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BCK/BCI–RELATIONS

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Abstract. We introduce the notion of a BCK/BCI-relation which is a generalization of a homomorphism of BCK/BCI-algebras, and investigates some properties related to the subalgebras.

1. Introduction

The notion of BCK-algbera was proposed by Imai and Iséki in 1966. In the same year, Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algbera. In this paper, as a generalization of a homomorphism of BCK/BCI-algebras, the concept of BCK/BCI-relations is introduced, and then basic properties related to the subalgebras are investigated.

2. Preliminaries

We include some elementary aspects of BCK/BCI-algebras that are necessary for this paper, and for more details we refer to [3].

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. A BCI-algebra X satisfying the additional condition:

(V) $0 * x = 0$ for all $x \in X$

is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial ordering "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCK/BCI-algberas is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK/BCI-algberas, then$f(0_X) = 0_Y$, where 0_X and 0_Y are zero elements of X and Y, respectively.

3. BCK/BCI-relations

Definition 3.1. Let X and Y be BCK/BCI-algebras. A nonempty relation $\Theta \subseteq X \times Y$ is called a *BCK/BCI-relation* if

(R1) for every $x \in X$ there exists $y \in Y$ such that $x\Theta y$,
(R2) $x\Theta a$ and $y\Theta b$ imply $(x * y)\Theta(a * b)$.

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We usually denote such relation by $\Theta : X \to Y$. It is clear from (R1) and (R2) that $0_X \Theta 0_Y$.

Example 3.2. (1) Consider a BCK-algebra $X = \{0, a, b, c\}$ possessing the following Cayley table (see [3]):

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a relation $\Theta : X \to X$ as follows: $0\Theta 0$, $0\Theta a$, $0\Theta b$, $a\Theta 0$, $a\Theta a$, $a\Theta b$, $b\Theta 0$, $c\Theta 0$. It is easy to verify that Θ is a BCK-relation.

(2) Consider a BCI-algebra $X = \{0, a, b, c\}$ having the following Cayley table (see [4]):

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a relation $\Omega : X \to X$ as follows: $0\Omega 0$, $0\Omega b$, $a\Omega 0$, $a\Omega b$, $b\Omega 0$, $b\Omega b$, $c\Omega 0$, $c\Omega b$. It is easy to verify that Ω is a BCI-relation.

(3) Consider two BCI-algebras $X = \{0, 1, 2\}$ and $Y = \{0, a, b\}$ having the following Cayley tables (see [1]):

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

$*$	0	a	b
0	0	a	b
a	a	0	b
b	b	a	0

Let $\Psi : X \to Y$ be a relation defined by $0\Psi 0$, $0\Psi b$, $1\Psi 0$, $1\Psi b$, $2\Psi 0$, $2\Psi b$. It is easy to verify that Ψ is a BCI-relation. If a relation $\Phi : X \to Y$ is defined by $0\Phi 0$, $1\Phi 0$, $2\Phi 0$, then Φ is a BCI-relation which is a homomorphism.

Theorem 3.3. *Every homomorphism of BCK/BCI-algebras is a BCK/BCI-relation.*

Proof. Let $\Theta : X \to Y$ be a homomorphism of BCK/BCI-algebras. Clearly, Θ satisfies conditions (R1) and (R2). $\square$

The following example shows that the converse of Theorem 3.3 need not be true.

Example 3.4. In Example 3.2, the BCK/BCI-relation Θ and Ω are not homomorphisms.

Let $\Theta : X \to Y$ be a BCK/BCI-relation. For any $x \in X$ and $y \in Y$, let

$$\Theta[x] := \{y \in Y \mid x\Theta y\} \quad \text{and} \quad \Theta^{-1}[y] := \{x \in X \mid x\Theta y\}.$$

Note that $\Theta[x]$ and $\Theta^{-1}[y]$ are not subalgebras of Y and X, respectively as seen in the following example.

Example 3.5. Let $X = \{0, a, b\}$ be a BCI-algebra with Cayley table as follows (see [1]):

$*$	0	a	b
0	0	a	b
a	a	0	b
b	b	a	0

Define a relation $\Theta : X \to X$ as follows: $0\Theta 0$, $a\Theta b$, $b\Theta a$. It is routine to show that Θ is a BCI-relation, and that $\Theta^{-1}[b] = \{a\}$ (resp. $\Theta[a] = \{b\}$) is not a subalgebra of X (resp. Y).

Theorem 3.6. *For any BCK/BCI-relation $\Theta : X \to Y$, we have*
(i) $\Theta[0_X]$, *called the zero image of Θ, is a subalgebra of Y.*
(ii) $\Theta^{-1}[0_Y]$, *called the kernel of Θ and denoted by* $\mathrm{Ker}\Theta$, *is a subalgebra of X.*

Proof. (i) Let $y_1, y_2 \in \Theta[0_X]$. Then $0_X\Theta y_1$ and $0_X\Theta y_2$. It follows from (R2) and (III) that $0_X\Theta(y_1 * y_2)$, that is, $y_1 * y_2 \in \Theta[0_X]$.
(ii) Let $x_1, x_2 \in \mathrm{Ker}\Theta$. Then $x_1\Theta 0_Y$ and $x_2\Theta 0_Y$. By using (R2) and (III), we get $(x_1 * x_2)\Theta 0_Y$ and so $x_1 * x_2 \in \mathrm{Ker}\Theta$. This completes the proof. $\square$

Proposition 3.7. *Let $\Theta : X \to Y$ be a BCK/BCI-relation.*
(i) *If $\Theta[a] \cap \Theta[b] \neq \emptyset$ where $a, b \in X$, then $a * b \in \mathrm{Ker}\Theta$.*
(ii) *If $\Theta^{-1}[u] \cap \Theta^{-1}[v] \neq \emptyset$ where $u, v \in Y$, then $u * v \in \Theta[0_X]$.*

Proof. (i) Let $a, b \in X$ be such that $\Theta[a] \cap \Theta[b] \neq \emptyset$. Taking $y \in \Theta[a] \cap \Theta[b]$, we have $a\Theta y$ and $b\Theta y$. It follows from (R2) and (III) that $(a * b)\Theta(y * y) = (a * b)\Theta 0_Y$ so that $a * b \in \mathrm{Ker}\Theta$.
(ii) Let $x \in \Theta^{-1}[u] \cap \Theta^{-1}[v]$. Then $x\Theta u$ and $x\Theta v$. Using (R2) and (III), we obtain $(x * x)\Theta(u * v) = 0_X\Theta(u * v)$, i.e., $u * v \in \Theta[0_X]$. This completes the proof. $\square$

Theorem 3.8. *Let $\Theta : X \to Y$ be a BCK/BCI-relation and let S be a subalgebra of X. Then*

$$\Theta[S] := \{y \in Y \mid x\Theta y \text{ for some } x \in S\}$$

is a subalgebra of Y.

Proof. Clearly, $\Theta[S] \neq \emptyset$ since $0_X\Theta 0_Y$. Let $y_1, y_2 \in \Theta[S]$. Then $x_1\Theta y_1$ and $x_2\Theta y_2$ for some $x_1, x_2 \in S$. Using (R2), we obtain $(x_1 * x_2)\Theta(y_1 * y_2)$ which implies that $y_1 * y_2 \in \Theta[S]$ since $x_1 * x_2 \in S$. Therefore $\Theta[S]$ is a subalgebra of Y. $\square$

Corollary 3.9. *Let $\Theta : X \to Y$ be a BCK/BCI-relation. Then*
(i) $\Theta[X]$ *is a subalgebra of Y.*
(ii) $\Theta[X] := \bigcup_{x \in X} \Theta[x]$.
(iii) *The zero image of Θ is a subalgebra of* $\Theta[X]$.

Proof. (i) and (ii) are straightforward.
(iii) Let $a, b \in \Theta[0_X]$. Then $0_X\Theta a$ and $0_X\Theta b$, and hence $0_X\Theta(a * b)$, i.e., $a * b \in \Theta[0_X]$. Therefore $\Theta[0_X]$ is a subalgbera of $\Theta[X]$. $\square$

Theorem 3.10. *Let $\Theta : X \to Y$ be a BCK/BCI-relation and let T be a subalgebra of Y. Then*

$$\Theta^{-1}[T] := \{x \in X \mid x\Theta y \text{ for some } y \in T\}$$

is a subalgebra of X.

Proof. Obviously, $\Theta^{-1}[T] \neq \emptyset$ since $0_X\Theta 0_Y$. Let $x_1, x_2 \in \Theta^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1\Theta y_1$ and $x_2\Theta y_2$. Note that $y_1 * y_2 \in T$ since T is a subalgbera of Y.

It follows from (R2) that $(x_1 * x_2)\Theta(y_1 * y_2)$ so that $x_1 * x_2 \in \Theta^{-1}[T]$. Hence $\Theta^{-1}[T]$ is a subalgebra of X. $\square$

Corollary 3.11. *Let $\Theta : X \to Y$ be a BCK/BCI-relation. Then*
(i) $\Theta^{-1}[Y]$ *is a subalgebra of* X.
(ii) $\Theta^{-1}[Y] := \bigcup_{y \in Y} \Theta^{-1}[y]$.
(iii) *The kernel of* Θ *is a subalgebra of* $\Theta^{-1}[Y]$.

Proof. (i) and (ii) are straightforward.

(iii) Let $x, y \in \mathrm{Ker}\Theta$. Then $x\Theta 0_Y$ and $y\Theta 0_Y$. It follows from (R2) and (III) that $(x * y)\Theta(0_Y * 0_Y) = (x * y)\Theta 0_Y$ so that $x * y \in \mathrm{Ker}\Theta$. Hence $\mathrm{Ker}\Theta$ is a subalgebra of $\Theta^{-1}[Y]$. This completes the proof. $\square$

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