# ON PRIME AND SEMIPRIME IDEALS IN GAMMA-SEMINEAR-RINGS

Kyung Ho Kim

Received August 14, 2000; revised February 13, 2001

Abstract. In this paper, we introduce the notion of prime and semiprime ideal and characterize prime and semiprime ideals in a $\Gamma$-seminear-ring. Among them, for any $\Gamma$-seminear-ring, an ideal is semiprime if and only if it is the intersection of all primes containg it. Moreover, an ideal of a $\Gamma$-seminear-ring is prime if and only if it is semiprime and strongly irreducible.

## 1. Introduction

In 1967, W. G. van Hoorn and B . van Rootselar introduced the concept of seminear-ring. In [2], we introduced the the notion of $\Gamma$-seminear-ring as a generalization of seminear-ring. In this paper, we introduce the notion of prime, semiprime ideal and $m$-system in $\Gamma$-seminear-ring. Using the notion of prime and semiprime ideals, we characterize prime and semiprime ideals in a $\Gamma$-seminear-ring. Among them, for any $\Gamma$-seminear-ring, an ideal is semiprime if and only if it is the intersection of all primes containg it. Moreover, an ideal of a $\Gamma$-seminear-ring is prime if and only if it is semiprime and strongly irreducible in $\Gamma$-seminear-ring.

## 2. Preliminaries

We first recall some basic concepts for the sake of completeness. Recall that a non-empty set $R$ with two binary operations "+"(addition) and "$\cdot$" (multiplication) is called a *seminear-ring* , if it satisfies the following axioms:

(i) $(R, +)$ and $(R, \cdot)$ are semigroups,

(ii) $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a right seminear-ring because it satisfies the right distributive law. We will use the word "seminear-ring" to mean "right seminear-ring". We denote $xy$ instead of $x \cdot y$.

## 3. Prime and semiprime ideals

We begin by defining the notion of a $\Gamma$-seminear-ring.

**Definition 3.1 ([2]).** A $\Gamma$-*seminear-ring* is a triple $(R, +, \Gamma)$ where

(i) $\Gamma$ is a non-empty set of binary operators on $R$ such that for each $\alpha \in \Gamma$, $(R, +, \alpha)$ is a seminear-ring,

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

**Example 3.2.** Let $R$ be the set of all integers of the form $4n + 3$ and $\Gamma$ the set of all integers of the form $4n + 1$. If " + " is an usual sum of integers and $x\alpha y = x + \alpha + y$ for $x, y \in R$ and $\alpha \in \Gamma$, then $(R, +, \Gamma)$ is a $\Gamma$-seminear-ring.

---

*2000 Mathematics Subject Classification.* Primary 16Y99.

*Key words and phrases.* $\Gamma$-seminear-ring, prime and semiprime ideal, $m$-system, irreducible.

**Definition 3.3 ([2]).** Let $R$ be a $\Gamma$-seminear-ring. A subsemigroup $A$ of $(R, +)$ is called a *left* (resp. *right*) *ideal* of $R$ if $R\Gamma A \subseteq A$ (resp. $A\Gamma R \subseteq A$). A left and right ideal is called an *ideal*.

Let $R$ be a $\Gamma$-seminear-ring and $I, J \subseteq R$. Denote $I\Gamma J = \{a\alpha b | a, b \in R \text{ and } \alpha \in \Gamma\}$.

**Definition 3.4 ([2]).** Let $R$ and $R'$ be $\Gamma$-seminear-rings. A mapping $f : R \to R'$ is called a $\Gamma$-*seminear-ring homomorphism* (briefly, $\Gamma$-*homomorphism*) if $f(x+y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in R$ and $\gamma \in \Gamma$.

**Lemma 3.5.** *Let $R$ and $R'$ be a $\Gamma$-seminear-ring $S$ and $f : R \to R'$ be a $\Gamma$-seminear-ring homomorphism. Then*

(i) $(I\Gamma J)\Gamma K = I\Gamma(J\Gamma K)$ *for all* $I, J, K \subseteq R$.
(ii) $f(I_1\Gamma I_2) = f(I_1)\Gamma f(I_2)$ *for all* $I_1, I_2 \subseteq R$.
(iii) $f^{-1}(J_1)\Gamma f^{-1}(J_2) \subset f^{-1}(J_1\Gamma J_2)$ *for all* $J_1, J_2 \in R'$.

*Proof.* The proof is easy. $\square$

**Definition 3.6.** Let $R$ be a $\Gamma$-seminear-ring. A proper ideal $P$ of $R$ is called *prime* if for any ideals $I$ and $J$, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

**Definition 3.7.** Let $S$ be a subset of a $\Gamma$-seminear-ring $R$. Then

(i) The right (resp. left) ideal generated by $S$ is the smallest right (resp. left) ideal containing $S$ and is denoted by $(S)_r$ (resp. $(S)_l$).
(ii) The ideal generated by $S$ is the smallest ideal containing $S$ and is denoted by $(S)$

For ecch $r$ of a $\Gamma$-seminear-ring, the smallest ideal containing $r$ is called the *principal ideal* generated by $r$ and is denoted by $(r)$.

**Proposition 3.8.** *Let $P$ be a proper ideal of a $\Gamma$-seminear-ring $R$. Then the following statements are equivalent.*

(i) *$P$ is prime*
(ii) *For ideals $I$ and $J$ of $R$, $(I\Gamma J) \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.*
(iii) *For elements $i$ and $j$ in $R$, $i \notin P$ and $j \notin P$ implies $(i)\Gamma(j) \nsubseteq P$.*

*Proof.* Clearly (i) and (ii) are equivalent.

(i)$\Rightarrow$ (iii): Let $P$ be a prime, $i \notin P$ and $j \notin P$. Suppose $(i)\Gamma(j) \subseteq P$. Then $(i) \subseteq P$ or $(j) \subseteq P$. So $i \in P$ or $j \in P$. This is a contradiction. Thus $(i)\Gamma(j) \nsubseteq P$.

(iii)$\Rightarrow$ (i): Assume that $I \nsubseteq P$ and $J \nsubseteq P$. Then there exists $i \in I\backslash P$ and $j \in J\backslash P$. So $(i)\Gamma(j) \subseteq I\Gamma J$ but $(i)\Gamma(j) \nsubseteq P$ by (iii). Thus $I\Gamma J \nsubseteq P$. $\square$

**Proposition 3.9.** *Let $P$ be a proper ideal of a $\Gamma$-seminear-ring $R$. If $\{a\alpha r\beta b | r \in R, \alpha, \beta \in \Gamma\} \subseteq P$ if and only if $a \in P$ or $b \in P$, then $P$ is prime.*

*Proof.* Let $H$ and $K$ be ideals of $R$ with $H\Gamma K \subseteq P$. Assume that $H \nsubseteq P$ and let $a \in H\backslash P$. Then for any $b \in K$, $\{a\alpha r\beta b | r \in R \text{ and } \alpha, \beta \in \Gamma\} \subseteq H\Gamma K \subseteq P$. Since $a \notin P$, $b \in P$. So $K \subseteq P$ and hence $P$ is prime. $\square$

**Proposition 3.10.** *Let $\{P_\alpha\}_{\alpha \in A}$ be a family of prime ideals which are totally ordered by set inclusion. Then $\cap_{\alpha \in A} P_\alpha$ is prime.*

*Proof.* Let $I$ and $J$ be ideals of $R$. If $I\Gamma J \subseteq \cap_{\alpha \in A} P_\alpha$, then $I\Gamma J \subseteq P_\alpha$, for all $\alpha \in A$. Assume that there exists $\alpha \in A$ such that $I \nsubseteq P_\alpha$. Then $J \subseteq P_\alpha$ and so $J \subseteq P_\beta$ for all $\beta \geq \alpha$. Suppose that there exist $\gamma < \alpha$ such that $J \nsubseteq P_\gamma$. Then $I \subseteq P_\gamma$ and so $I \subseteq P_\alpha$, This is impossible. Thus $J \subseteq P_\beta$ for any $\beta \in A$. Hence $\cap_{\alpha \in A} P_\alpha$ is prime. $\square$

**Proposition 3.11.** *Let $I$ be an ideal of a $\Gamma$-seminear-ring $R$ with $R+I \subseteq I$ and $I+R \subseteq I$. Let $P$ be a proper ideal of $R$ containing $I$. If $\pi : R \to R/I$ is the canonical epimorphism, then $P$ is prime if and only if $\pi(P)$is prime.*

*Proof.* Assume that $P$ is prime in $R$, $J_1$ and $J_2$ be ideals in $R/I$ such that $J_1\Gamma J_2 \subseteq \pi(P)$. Let $\pi^{-1}(J_1) = I_1$ and $\pi^{-1}(J_2) = I_2$. Then $I_1\Gamma I_2 = \pi^{-1}(J_1)\Gamma\pi^{-1}(J_2) \subseteq \pi^{-1}(J_1\Gamma J_2) \subseteq \pi^{-1}(\pi(P)) = P$. Since $P$ is prime, $I_1 \subseteq P$ or $I_2 \subseteq P$. So, $J_1 = \pi(\pi^{-1}(J_1)) = \pi(I_1) \subseteq \pi(P)$ or $J_2 = \pi(\pi^{-1}(J_2)) = \pi(J_2) \subseteq \pi(P)$. Thus $\pi(P)$ is prime. Conversely, let $\pi(P)$ be prime and let $I_1, I_2$ be ideals of $R$ such that $I_1\Gamma I_2 \subseteq P$. Then $\pi(I_1)\Gamma\pi(I_2) = \pi(I_1\Gamma I_2) \subseteq \pi(P)$. Since $\pi(P)$ is prime, $\pi(I_1) \subseteq \pi(P)$ or $\pi(I_2) \subseteq \pi(P)$. So $I_1 \subseteq P$ or $I_2 \subseteq P$. Thus $P$ is prime. $\square$

**Definition 3.12.** Let $R$ be a $\Gamma$-seminear-ring. A nonempty subset $M$ of $R$ is called an *$m$-system* if for $a, b \in M$, there exist $a_1 \in (a), b_1 \in (b)$ and $\alpha \in \Gamma$ such that $a_1\alpha b_1 \in M$.

**Proposition 3.13.** *Let $P$ be a proper ideal of a $\Gamma$-seminear-ring $R$. Then $P$ is prime if and only if $R\backslash P$ is an $m$-system.*

*Proof.* Assume that $P$ is prime. Let $a \in R\backslash P$ and $b \in R\backslash P$. Then $(a)\Gamma(b) \nsubseteq P$. So, there exist $a_1 \in (a), b_1 \in (b)$ and $\alpha \in \Gamma$ such that $a_1\alpha b_1 \notin P$, i.e., $a_1\alpha b_1 \in R\backslash P$. Thus $R\backslash P$ is an $m$-system. Conversely, if $R\backslash P$ is an $m$-system and let $a \in R\backslash P$ and $b \in R\backslash P$. Then there exist $a_1 \in (a), b \in (b)$ and $\alpha \in \Gamma$ such that $a_1\alpha b_1 \in R\backslash P$. Thus $(a)\Gamma(b) \nsubseteq P$ and hence $P$ is prime. $\square$

**Definition 3.14.** Let $R$ be a $\Gamma$-seminear-ring. Then $A \subseteq R$ is said to be *subtractive* if $a \in A$ and $a + b \in A$ imply $b \in A$.

**Lemma 3.15.** *Let $R$ be a $\Gamma$-seminear-ring whose ideals are subtractive. Let $P$ be a proper ideal of $R$. Then $P$ is prime if and only if for any ideals $I, J$ of $R$, $P \subset I$ and $P \subset J$ imply $I\Gamma J \nsubseteq P$.*

*Proof.* Assume that for any ideals $I, J$ of $R$, $P \subset I$ and $P \subset J$ imply $I\Gamma J \nsubseteq P$. Let $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $i \in I\backslash P$ and $j \in J\backslash P$ and so $P \subset P + (i)$. By hypothesis, $(P + (i))\Gamma(P + (j)) \nsubseteq P$ and so there exist $i' \in (i), j' \in (j)$ and $p, p' \in P$ and $\alpha \in \Gamma$ such that $(p + i')\alpha(p' + j') \notin P$. Since $p\alpha(p' + j') \in P$, $i'\alpha(p' + j') \notin P$. And since $P$ is an ideal, $i' \notin P$, and $p' + j' \notin P$. So $i' \notin P$ and $j' \notin P$ because $P$ is subtractive. Thus $(i')\Gamma(j') \nsubseteq P$. But $(i')\Gamma(j') \subseteq I\Gamma J$. So $I\Gamma J \nsubseteq P$. Hence $P$ is prime. The converse is obvious. $\square$

**Theorem 3.16.** *Let $M$ be a $m$-system of a $\Gamma$-seminear-ring $R$ whose ideals are subtractive. Let $I$ be an ideal with $I \cap M = \emptyset$. Then there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap M = \emptyset$.*

*Proof.* Let $\mathcal{I} = \{J | J$ is an ideal of $R, I \subseteq J$ and $J \cap M \neq \emptyset\}$ . Then $\mathcal{I} \neq \emptyset$. Let $\{J_\alpha\}_{\alpha\in A}$ be a chain in $\mathcal{I}$ under set inclusion. Then $I \subseteq \cap_{\alpha\in A} J_\alpha$ and $(\cup_{\alpha\in A} J_\alpha) \cap M = \cup_{\alpha\in A}(J_\alpha \cap M) = \emptyset$. So $\cup_{\alpha\in A} J_\alpha \in \mathcal{I}$. By Zorn's Lemma, $\mathcal{I}$ has a maximal element $P$. Now we claim that $P$ is prime. If $P \subset K_1$ and $P \subset K_2$, then there exist $k_1 \in K_1 \cap M, k_2 \in K_2 \cap M$ and $\alpha \in \Gamma$ such that $(k_1)\alpha(k_2) \subseteq K_1\Gamma K_2$ and there exist $k'_1 \in (k_1)$ and $k'_2 \in (k_2)$ such that $k'_1\alpha k'_2 \in M$. So $k'_1\alpha k'_2 \in K_1\Gamma K_2 \cap M$. Since $P \cap M = \emptyset$, $(K_1\Gamma K_2) \nsubseteq P$. Hence $P$ is prime. $\square$

**Definition 3.17.** A $\Gamma$-seminear-ring $R$ containing 0 is called a *prime $\Gamma$-seminear-ring* if $\{0\}$ is a prime ideal.

**Example 3.18.** Let $(R, +)$ be any $\Gamma$-semigroup with identity 0. For $a, b \in R$ and $\alpha \in \Gamma$, define $a\alpha b = a$ if $b \neq 0$ and $a\alpha b = 0$ if $b = 0$. Then $(R, +, \cdot)$ is a $\Gamma$-seminear-ring. Indeed it is prime. Let $I$ and $J$ be ideals such that $I \neq 0$ and $J \neq 0$. Then there exist $i \in I\backslash\{0\}, j \in J\backslash\{0\}$. So $i\alpha j = i \neq 0$. Thus $I\Gamma J \neq 0$ and hence $\{0\}$ is prime. $\square$

**Definition 3.19.** Let $R$ be a $\Gamma$-seminear-ring. An ideal $Q$ is said to be *semiprime* if for any ideal of $I$ of $R$, $I\Gamma I \subseteq Q$ implies $I \subseteq Q$. A nonempty subset $S$ is said an *sp-system* if for every $s \in S$, there exist $s_1, s_2 \in (s)$ ans $\alpha \in \Gamma$ such that $s_1 \alpha s_2 \in S$.

Clearly, every prime ideal is semiprime and each $m$-system is an $sp$-system.

**Proposition 3.20.** *Let $R$ be a $\Gamma$-seminear-ring and $Q$ an ideal of $R$. Then $Q$ is semiprime if and only if $R \backslash Q$ is an sp-system.*

*Proof.* Assume that $Q$ is semiprime. Let $a \in R\backslash Q$. Then $(a) \nsubseteq Q$ and so $(a)\Gamma(a) \nsubseteq Q$. Thus there exist $a_1, a_2 \in (a)$ and $\alpha \in \Gamma$ such that $a_1 \alpha a_2 \notin Q$. Hence $R\backslash Q$ is an $sp$-system. Conversely, assume that $R\backslash Q$ is an $sp$-system. Let $I$ be an ideal with $I\Gamma I \subseteq Q$. Suppose that $I \nsubseteq Q$. Then there exist $s \in I\backslash Q \subseteq R\backslash Q$. Since $s_1 \alpha s_2 \in (s)\Gamma(s) \subseteq I\Gamma I, I\Gamma I \nsubseteq Q$. this is impossible. So $I \subseteq Q$ and hence $Q$ is semiprime. □

**Remark 1.** *Let $\{S_\alpha\}_{\alpha \in A}$ be a family of sp-systems of a $\Gamma$-seminear-ring $R$. If $s \in \cup_{\alpha \in A} S_\alpha$, then $s \in S_\alpha$ for some $\alpha \in A$. Since $S_\alpha$ is an sp-system, there exist $s_1, s_2 \in S_\alpha \subseteq \cup_{\alpha \in A} S_\alpha$. Thus $\cup_{\alpha \in A} S_\alpha$ is an sp-system.*

**Lemma 3.21.** *Let $S$ be a nonempty subset of a $\Gamma$-seminear-ring $R$. Then $S$ is an sp-system if and only if $S = \cup_{\alpha \in A} S_\alpha$, where ${S_\alpha}'s$ are $m$-systems of $R$.*

*Proof.* Assume that $S$ is an $sp$-system and $s_0 \in S$. Then there exist ${s_0}^1, {s_0}^2 \in (s_0)$ and $\alpha \in \Gamma$ such that $s_1 = {s_0}^1 \alpha {s_0}^2 \in S$. And for $s_1$, there exist ${s_1}^1, {s_1}^2 \in (s_1)$ and $\beta \in \Gamma$ such that $s_2 = {s_1}^1 \beta {s_1}^2 \in S$. Continuing this process, we can get a sequence $s_0, s_1, s_2, \cdots$. We claim that $M = \{s_0, s_1, s_2, \cdots.\}$ is an $m$-system. Let $s_i, s_j \in M$. We may assume that $i < j$ without loss of generality. Then $(s_j) \subseteq (s_i)$. Take ${s_j}^1, {s_j}^2 \in (s_j)$ and $\gamma \in \Gamma$. Then ${s_j}^1 \gamma {s_j}^2 = s_{j+1} \in M$. Thus $M$ is an $m$-system. The converse is clear. □

**Theorem 3.22.** *Let $Q$ be an ideal in a $\Gamma$-seminear-ring $R$. Then $Q$ is semiprime if and only if $Q$ is an intersection of all prime ideals $P_\alpha (\alpha \in A)$ containing $Q$.*

*Proof.* Assume that $Q$ is semiprime and let $S = R\backslash Q$. Then $S$ is an $sp$-system. By Lemma 3.21, $S = \cup_{\beta \in B} S_\beta$ for some $m$-system $S_\beta$. Since for each $\beta \in B$, $S_\beta \subseteq S, P_\beta = R\backslash S_\beta$ is prime containing $Q$ and so $Q \subseteq \cap_{\alpha \in A} P_\alpha \subseteq \cap_{\beta \in B} P_\beta = \cap_{\beta \in B}(R\backslash S_\beta) = R \backslash \cup_{\beta \in B} S_\beta = R\backslash S = Q$. Thus $Q$ is an intersection of $P_\alpha$. Conversely, let $I$ be an ideal in $R$ with $I\Gamma I \subseteq Q$. Then $I\Gamma I \subseteq P_\alpha$ for all $\alpha \in A$. Since $P_\alpha$ is prime, $I \subseteq P_\alpha$ for all $\alpha \in A$ and so $I \subseteq Q$. Thus $Q$ is semiprime. □

**Definition 3.23.** An ideal $I$ of a $\Gamma$-seminear-ring $R$ is said to be *irreducible* if for any ideals $H, K$ in $R$, $I = H \cap K$ implies $I = H$ or $I = K$. $I$ is *strongly irreducible* if $H \cap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$. A nonempty subset $A$ of $R$ is an *$i$-system* if for any $a, b \in A, (a) \cap (b) \cap A \neq \emptyset$.

**Remark 2.** *Let $M$ be an $m$-system of $R$ and $a, b \in M$. Then there exist $a_1 \in (a)$ and $b_1 \in (b)$ and $\alpha \in \Gamma$ such that $a_1 \alpha b_1 \in (a) \cap (b) \cap M$. Hence every $m$-system is an $i$-system.*

**Proposition 3.24.** *The following conditions on an ideal $I$ in a $\Gamma$-seminear-ring $R$ are equivalent.*

(i) *$I$ is strongly irreducible*
(ii) *If $a, b \in R$ such that $(a) \cap (b) \subseteq I$, then $a \in I$ or $b \in I$.*
(iii) *$R\backslash I$ is an $i$-system.*

*Proof.* (i)⇒(ii): It is clear.

(ii)⇒ (iii): Let $a, b \in R\backslash I$. Suppose that $(a) \cap (b) \cap R\backslash I = \emptyset$. Then $(a) \cap (b) \subseteq I$. By (ii), we have $a \in I$ or $b \in I$. It is a contradiction. Therefore $(a) \cap (b) \cap R\backslash I \neq \emptyset$. Thus $R\backslash I$ is an $i$-system.

(iii)$\Rightarrow$ (i): Let $H, K$ be ideals of $R$ not contained in $I$. Then there exist $a \in H \backslash I$ and $b \in K \backslash I$. By (iii), $(a) \cap (b) \cap R \backslash I \neq \emptyset$. Therefore there exist $c \in (a) \cap (b)$ and $c \notin I$. Hence $H \cap K \nsubseteq I$ and $I$ is strongly irreducible. $\square$

**Theorem 3.25.** *A proper ideal $P$ of a $\Gamma$-seminear-ring $R$ is prime if and only if it is semiprime and strongly irreducible.*

*Proof.* If $P$ is prime, it is semiprime. Moreover, if $K, H$ are ideals of $R$ such that $H \cap K \subseteq P$, then $H \Gamma K \subseteq P$, then $H \Gamma K \subseteq H \cap K \subseteq P$. Since $P$ is prime, then $H \subseteq P$ or $K \subseteq P$ and so $P$ is strongly irreducible. Conversely, assume that $P$ is semiprime and strongly irreducible. If $H$ and $K$ are ideals of $R$ such that $H \Gamma K \subseteq P$, then $(H \cap K)\Gamma(H \cap K) \subseteq H \Gamma K \subseteq P$. Since $P$ is semiprime, $H \cap K \subseteq P$. By the strongly irreducible, we have $H \subseteq P$ or $K \subseteq P$. Thus $P$ is prime. $\square$

## References

[1] G. Pilz, *Near-rings*, North-Holland, Amsterdam, 1983.
[2] Y. B. Jun and K. H. Kim, *On structures of gamma-seminear-rings*, (submitted).
[2] H. J. Weinert, *Seminear-rings, seminear-field and their semigroup theoretic background*, Semigroup Forum **24** (1982), 235-254.

K. H. Kim
Department of Mathematics
Chungju National University
Chungju 380-702, Korea
E-mail: ghkim@gukwon.chungju.ac.kr