d(x,y)+d(x',y))^{1-\gamma} \\ &\le K_3\left(K_1 + \frac{3}{2}\right)^{1-\gamma} d(x,x')^\gamma d(x,y)^{1-\gamma}, \end{aligned}$$

and

$$\frac{d(x,y)}{2K_1} \le u \le d(x,y).$$

Hence

$$(v-u)\frac{\phi(u)}{u^{Q+1}} \le C d(x,x')^\gamma \frac{\phi(d(x,y))}{d(x,y)^{Q+\gamma}}.$$

Therefore we have (3.2). □

The following is used in the proof of Theorem 1.1. For all balls B and for all integrable functions f on B,

$$\frac{1}{\mu(B)}\int_B |f(y) - f_B|\,d\mu(y) \le 2\inf_c \frac{1}{\mu(B)}\int_B |f(y) - c|\,d\mu(y). \tag{3.3}$$

4. Proofs of Theorems

Proof of Theorem 1.3. By Lemma 3.2 and Remark 3.1 we have an N-function Φ with the property (1.6). For $r > 0$, let

$$\begin{aligned} J_1 &= \int_{d(x,y)<r} f(y)\frac{\phi(d(x,y))}{d(x,y)^Q}\,d\mu(y) \quad \text{and} \\ J_2 &= \int_{d(x,y)\ge r} f(y)\frac{\phi(d(x,y))}{d(x,y)^Q}\,d\mu(y). \end{aligned}$$

Since $\phi(r)/r^Q$ is decreasing,

$$|J_1| \le Mf(x)\int_{d(x,y)<r} \frac{\phi(d(x,y))}{d(x,y)^Q}\,d\mu(y), \tag{4.1}$$

where M is the Hardy-Littlewood maximal function (see Stein[10, p.57]). By (1.2) and (1.4) we have

$$\int_{r_j \le d(x,y) < 2r_j} \frac{\phi(d(x,y))}{d(x,y)^Q}\, d\mu(y) \le \frac{\phi(r_j)}{{r_j}^Q}\mu(B(x,2r_j)) \le C\phi(r_j) \le C' \int_{r_j}^{2r_j} \frac{\phi(t)}{t}\, dt, \quad r_j = 2^{-j}r,\ j = 1,2,\dots. \tag{4.2}$$

From (4.1) and (4.2) it follows that

$$|J_1| \le CMf(x)\int_0^r \frac{\phi(t)}{t}\, dt. \tag{4.3}$$

Next we estimate $|J_2|$. Let $1/s + 1/s' = 1$. Let $\chi_{B(x,r)^c}$ be the characteristic function of $B(x,r)^c$. By Hölder's inequality we have

$$|J_2| \le \|f\|_s \left\| \frac{\phi(d(x,\cdot))}{d(x,\cdot)^Q}\chi_{B(x,r)^c}(\cdot)\right\|_{s'} = \|f\|_s \left(\int_{d(x,y)\ge r} \left(\frac{\phi(d(x,y))}{d(x,y)^Q}\right)^{s'} d\mu(y)\right)^{1/s'}. \tag{4.4}$$

By (1.2) and (1.4) we have

$$\int_{r_j \le d(x,y) < 2r_j} \left(\frac{\phi(d(x,y))}{d(x,y)^Q}\right)^{s'} d\mu(y) \le \left(\frac{\phi(r_j)}{{r_j}^Q}\right)^{s'} \mu(B(x,2r_j)) \le C\left(\frac{\phi(r_j)}{{r_j}^{Q/s}}\right)^{s'} \le C' \int_{r_j}^{2r_j} \left(\frac{\phi(t)}{t^{Q/s}}\right)^{s'} t^{-1}\, dt, \quad r_j = 2^j r,\ j = 0,1,2,\dots. \tag{4.5}$$

By Lemma 3.1 we have

$$\left(\int_r^\infty \left(\frac{\phi(t)}{t^{Q/s}}\right)^{s'} t^{-1}\, dt\right)^{1/s'} \le C\frac{\phi(r)}{r^{Q/s}}. \tag{4.6}$$

From (4.4), (4.5) and (4.6) it follows that

$$|J_2| \le C\|f\|_s \frac{\phi(r)}{r^{Q/s}}. \tag{4.7}$$

By (4.3) and (4.7) we have

$$|I_\phi f(x)| \le C\left(Mf(x) + \|f\|_s \frac{1}{r^{Q/s}}\right)\int_0^r \frac{\phi(t)}{t}\, dt.$$

We note that there exists a constant $C_s > 0$ such that

$$\|Mf\|_s \le C_s\|f\|_s, \quad \text{for} \quad f \in L^s(X).$$

Set $r = (1/\sigma)^{s/Q}$ and $\sigma = Mf(x)/(C_s\|f\|_s)$. Then

$$Mf(x) + \|f\|_s\frac{1}{r^{Q/s}} = \left(1 + \frac{1}{C_s}\right)Mf(x),$$

and

$$\int_0^r \frac{\phi(t)}{t}\, dt \le Cr^{Q/s}\Phi^{-1}\left(\frac{1}{r^Q}\right) = C\frac{\Phi^{-1}(\sigma^s)}{\sigma}.$$

Therefore

$$|I_\phi f(x)| \le CMf(x)\frac{\Phi^{-1}(\sigma^s)}{\sigma} = C\Phi^{-1}\left(\left(\frac{Mf(x)}{C_s\|f\|_s}\right)^s\right)\|f\|_s,$$

i.e.

$$\Phi\left(\frac{I_\phi f(x)}{C\|f\|_s}\right) \leq \left(\frac{Mf(x)}{C_s\|f\|_s}\right)^s .$$

This shows

$$\int_X \Phi\left(\frac{I_\phi f(x)}{C\|f\|_s}\right) d\mu(x) \leq 1,$$

and

$$\|I_\phi f(x)\|_\Phi \leq C\|f\|_s.$$

Proof of Theorem 1.1. Fix $x \in X$; we will estimate $F^\sharp_\psi(x)$. Let $B = B(a,r)$ be a ball containing x and $\tilde{B} = B(a, 2K_1 r)$. Let χ be the characteristic function of $\tilde{B}$. Set $F = F_1 + F_2$ with $F_1 = I_\phi(f\chi)$ and $F_2 = I_\phi(f(1-\chi))$.

To estimate $(F_1)^\sharp_\psi(x)$, let $1 < s < p$. By Theorem 1.3 we have an N-function Φ with the property (1.6) and

$$\|I_\phi f\|_\Phi \leq C\|f\|_s. \tag{4.8}$$

Let Ψ be the complement of Φ. From (2.1), (2.3), (1.6), (1.4) and (4.8), it follows that

$$\begin{aligned}\frac{1}{r^Q\psi(r)}\int_B |I_\phi(f\chi)(z)|\,d\mu(z) &\leq \frac{2}{r^Q\psi(r)}\|\chi_B\|_\Psi \|I_\phi(f\chi)\|_\Phi \\ &\leq \frac{2}{r^Q\psi(r)}\mu(B)\Phi^{-1}\left(\frac{1}{\mu(B)}\right)\|I_\phi(f\chi)\|_\Phi \leq \frac{C}{r^{Q/s}}\|f\chi\|_s \\ &= C\left(\frac{1}{r^Q}\int_{\tilde{B}}|f(z)|^s\,d\mu(z)\right)^{1/s} \leq C' M_s(f)(x),\end{aligned}$$

where $M_s(f) = [M(|f|^s)]^{1/s}$. By (3.3) we have

$$(F_1)^\sharp_\psi(x) \leq C M_s(f)(x). \tag{4.9}$$

Second we estimate $(F_2)^\sharp_\psi(x)$. Observe that

$$I_\phi(f(1-\chi))(z) - I_\phi(f(1-\chi))(a) = \int_{(\tilde{B})^c} f(y)\left(\frac{\phi(d(z,y))}{d(z,y)^Q} - \frac{\phi(d(a,y))}{d(a,y)^Q}\right) d\mu(y),$$

then by Lemma 3.3 we have

$$\begin{aligned}&\int_B |I_\phi(f(1-\chi))(z) - I_\phi(f(1-\chi))(a)|\,d\mu(z) \\ &\qquad \leq C\int_B d(a,z)^\gamma\left(\int_{(\tilde{B})^c}\frac{\phi(d(a,y))|f(y)|}{d(a,y)^{Q+\gamma}}\,d\mu(y)\right) d\mu(z).\end{aligned} \tag{4.10}$$

To estimate the inner integral we write

$$\begin{aligned}\int_{(\tilde{B})^c} \frac{\phi(d(a,y))|f(y)|}{d(a,y)^{Q+\gamma}}\, d\mu(y) &\le \sum_{k=1}^{\infty} \int_{2^k r \le d(a,y) < 2^{k+1} r} \frac{\phi(2^k r)|f(y)|}{(2^k r)^{Q+\gamma}}\, d\mu(y) \\ &\le \sum_{k=1}^{\infty} (2^{k+1} r)^Q \frac{\phi(2^k r)}{(2^k r)^{Q+\gamma}} \frac{1}{(2^{k+1} r)^Q} \int_{B(a, 2^{k+1} r)} |f(y)|\, d\mu(y) \\ &\le C \left(\sum_{k=1}^{\infty} \frac{\phi(2^k r)}{(2^k r)^{\gamma}} \right) Mf(x) \le C' \left(\sum_{k=1}^{\infty} \int_{2^{k-1} r}^{2^k r} \frac{\phi(t)}{t^{1+\gamma}}\, dt \right) Mf(x) \\ &= C' \left(\int_r^{\infty} \frac{\phi(t)}{t^{1+\gamma}}\, dt \right) Mf(x) \le C' \frac{\psi(r)}{r^{\gamma}} Mf(x).\end{aligned}$$

Using the estimate (4.10) and (3.3) we get

$$(F_2)^{\sharp}_{\psi}(x) \le CMf(x) \le CM_s(f)(x). \tag{4.11}$$

By (4.9), (4.11) and the fact that the sharp function operator is subadditive, we have

$$F^{\sharp}_{\psi}(x) \le CM_s(f)(x).$$

Finally, using the strong type p/s of M we have

$$\|F^{\sharp}_{\psi}\|_p \le C\|f\|_p.$$

This concludes the proof of Theorem 1.1.

5. Examples

For functions $\theta, \kappa : (0,\infty) \to (0,\infty)$, we denote $\theta(r) \sim \kappa(r)$, $u < r < v$, if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r), \quad u < r < v.$$

First we give examples of ψ in (1.3). Let $0 \le \alpha_i < \infty$ and $-\infty < \beta_i < \infty$ $(i = 1, 2)$. For constants r_1 and r_2 $(0 < r_1 < 1/e, e < r_2)$, let

$$\phi(r) = \begin{cases} k_1 r^{\alpha_1} (1/\log(1/r))^{\beta_1}, & 0 < r < r_1, \\ 1, & r_1 \le r \le r_2, \\ k_2 r^{\alpha_2} (\log r)^{\beta_2}, & r_2 < r < \infty, \end{cases} \tag{5.1}$$

where $k_1 = ({r_1}^{\alpha_1}(1/\log(1/r_1))^{\beta_1})^{-1}$ and $k_2 = ({r_2}^{\alpha_2}(\log r_2)^{\beta_2})^{-1}$.

If $\alpha_1,\ \alpha_2 > 0$, then

$$\int_0^r \frac{\phi(t)}{t}\, dt \sim \phi(r).$$

If $\alpha_1,\ \alpha_2 < \gamma$, then

$$r^{\gamma} \int_r^{\infty} \frac{\phi(t)}{t^{1+\gamma}}\, dt \sim \phi(r).$$

If $\alpha_1 = 0$ and $\beta_1 > 1$, i.e., $\phi(r) = k_1(1/\log(1/r))^{\beta_1}$, $0 < r < r_1$, then

$$r^{\gamma} \int_r^{\infty} \frac{\phi(t)}{t^{1+\gamma}}\, dt \sim \phi(r) \le C \int_0^r \frac{\phi(t)}{t}\, dt = C'(1/\log(1/r))^{\beta_1 - 1}, \quad 0 < r < r_1,$$

i.e.,

$$\psi(r) \sim (1/\log(1/r))^{\beta_1 - 1}, \quad 0 < r < r_1.$$

If $\alpha_2 = \gamma$, $\beta_2 < -1$, i.e., $\phi(r) = k_2 r^\gamma (\log r)^{\beta_2}$, $r > r_2$, then

$$\int_0^r \frac{\phi(t)}{t}\,dt \sim \phi(r) \le Cr^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}}\,dt = C' r^\gamma (\log r)^{\beta_2+1},\ r > r_2,$$

i.e.,

$$\psi(r) \sim r^\gamma (\log r)^{\beta_2+1}, \quad r > r_2.$$

The following example shows that we cannot replace $\int_0^r (\phi(t)/t)\,dt$ by $\phi(r)$ in Theorem 1.3. Let $X = \mathbb{R}^n$, $1 < s < \infty$ and ϕ is as in (5.1) with $\alpha_1 = 0$, $\beta_1 > 1$ and $0 < \alpha_2 < n/s$. Let $0 < \epsilon < n/s - \alpha_2$. Choose r_1 and r_2 so that ϕ is increasing and that $\phi(r)/r^{n/s-\epsilon}$ are decreasing. For $1 < \delta < s$, let

$$f(x) = \begin{cases} (1/|x|)^{n/s}(1/\log(1/|x|))^{\delta/s}, & |x| < r_1, \\ 0, & |x| \ge r_1, \end{cases} \quad x \in \mathbb{R}^n.$$

Then $f \in L^s(\mathbb{R}^n)$. From Theorem 1.3 it follows that there exists an N-function Φ such that

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \sim \frac{1}{r^{n/s}} \int_0^r \frac{\phi(t)}{t}\,dt,$$

and that $I_\phi f \in L^\Phi(\mathbb{R}^n)$. However, if there exists an N-function Φ_1 such that

$$\Phi_1^{-1}\left(\frac{1}{r^n}\right) \sim \frac{1}{r^{n/s}}\phi(r),$$

then $I_\phi f \notin L^{\Phi_1}(\mathbb{R}^n)$. Actually, if $|x| < r_1/2$ and $|y| < |x|/2$, then $|x|/2 \le |x-y| \le 3|x|/2$ and $f(x) \sim f(x-y)$. Hence,

$$\begin{aligned} I_\phi f(x) &\ge \int_{|y|\le|x|/2} f(x-y)\frac{\phi(|y|)}{|y|^n}\,dy \\ &\ge Cf(x)\int_{|y|\le|x|/2} \frac{\phi(|y|)}{|y|^n}\,dy \ge C' f(x)(1/\log(2/|x|))^{\beta_1-1} \\ &\ge C''(1/|x|)^{n/s}(1/\log(1/|x|))^{\beta_1} \sim \Phi_1^{-1}\left(\frac{1}{|x|^n}\right), \quad |x| < r_1/2. \end{aligned}$$

Since $\Phi_1(r) \le \Phi_1(2r) \le C\Phi_1(r)$, for any $\lambda > 0$, there exists a constant $\lambda' > 0$ such that

$$\Phi_1\left(\frac{I_\phi f(x)}{\lambda}\right) \ge \frac{1}{\lambda'}\frac{1}{|x|^n}, \quad |x| < \frac{r_1}{2}.$$

Therefore $I_\phi f \notin L^{\Phi_1}(\mathbb{R}^n)$.

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