PARALINDELÖF SUBSPACES IN PRODUCTS OF TWO ORDINALS

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Abstract. Let ρ and σ be ordinals with the order topologies. It is known from [KTY] that metacompactness, screenability and weak submetaLindelöfness are equivalent for every subspace of $\rho \times \sigma$. However there are a metacompact subspace of $(\omega_1+1)\times(\omega_2+1)$ which is not subparacompact, and a subparacompact subspace of $(\omega+1)\times(\omega_1+1)$ which is not paracompact, see [KTY, Example 4.2 and 4.4]. Moreover it is not difficult to show that these examples are not paraLindelöf. So it is natural to ask whether all paraLindelöf subspaces of $\rho\times\sigma$ are paracompact for every ordinals ρ and σ. In this paper, we will see that paraLindelöf subspaces of $(\rho+1)\times(\omega_1\cdot\omega)$ are paracompact for every ordinal ρ, where $\omega_1\cdot\omega$ denotes the ordinal number $\omega_1+\omega_1+\cdots(\omega$-times), see [Ku, I Definition 7.19]. Moreover we will show that paraLindelöf subspaces of $(\rho+1)^2$ are paracompact for every ordinal $\rho<\omega_1\cdot\omega_1$. And we will construct a non-paracompact subspace X of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$ which can be represented as the locally countable union of clopen paracompact subspaces.

All spaces are assumed to be regular T_1. Let ρ and σ be ordinals with the order topologies. It is known from [KTY] that metacompactness, screenability and weak submetaLindelöfness are equivalent for every subspace of $\rho\times\sigma$. So at least, such paraLindelöf subspaces are metacompact. However there are a metacompact subspace of $(\omega_1+1)\times(\omega_2+1)$ which is not subparacompact, and a subparacompact subspace of $(\omega+1)\times(\omega_1+1)$ which is not paracompact, see [KTY, Example 4.2 and 4.4]. Moreover it is not difficult to show that these examples are not paraLindelöf. In this connection, it is known in [KY] that for subspaces $A\subset\rho$ and $B\subset\sigma$, $A\times B$ is paracompact iff A and B are paracompact. Since, by [Be], paraLindelöf GO-spaces are paracompact, for subspaces $A\subset\rho$ and $B\subset\sigma$, $A\times B$ is paraLindelöf iff $A\times B$ is paracompact. So it is natural to ask whether all paraLindelöf subspaces of $\rho\times\sigma$ are paracompact for every ordinals ρ and σ. In this paper, we will see that paraLindelöf subspaces of $(\rho+1)\times(\omega_1\cdot\omega)$ are paracompact for every ordinal ρ, where $\omega_1\cdot\omega$ denotes the ordinal number $\omega_1+\omega_1+\cdots(\omega$-times), see [Ku, I Definition 7.19]. Moreover we will show that paraLindelöf subspaces of $(\rho+1)^2$ are paracompact for every ordinal $\rho<\omega_1\cdot\omega_1$. And we will construct a non-paracompact subspace X of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$ which can be represented as the locally countable union of clopen paracompact subspaces.

We recall basic definitions and introduce specific notation from [KTY].

In our discussion, for some technical reasons, we always assume $X\subset(\rho+1)\times(\sigma+1)$ for some suitably large ordinals ρ and σ. Moreover, in general, the letters μ and ν stand for limit ordinals with $\mu\le\rho$ and $\nu\le\sigma$. For each $A\subset\rho+1$ and $B\subset\sigma+1$ put

$$X_A = A\times(\sigma+1)\cap X,\ X^B=(\rho+1)\times B\cap X,$$

and

$$X_A^B = X_A\cap X^B.$$

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For each $\alpha \le \rho$ and $\beta \le \sigma$, put

$$V_\alpha(X) = \{\beta \le \sigma : \langle \alpha, \beta \rangle \in X\},$$

$$H_\beta(X) = \{\alpha \le \rho : \langle \alpha, \beta \rangle \in X\}.$$

$\operatorname{cf}\mu$ denotes the cofinality of the ordinal μ. When $\operatorname{cf}\mu \ge \omega_1$, a subset S of μ is called *stationary in* μ if it intersects all cub (i.e., closed and unbounded) sets in μ. Moreover for each $A \subset \mu$, $\operatorname{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in μ. For convenience, we consider $\sup \emptyset = -1$ and -1 is the immediate predecessor of the ordinal 0. Therefore $\operatorname{Lim}_\mu(A)$ is cub in μ whenever A is unbounded in μ. We will simply denote $\operatorname{Lim}_\mu(A)$ by $\operatorname{Lim}(A)$ if the situation is clear in its context. In particular, assume C is a cub set in μ with $\operatorname{cf}\mu \ge \omega$, then $\operatorname{Lim}(C) \subset C$. In this case, we define $\operatorname{Succ}(C) = C \setminus \operatorname{Lim}(C)$, and $\mathrm{p}_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that, for each $\alpha \in C$, $\mathrm{p}_C(\alpha) \in C \cup \{-1\}$, and $\mathrm{p}_C(\alpha) < \alpha$ iff $\alpha \in \operatorname{Succ}(C)$. So $\mathrm{p}_C(\alpha)$ is the immediate predecessor of α in $C \cup \{-1\}$ whenever $\alpha \in \operatorname{Succ}(C)$. Moreover observe that $\mu \setminus C$ is the union of the pairwise disjoint collection $\{(\mathrm{p}_C(\alpha), \alpha) : \alpha \in \operatorname{Succ}(C)\}$ of open intervals of μ and that $\mu \setminus \operatorname{Lim}(C)$ is the union of the pairwise disjoint collection $\{(\mathrm{p}_C(\alpha), \alpha] : \alpha \in \operatorname{Succ}(C)\}$ of clopen intervals of μ. For short, let denote $\operatorname{Lim} = \operatorname{Lim}(\omega_1)$ and $\operatorname{Succ} = \operatorname{Succ}(\omega_1)$.

Let κ be a regular uncountable cardinal and $A \subset \kappa$. Assume that a cub set C_γ is assigned for each $\gamma \in A$. Then, by the argument of [Ku, II 6.14], the diagonal intersection

$$\triangle_{\gamma \in A} C_\gamma = \{\delta < \kappa : \forall \gamma \in A \cap \delta (\delta \in C_\gamma)\}$$

is cub in κ.

A strictly increasing function $M : \operatorname{cf}\mu + 1 \to \mu + 1$ is said to be a *normal function for* μ if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma \le \operatorname{cf}\mu$ and $M(\operatorname{cf}\mu) = \mu$. Observe that, if $\operatorname{cf}\mu \ge \omega_1$, then two normal functions for μ coincide on a cub set of $\operatorname{cf}\mu$. Note that a normal function for μ always exists if $\operatorname{cf}\mu \ge \omega$. So we always fix a normal function M for each ordinal μ with $\operatorname{cf}\mu \ge \omega$. In particular, if μ is regular, i.e. $\operatorname{cf}\mu = \mu$, then we can fix the identity map on $\mu + 1$ as the normal function. Then M carries $\operatorname{cf}\mu + 1$ homeomorphically to the range $\operatorname{ran} M$ of M and $\operatorname{ran} M$ is closed in $\mu + 1$. Note that for all $S \subset \mu$ with $\operatorname{cf}\mu \ge \omega_1$, S is stationary in μ if and only if $M^{-1}(S)$ is stationary in $\operatorname{cf}\mu$. For convenience, we define $M(-1) = -1$.

A space Y is said to be *paraLindelöf* if every open cover of Y has a locally countable open refinement.

The next lemma follows from [KTY, Lemma 2.2].

Lemma 1. *Let Y be a subspace of $\rho + 1$ for some ordinal ρ. Then the following are equivalent:*

(A) *Y is paracompact,*
(B) *Y is paraLindelöf,*
(C) *For every $\mu \in (\rho + 1) \setminus Y$ with $\operatorname{cf}\mu \ge \omega_1$, $Y \cap \mu$ is not stationary in μ.*

Lemma 2. *Let X be a paraLindelöf subspace of $(\mu + 1) \times (\nu + 1)$ and let E be a closed subset which is disjoint from $X^{\{\nu\}}$. If $\operatorname{cf}\nu \ge \omega_1$, then E and $X^{\{\nu\}}$ are separated by disjoint open subsets of X.*

Proof. Let $\mathcal{U} = \{X \setminus E\} \cup \{X^{[0, N(\delta)]} : \delta < \operatorname{cf}\nu\}$. Then $\mathcal{U}$ is an open cover of X. Since X is paraLindelöf, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta < \operatorname{cf}\nu\}$ of $\mathcal{U}$, where "precise" means $W \subset X \setminus E$ and $W(\delta) \subset X^{[0, N(\delta)]}$ for each $\delta < \operatorname{cf}\nu$. Let

$G = \bigcup_{\delta < \operatorname{cf} \nu} W(\delta)$, then $E \subset G$. It suffices to show $X^{\{\nu\}} \subset X \setminus \operatorname{Cl} G$. Let $\langle \alpha, \nu \rangle \in X^{\{\nu\}}$. Take $f(\alpha) < \alpha$ and $g(\alpha) < \nu$ such that

$$D_\alpha = \{\delta < \operatorname{cf} \nu : X_{(f(\alpha), \alpha]}^{(g(\alpha), \alpha]} \cap W(\delta) \neq \emptyset\}$$

is countable. Let $\nu' = \max\{g(\alpha), \sup D_\alpha\}$. Then $\nu' < \nu$ and $X_{(f(\alpha), \alpha]}^{(\nu', \nu]} \cap G = \emptyset$. This implies $\langle \alpha, \nu \rangle \in X \setminus \operatorname{Cl} G$. $\square$

Theorem 3. *Let X be a paraLindelöf subspace of $(\mu+1) \times (\nu+1)$. Assume that $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are paracompact for each $\mu' < \mu$ and $\nu' < \nu$. Then in either cases of the following, X is paracompact.*

(a) $\operatorname{cf} \mu \leq \nu$, *moreover either ν is regular uncountable or $\nu < \omega_1$.*
(b) $\nu < \operatorname{cf} \mu$.

Proof. Assume that X is not paracompact. Then it follows from Lemma 1 that μ and ν are limit ordinals.

Claim 1. $\langle \mu, \nu \rangle \notin X$.

Proof. Assume $\langle \mu, \nu \rangle \in X$. Let $\mathcal{U}$ be an open cover of X. Take $U \in \mathcal{U}$ with $\langle \mu, \nu \rangle \in U$, moreover take $\mu' < \mu$ and $\nu' < \nu$ such that $X_{(\mu', \mu]}^{(\nu', \nu]} \subset U$. Since, $X_{[0,\mu']} \cup X^{[0,\nu']}$ is a paracompact clopen subspace, it is not difficult to construct a locally finite open refinement of $\mathcal{U}$, a contradiction.

Moreover we have $\operatorname{cf} \mu \geq \omega_1$ or $\operatorname{cf} \nu \geq \omega_1$. Indeed, if $\operatorname{cf} \mu = \operatorname{cf} \nu = \omega$, then $X = \bigoplus_{n \in \omega} X_{(M(n-1), M(n)]} \cup \bigoplus_{n \in \omega} X_{(N(n-1), N(n)]}$ is the union of countably many clopen paracompact subspaces. Therefore X is paracompact, a contradiction.

We will consider several cases. In all cases, we will derive contradictions. First we consider the case (a).

Case (a). $\operatorname{cf} \mu \leq \nu$, moreover either ν is regular uncountable or $\nu < \omega_1$.

In this case, we may assume that $\operatorname{cf} \mu \leq \nu$ and ν is regular uncountable. Because, if $\operatorname{cf} \mu \leq \nu < \omega_1$, then $\operatorname{cf} \mu = \operatorname{cf} \nu = \omega$, a contradiction.

There are three subcases to consider.

Subcase (a-1). $\operatorname{cf} \mu = \omega$.

Since X is paraLindelöf, by Lemma 1, we can take a cub set D in ν disjoint from $V_\mu(X)$. Then $X_{\{\mu\}}$ and $X^{D \cup \{\nu\}}$ are disjoint closed subsets. In particular, $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are disjoint closed subsets. Applying Lemma 2, take an open set G of X such that $X_{\{\mu\}} \subset G$ and $\operatorname{Cl} G \cap X^{\{\nu\}} = \emptyset$. Let $\mathcal{U} = \{X \setminus \operatorname{Cl} G\} \cup \{X^{[0,\delta)} : \delta < \nu\}$. Then $\mathcal{U}$ is an open cover of X. Since X is paraLindelöf, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta < \nu\}$ of $\mathcal{U}$. For each $\beta \in V_\mu(X)$, we can take $f(\beta) < \mu$ and $g(\beta) < \beta$ such that $(g(\beta), \beta] \cap D = \emptyset$, $H(\beta) = X_{(f(\beta), \mu]}^{(g(\beta), \beta]} \subset G$ and $\{\delta < \nu : W(\delta) \cap H(\beta) \neq \emptyset\}$ is countable. Let $H = \bigcup_{\beta \in V_\mu(X)} H(\beta)$. Then $X_{\{\mu\}} \subset H \subset G$ is obvious.

Claim 2. *$S = \{\delta \in D : X^{\{\delta\}} \cap \operatorname{Cl} H \neq \emptyset\}$ is not stationary in ν.*

Proof. Assume that S is stationary in ν. For each $\delta \in S$, take $h(\delta) < \mu$ with $\langle h(\delta), \delta \rangle \in \operatorname{Cl} H$. Since $\mathcal{W}$ is an open cover, there is $\psi(\delta) < \nu$ with $\langle h(\delta), \delta \rangle \in W(\psi(\delta))$. Note that $\psi(\delta) > \delta$ because of $W(\psi(\delta)) \subset X^{[0,\psi(\delta))}$. Since $(g(\beta), \beta] \cap D = \emptyset$ for each $\beta \in V_\mu(X)$ and $W(\psi(\delta))$ is a neighborhood of $\langle h(\delta), \delta \rangle \in \operatorname{Cl} H$, we can find $\beta(\delta) \in V_\mu(X)$ with $\beta(\delta) < \delta$ such that

$W(\psi(\delta)) \cap H(\beta(\delta)) \neq \emptyset$. For each $\delta \in \nu \backslash S$, define $\psi(\delta) = 0$. Then $D' = \{\delta < \nu : \forall \delta' < \delta(\psi(\delta') < \delta)\}$ is cub in ν. By the PDL(Pressing Down Lemma), there is a stationary set $S' \subset S \cap D'$ in ν and $\beta \in V_\mu(X)$ such that $\beta(\delta) = \beta$ for each $\delta \in S'$. Since $\delta < \psi(\delta)$ for each $\delta \in S'$ and $S' \subset D'$, the members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Therefore $H(\beta)$ meets uncountably many $W(\psi(\delta))$'s, $\delta \in S'$, a contradiction.

Applying Claim 2, take a cub set $E \subset D$ in ν with $E \cap S = \emptyset$. Then, since $H \subset G$ and $\mathrm{Cl}\, G \cap X^{\{\nu\}} = \emptyset$, $X_{\{\mu\}}$ and $X^{E \cup \{\nu\}}$ are separated by H and $X \setminus \mathrm{Cl}\, H$. Since

$$X \setminus H \subset X \setminus X_{\{\mu\}} \subset \bigoplus_{n \in \omega} X_{(M(n-1), M(n)]} \text{ and}$$

$$\mathrm{Cl}\, H \subset X \setminus X^{E \cup \{\nu\}} \subset \bigoplus_{\delta \in \mathrm{Succ}(E)} X^{(\mathrm{p}_E(\delta), \delta]},$$

$X = (X \setminus H) \cup \mathrm{Cl}\, H$ is the union of two paracompact closed subspaces. So X is paracompact, a contradiction.

Subcase (a-2). $\omega_1 \leq \mathrm{cf}\, \mu < \nu$.

Since $\langle \mu, \nu \rangle \notin X$ and $\mathrm{cf}\, \mu \geq \omega_1$, $H_\nu(X)$ is not stationary in μ, so we can take a cub set C in $\mathrm{cf}\, \mu$ such that $M(C) \cap H_\nu(X) = \emptyset$. Similarly, for each $\gamma \in C \cup \{\mathrm{cf}\, \mu\}$, since $V_{M(\gamma)}(X)$ is not stationary in ν, we can take a cub set D_γ in ν disjoint from $V_{M(\gamma)}(X) = \emptyset$. Put $D = \bigcap_{\gamma \in C \cup \{\mathrm{cf}\, \mu\}} D_\gamma$. Then, since $\mathrm{cf}\, \mu < \nu = \mathrm{cf}\, \nu$, D is a cub set in ν and $X_{M(C) \cup \{\mu\}}$ and $X^{D \cup \{\nu\}}$ are disjoint closed subsets. In particular, $X_{M(C) \cup \{\mu\}}$ and $X^{\{\nu\}}$ are disjoint closed subsets. By Lemma 2, we can take an open subset G such that $X_{M(C) \cup \{\mu\}} \subset G$ and $\mathrm{Cl}\, G \cap X^{\{\nu\}} = \emptyset$. Since $\mathcal{U} = \{X \setminus \mathrm{Cl}\, G\} \cup \{X^{[0,\delta)} : \delta < \nu\}$ is an open cover of the paraLindelöf space X, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta \in \nu\}$ of $\mathcal{U}$. For each $\gamma \in C \cup \{\mathrm{cf}\, \mu\}$ and each $\beta \in V_{M(\gamma)}(X)$, we can take $f(\gamma, \beta) < M(\gamma)$ and $g(\gamma, \beta) < \beta$ such that $(g(\gamma, \beta), \beta] \cap D = \emptyset$, $H(\gamma, \beta) = X^{(g(\gamma,\beta),\beta]}_{(f(\gamma,\beta),M(\gamma)]} \subset G$ and $\{\delta < \nu : W(\delta) \cap H(\gamma, \beta) \neq \emptyset\}$ is countable. Let

$$H = \bigcup_{\gamma \in C \cup \{\mathrm{cf}\, \mu\}, \beta \in V_{M(\gamma)}(X)} H(\gamma, \beta).$$

Then $X_{M(C) \cup \{\mu\}} \subset H \subset G$ is obvious.

Claim 3. *$S = \{\delta \in D : X^{\{\delta\}} \cap \mathrm{Cl}\, H \neq \emptyset\}$ is not stationary in ν.*

Proof. Assume that S is stationary in ν. For each $\delta \in S$ take $h(\delta) < \mu$ with $\langle h(\delta), \delta \rangle \in \mathrm{Cl}\, H$ and take $\psi(\delta) < \nu$ with $\langle h(\delta), \delta \rangle \in W(\psi(\delta))$. Note $\psi(\delta) > \delta$ because of $W(\psi(\delta)) \subset X^{[0, \psi(\delta))}$. Since $(g(\gamma, \beta), \beta] \cap D = \emptyset$ for each $\gamma \in C \cup \{\mathrm{cf}\, \mu\}$ and $\beta \in V_{M(\gamma)}(X)$, we can take $\gamma(\delta) \in C \cup \{\mathrm{cf}\, \mu\}$ and $\beta(\delta) \in V_{M(\gamma(\delta))}(X)$ with $\beta(\delta) < \delta$ such that $W(\psi(\delta)) \cap H(\gamma(\delta), \beta(\delta)) \neq \emptyset$. As in Claim 2, noting $|C \cup \{\mathrm{cf}\, \mu\}| = \mathrm{cf}\, \mu < \nu$, by the PDL, we find a stationary set $S' \subset S$, $\beta < \nu$ and $\gamma \in C \cup \{\mathrm{cf}\, \mu\}$ such that $\beta(\delta) = \beta$ for each $\delta \in S'$, $\gamma(\delta) = \gamma$ and members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Then $H(\beta, \gamma)$ meets uncountably many $W(\psi(\delta))$'s, a contradiction.

Applying Claim 3, take a cub set $E \subset D$ in ν with $E \cap S = \emptyset$. Then, since $H \subset G$ and $\mathrm{Cl}\, G \cap X^{\{\nu\}} = \emptyset$, $X_{M(C) \cup \{\mu\}}$ and $X^{E \cup \{\nu\}}$ are separated by H and $X \setminus \mathrm{Cl}\, H$. Since

$$X \setminus H \subset X \setminus X_{M(C) \cup \{\mu\}} \subset \bigoplus_{\gamma \in \mathrm{Succ}(C)} X_{(M(\mathrm{p}_C(\gamma)), M(\gamma)]} \text{ and}$$

$$\operatorname{Cl} H \subset X \setminus X^{E \cup \{\nu\}} \subset \bigoplus_{\delta \in \operatorname{Succ}(E)} X^{(\mathrm{p}_E(\delta), \delta]},$$

$X \setminus H$ and $\operatorname{Cl} H$ are paracompact. Therefore $X = (X \setminus H) \cup \operatorname{Cl} H$ is paracompact, a contradiction.

Subcase (a-3). $\operatorname{cf} \mu = \nu$.

Note, in this case, $\omega_1 \leq \operatorname{cf} \mu = \operatorname{cf} \nu = \nu$. First we consider the special case that $X \subset \mu \times (\nu + 1)$. Since $\langle \mu, \nu \rangle \notin X$, $\Delta(X) = \{\gamma < \nu : \langle M(\gamma), \gamma \rangle \in X\}$ is homeomorphic to the closed subspace $X \cap \{\langle M(\gamma), \gamma \rangle : \gamma < \nu\}$ of X, $\Delta(X)$ is not stationary in ν. Take a cub set C in ν such that $C \cap \Delta(X) = \emptyset$.

Claim 4. $\mathcal{X} = \{X^{(\mathrm{p}_C(\gamma), \gamma]}_{(M(\mathrm{p}_C(\gamma)), M(\gamma)]} : \gamma \in \operatorname{Succ}(C)\}$ *is a discrete collection of clopen paracompact subspaces.*

Proof. By the assumption, it suffices to show that $\mathcal{X}$ is dicrete. Let $\langle \alpha, \beta \rangle \in X$. If $\alpha \notin M(C)$, then take $\gamma \in \operatorname{Succ}(C)$ such that $\alpha \in (M(\mathrm{p}_C(\gamma)), M(\gamma)]$. Then $X_{(M(\mathrm{p}_C(\gamma)), M(\gamma)]}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets at most one member of $\mathcal{X}$. Similarly if $\beta \notin C \cup \{\nu\}$, we can take a neighborhood of $\langle \alpha, \beta \rangle$ which meets at most one member of $\mathcal{X}$. So we may assume $\langle \alpha, \beta \rangle \in X^{C \cup \{\nu\}}_{M(C)}$. Take $\gamma(\alpha) \in C$ with $M(\gamma(\alpha)) = \alpha$. Since $C \cap \Delta(X) = \emptyset$, we have $M(\beta) \neq \alpha = M(\gamma(\alpha))$, so $\beta \neq \gamma(\alpha)$. If $\beta < \gamma(\alpha)$, then $X^{[0,\beta]}_{(M(\beta), \alpha]}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets no member of $\mathcal{X}$. If $\beta > \gamma(\alpha)$, then $X^{(\gamma(\alpha), \beta]}_{[0,\alpha]}$ is neighborhood of $\langle \alpha, \beta \rangle$ which meets no member of $\mathcal{X}$. This completes the proof of Claim 4.

Let

$$Y(0) = \{\langle \alpha, \beta \rangle \in X : \alpha > M(\beta)\} \setminus (\bigcup \mathcal{X}).$$

Then $Y(0)$ is clopen subspace of X. Because $Y(0)$ can be represented as $\{\langle \alpha, \beta \rangle \in X : \alpha \geq M(\beta)\} \setminus (\bigcup \mathcal{X})$. Similarly

$$Y(1) = \{\langle \alpha, \beta \rangle \in X : \alpha < M(\beta)\} \setminus (\bigcup \mathcal{X})$$

is a clopen subspace of X.

Claim 5. $Y(0)$ *is paracompact.*

Proof. Note, in this special case, $\mu \notin H_\delta(X)$ for each $\delta < \nu$. Therfore $H_\delta(X)$ is not stationary in μ for each $\delta < \nu$. Take a cub set C_δ in $\operatorname{cf} \mu$ such that $M(C_\delta) \cap H_\delta(X) = \emptyset$. Put

$$C' = C \cap \Delta_{\delta < \nu} C_\delta$$

We shall show $X_{M(C')} \cap Y(0) = \emptyset$. Assume on the countary that $\langle \alpha, \beta \rangle \in X_{M(C')} \cap Y(0)$. Take $\gamma(\alpha) \in C'$ with $M(\gamma(\alpha)) = \alpha$. By the definiton of $Y(0)$, we have $M(\beta) < \alpha = M(\gamma(\alpha))$, so $\beta < \gamma(\alpha)$. It follows from $\beta < \gamma(\alpha) \in C' \subset \Delta_{\delta < \nu} C_\delta$ that $\gamma(\alpha) \in C_\beta$. So $\alpha = M(\gamma(\alpha)) \in M(C_\beta) \cap H_\beta(X)$, a contradiction. Hence $X_{M(C')} \cap Y(0) = \emptyset$. Since $Y(0)$ is clopen and

$$Y(0) \subset X \setminus X_{M(C')} \subset \bigoplus_{\gamma \in \operatorname{Succ}(C')} X_{(M(\mathrm{p}_{C'}(\gamma)), M(\gamma)]},$$

$Y(0)$ is paracompact.

Since $X = Y(0) \bigoplus (\bigcup \mathcal{X}) \bigoplus Y(1)$ is not paracompact but $\bigcup \mathcal{X}$ and $Y(0)$ are paracompact, $Y(1)$ is not paracompact. By considering $Y(1)$ as X, we may now assume that

$$X \subset \{\langle \alpha, \beta \rangle \in \mu \times (\nu + 1) : \alpha < M(\beta)\}$$

is paraLindelöf but not paracompact, moreover $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are paracompact for each $\mu' < \mu$ and $\nu' < \nu$.

Since X is paraLindelöf, $H_\nu(X)$ is not stationary in μ. So take a cub set C' in $\mathrm{cf}\,\mu$ such that $M(C') \cap H_\nu(X) = \emptyset$. Similarly for each $\gamma \in C'$, we can take a cub set C_γ in ν such that $C_\gamma \cap V_{M(\gamma)}(X) = \emptyset$. Put

$$C = C' \cap \Delta_{\gamma \in C'} C_\gamma.$$

Claim 6. $X_{M(C)} \cap X^{C \cup \{\nu\}} = \emptyset$.

Proof. Assume on the contrary that $\langle \alpha, \beta \rangle \in X_{M(C)} \cap X^{C \cup \{\nu\}}$. Since $M(C') \cap H_\nu(X) = \emptyset$ and $\alpha \in M(C) \subset M(C')$, we have $\beta \neq \nu$ so $\beta \in C \subset C'$. Take $\gamma(\alpha) \in C$ with $M(\gamma(\alpha)) = \alpha$. Since $\alpha < M(\beta)$, we have $\gamma(\alpha) < \beta$. It follows from

$$\gamma(\alpha) < \beta \in C \subset \Delta_{\gamma \in C'} C_\gamma$$

that $\beta \in C_{\gamma(\alpha)}$. So $\beta \in C_{\gamma(\alpha)} \cap V_\alpha(X) = C_{\gamma(\alpha)} \cap V_{M(\gamma(\alpha))}$, a contradiction.

By Claim 6, in particular, $X_{M(C)}$ and $X^{\{\nu\}}$ are disjoint closed subsets. By Lemma 2, we can find an open subset G such that $X_{M(C)} \subset G$ and $\mathrm{Cl}\, G \cap X^{\{\nu\}} = \emptyset$. Since $\mathcal{U} = \{X \setminus \mathrm{Cl}\, G\} \cup \{X^{[0,\delta)} : \delta < \nu\}$ is an open cover of the paraLindelöf space X, there is a precise locally countable open refinement $\mathcal{W} = \{W\} \cup \{W(\delta) : \delta < \nu\}$ of $\mathcal{U}$. For each $\gamma \in C$ and each $\beta \in V_{M(\gamma)}(X)$, we can find $f(\gamma, \beta) < M(\gamma)$ and $g(\gamma, \beta) < \beta$ such that $(g(\gamma, \beta), \beta] \cap C = \emptyset$, $H(\gamma, \beta) = X^{(g(\gamma,\beta),\beta]}_{(f(\gamma,\beta),M(\gamma)]} \subset G$ and $\{\delta < \nu : W(\delta) \cap H(\gamma, \beta) \neq \emptyset\}$ is countable. Let

$$H = \bigcup_{\gamma \in C, \beta \in V_{M(\gamma)}(X)} H(\gamma, \beta)$$

Then $X_{M(C)} \subset H \subset G$ is obvious.

Claim 7. $S = \{\delta \in C : X^{\{\delta\}} \cap \mathrm{Cl}\, H \neq \emptyset\}$ *is not stationary in* ν.

Proof. Assume that S is stationary in ν. For each $\delta \in S$ take $h(\delta) < \mu$ with $\langle h(\delta), \delta \rangle \in \mathrm{Cl}\, H$ and take $\psi(\delta) \in \nu$ with $\langle h(\delta), \delta \rangle \in W(\psi(\delta))$. As in Claim 3, we can take $\gamma(\delta) \in C$ and $\beta(\delta) \in V_{M(\gamma(\delta))}(X)$ with $\beta(\delta) < \delta$ such that $W(\psi(\delta)) \cap H(\gamma(\delta), \beta(\delta)) \neq \emptyset$. Since $X \subset \{\langle \alpha, \beta \rangle \in \mu \times (\nu + 1) : \alpha < M(\beta)\}$, we have $M(\gamma(\delta)) < M(\beta(\delta))$. Hence $\gamma(\delta) < \beta(\delta) < \delta$. Applying the PDL twice, we can find a stationary set $S' \subset S$ in ν, $\gamma \in C$ and $\beta \in V_{M(\gamma)}(X)$ such that $\gamma(\delta) = \gamma$ and $\beta(\delta) = \beta$ for each $\delta \in S'$, moreover members of $\{\psi(\delta) : \delta \in S'\}$ are all distinct. Then $H(\gamma, \beta)$ meets uncountably many $W(\psi(\delta))$'s, a contradiction.

Applying Claim 7, take a cub set $E \subset C$ in ν with $E \cap S = \emptyset$. Then, by a similar argument of one after Claim 3, we can see that X is paracompact, a contradiction.

Next we consider the general case, that is, $X \subset (\mu + 1) \times (\nu + 1)$. Since X is paraLindelöf, $V_\mu(X)$ is not stationary in ν, so we can take a cub set D in ν such that $D \cap V_\mu(X) = \emptyset$. Then $X^{D \cup \{\nu\}}$ and $X_{\{\mu\}}$ are disjoint closed subsets. By Lemma 2, there is a open subset G such that $X^{D \cup \{\nu\}} \subset G$ and $\mathrm{Cl}\, G \cap X_{\{\mu\}} = \emptyset$. Then

$$X \setminus G \subset X \setminus X^{D \cup \{\nu\}} \subset \bigoplus_{\delta \in \mathrm{Succ}(D)} X^{(\mathrm{p}_D(\delta), \delta]} \text{ and}$$

and

$$\mathrm{Cl}\, G \subset X \setminus X_{\{\mu\}} \subset \mu \times (\nu + 1).$$

By the special case, $\mathrm{Cl}\,G$ is paracompact. Therefore $X = (X \setminus G) \cup \mathrm{Cl}\,G$ is paracompact, a contradiction.

Case (b). $\nu < \mathrm{cf}\,\mu$.

There are two subcases.

Subcase (b-1). $\omega_1 \leq \mathrm{cf}\,\nu$.

Since $\mathrm{cf}\,\mu > \nu \geq \mathrm{cf}\,\nu \geq \omega_1$, we can find a cub set D in $\mathrm{cf}\,\nu$ such that $N(D) \cap V_\mu(X) = \emptyset$. Then $X^{N(D)\cup\{\nu\}}$ and $X_{\{\mu\}}$ are disjoint closed subsets. Applying Lemma 2, take an open set G such that $X^{N(D)\cup\{\nu\}} \subset G$ and $\mathrm{Cl}\,G \cap X_{\{\mu\}} = \emptyset$.

Claim 8. *$S = \{\gamma < \mathrm{cf}\,\mu : X_{\{M(\gamma)\}} \cap \mathrm{Cl}\,G \neq \emptyset\}$ is not stationary in $\mathrm{cf}\,\mu$.*

Proof. Assume that S is stationary in $\mathrm{cf}\,\mu$. For each $\gamma \in S$, fix $h(\gamma) \leq \nu$ with $\langle M(\gamma), h(\gamma)\rangle \in \mathrm{Cl}\,G$. By $\nu < \mathrm{cf}\,\mu$ and the PDL, we can find a stationary set $S' \subset S$ in $\mathrm{cf}\,\mu$ and $\nu' \leq \nu$ such that $h(\gamma) = \nu'$ for each $\gamma \in S'$. Then $M(S') \subset H_{\nu'}(X)$, therefore $H_{\nu'}(X) \cap \mu$ is stationary in μ. Since X is paraLindelöf and $H_{\nu'}(X) \cap \mu$ is stationary in μ, we have $\mu \in H_{\nu'}(X)$. So

$$\langle \mu, \nu' \rangle \in \mathrm{Cl}\{\langle M(\gamma), \nu'\rangle : \gamma \in S'\} \cap X_{\{\mu\}} \subset \mathrm{Cl}\,G \cap X_{\{\mu\}},$$

a contradiction.

By Claim 8, we can take a cub set C in $\mathrm{cf}\,\mu$ such that $C \cap S = \emptyset$. Then we have $X^{N(D)\cup\{\nu\}} \subset G$ and $X_{M(C)\cup\{\mu\}} \cap \mathrm{Cl}\,G = \emptyset$. Since

$$X \setminus G \subset X \setminus X^{N(D)\cup\{\nu\}} \subset \bigoplus_{\delta \in \mathrm{Succ}(D)} X^{(N(\mathrm{p}_D(\delta)), N(\delta)]} \text{ and}$$

$$\mathrm{Cl}\,G \subset X \setminus X_{M(C)\cup\{\mu\}} \subset \bigoplus_{\gamma \in \mathrm{Succ}(C)} X_{(M(\mathrm{p}_C(\gamma)), M(\gamma)]},$$

$X = (X \setminus G) \cup \mathrm{Cl}\,G$ is paracompact, a contradicion.

Subcase (b-2). $\mathrm{cf}\,\nu = \omega$.

Note that in this case, we have $\omega_1 \leq \mathrm{cf}\,\mu$. By Lemma 2, take an open set G with $X^{\{\nu\}} \subset G$ and $\mathrm{Cl}\,G \cap X_{\{\mu\}} = \emptyset$.

By a similar argument of Claim 8, $S = \{\gamma < \mathrm{cf}\,\mu : X_{\{M(\gamma)\}} \cap \mathrm{Cl}\,G \neq \emptyset\}$ is not stationary in $\mathrm{cf}\,\mu$. Then by a similar argument of one after Claim 2, we can show that X is paracompact, a contradiction. $\square$

Corollary 4. *Let X be a paraLindelöf subspace of $(\rho + 1) \times (\nu + 1)$ for a suitably large ordinal ρ such that $X^{[0,\nu']}$ is paracompact for each $\nu' < \nu$. If ν is regular uncountable or $\nu < \omega_1$, then X is paracompact.*

Proof. Assume that X is not paracompact and let

$$\mu = \min\{\mu' \leq \rho : X_{[0,\mu']} \text{ is not paracompact }\}.$$

Then $X_{[0,\mu]}$ is not paracompact. First assume ν is regular uncountable. Then by Theorem 3, in either cases of $\mathrm{cf}\,\mu \leq \nu$ or $\nu < \mathrm{cf}\,\mu$, $X_{[0,\mu]}$ is paracompact, a contradiction. Next assume $\nu < \omega_1$. Similarly by Theorem 3, $X_{[0,\mu]}$ is paracompact, a contradiction. $\square$

This Corollary immediately yields:

Corollary 5. *For a suitably large ordinal ρ, paraLindelöf subspaces of $(\rho+1)\times(\omega_1+1)$ are paracompact.*

Corollary 6. *For a suitably large ordinal ρ, paraLindelöf subspaces of $(\rho+1)\times(\omega_1\cdot\omega)$ are paracompact.*

Proof. Let X be a paraLindelöf subspace of $(\rho+1)\times(\omega_1\cdot\omega)$. Put $T_0=[0,\omega_1]$ and $T_n=(\omega_1\cdot n,\omega_1\cdot(n+1)]$ for each $n\in\omega$ with $n\geq 1$. Since T_n is homeomorphic to (ω_1+1), $X\cap(\rho+1)\times T_n$ is homeomorphic to a subspace of $(\rho+1)\times(\omega_1+1)$. By Corollary 5,

$$X=\bigoplus_{n\in\omega}[X\cap(\rho+1)\times T_n]$$

is the free union of paracompact clopen subspaces, so X is paracompact. □

Corollary 7. *For each $\rho<\omega_1\cdot\omega_1$, paraLindelöf subspaces of $(\rho+1)^2$ are paracompact.*

Proof. Assume that there is a paraLindelöf subspace X of $(\rho+1)^2$ which is not paracompact for some $\rho<\omega_1\cdot\omega_1$. Let

$$\mu=\min\{\mu'\leq\rho : X_{[0,\mu']} \text{ is not paracompact }\},$$

$$\nu=\min\{\nu'\leq\rho : X^{[0,\nu']}_{[0,\mu]} \text{ is not paracompact }\}.$$

Since $X^{[0,\nu]}_{[0,\mu]}$ is a clopen subspace of X, we may assume that $X=X^{[0,\nu]}_{[0,\mu]}$. Note that X is not paracompact, $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are paracompact for each $\mu'<\mu$ and $\nu'<\nu$. As in the proof of Theorem 3, we can show that $\langle\mu,\nu\rangle\notin X$ moreover that $\operatorname{cf}\mu\geq\omega_1$ or $\operatorname{cf}\nu\geq\omega_1$. So we may assume $\operatorname{cf}\nu=\omega_1$. Since $\nu\leq\rho<\omega_1\cdot\omega_1$, there is a $\xi\in$ Succ such that $\nu=\omega_1\cdot\xi$. Let $\xi-1$ be the immediate predecessor of ξ. Since $(\omega_1\cdot(\xi-1),\omega_1\cdot\xi]$ is homeomorphic to ω_1+1, by Corllary 6, $X^{(\omega_1\cdot(\xi-1),\omega_1\cdot\xi]}$ is a clopen paracompact subspace. Then $X=X^{[0,\omega_1\cdot(\xi-1)]}\cup X^{(\omega_1\cdot(\xi-1),\omega_1\cdot\xi]}$ is the union of two clopen paracompact subspaces, so X is paracompact, a contradiction. □

The reader should observe that, if Theorem 3 also holds for the following remaining case (c):

(c) $\operatorname{cf}\mu\leq\nu$, $\operatorname{cf}\nu<\nu$ and ν is uncountable.

then all paraLindelöf subspaces of products of two ordinals are paracompact.

However, we have no useful way to extend Corollary 6 for subspaces of $(\rho+1)\times(\omega_1\cdot\omega+1)$ and Corollary 7 for subspaces of $(\omega_1\cdot\omega_1)^2$. Now we conjecture the following.

Conjecture 8. There is a paraLindelöf subspace of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$ which is not paracompact. Or more generally, there is a paraLindelöf subspace of $(\omega_1\cdot\omega_1)^2$ which is not paracompact.

The main open problem on paraLindelöf spaces is the following, see [Wa, Problem 39]:

Problem 9. Are paraLindelöf spaces countably paracompact?

Now we see:

Proposition 10. *Let X be a paraLindelöf countably paracompact subspace of $(\mu+1)\times(\nu+1)$. Assume that $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are paracompact for each $\mu'<\mu$ and $\nu'<\nu$. Then X is paracompact in the following case:*

(c') $\operatorname{cf}\mu\leq\nu$, $\omega=\operatorname{cf}\nu<\nu$ *and ν is uncountable.*

Proof. Assume that X is not paracompact, then as usual, we can easily show that μ and ν are limit ordinals with $\operatorname{cf}\mu \geq \omega_1$ and $\langle \mu, \nu\rangle \notin X$. Take a cub set C in $\operatorname{cf}\mu$ with $M(C) \cap H_\nu(X) = \emptyset$. By Lemma 2, we can also take an open set G with $X^{\{\nu\}} \subset G \subset X \setminus X_{M(C)\cup\{\mu\}}$ and $\operatorname{Cl} G \cap X_{\{\mu\}} = \emptyset$. Let $\mathcal{W} = \{W\} \cup \{W(n) : n \in \omega\}$ be a precise locally finite open refinement of $\mathcal{U} = \{G\} \cup \{X^{[0,N(n)]} : n \in \omega\}$. $X_{M(C)\cup\{\mu\}} \subset \bigcup_{n\in\omega} W(n)$ is obvious. By the local finiteness of $\mathcal{W}$, $\operatorname{Cl}(\bigcup_{n\in\omega} W(n)) = \bigcup_{n\in\omega} \operatorname{Cl} W(n) \subset X \setminus X^{\{\nu\}}$. Then in a usual way, we can show that X is paracompact, a contradiction. $\square$

Using a similar argumet in Corollary 4 and 5, we see:

Corollary 11. *For a suitably large ordinal ρ and for each $\sigma < \omega_1 \cdot \omega_1$, paraLindelöf countably paracompact subspaces of $(\rho + 1) \times (\sigma + 1)$ are paracompact. In particular, paraLindelöf countably paracompact subspaces of $(\omega_1 \cdot \omega_1) \times (\sigma + 1)$ are paracompact for each $\sigma < \omega_1 \cdot \omega_1$*

Applying this corollary, we can show the positive answer of the conjecture 8 for $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ yields the negative answer of Problem 9, that is:

Corollary 12. *If there is a a paraLindelöf subspace X of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ which is not paracompact, then X is a paraLindelöf space which is not countably paracompact.*

However, strangely, we have also no useful way to show that paraLindelöf countably paracompact subspaces of $(\omega_1 \cdot \omega_1)^2$ are paracompact. Now we conjecture the following:

Conjecture 13. There is a paraLindelöf countably paracompact subspace of $(\omega_1 \cdot \omega_1)^2$ which is not paracompact.

Although the existence of a paraLindelöf non-paracompact subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ still remains open, now we give some informations about the existence of such a subspace.

From now on, consider the normal function M for $\omega_1 \cdot \omega_1$ by letting $M(\gamma) = \omega_1 \cdot \gamma$ for each $\gamma < \omega_1$.

Proposition 14. *Let X be a paraLindelöf subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$. Then for each $\alpha \in H_{\omega_1\cdot\omega}(X)$, there is $g(\alpha) < \omega$ such that $X_{\{\alpha\}}^{(\omega_1\cdot g(\alpha), \omega_1\cdot\omega]}$ is countable.*

Proof. For each $n \leq \omega$, since $H_{\omega_1\cdot n}(X)$ is not stationary in $\omega_1 \cdot \omega_1$, take a cub set C_n in ω_1 such that $M(C_n) \cap H_{\omega_1\cdot n}(X) = \emptyset$. Set $C = \bigcap_{n\leq\omega} C_n$ and $E = \{\omega_1 \cdot n : n \leq \omega\}$. Then $X_{M(C)}$ and X^E are disjoint closed sets of X. For each $\gamma < \omega_1$, let $U(\gamma) = X^{[0,\gamma]} \cup \bigcup_{1\leq n\in\omega} X^{(\omega_1\cdot n, \omega_1\cdot n+\gamma]}$. Take a precise locally countable open refinement $\mathcal{W} = \{W(\gamma) : \gamma < \omega_1\} \cup \{W\}$ of $\mathcal{U} = \{U(\gamma) : \gamma < \omega_1\} \cup \{X \setminus X_{M(C)}\}$. Let $\alpha \in H_{\omega_1\cdot\omega}(X)$. Since $\mathcal{W}$ is locally countable at $\langle \alpha, \omega_1 \cdot \omega\rangle$, there is $g(\alpha) < \omega$ such that $X_{\{\alpha\}}^{(\omega_1\cdot g(\alpha),\omega_1\cdot\omega]}$ meets at most countably many $W(\gamma)$'s. Then this $g(\alpha)$ works. $\square$

Proposition 15. *Assume that X is a paraLindelöf subspace of $(\omega_1 \cdot \omega_1) \times (\omega_1 \cdot \omega + 1)$ such that for some $\alpha_0 < \omega_1 \cdot \omega$, $H_{\omega_1\cdot\omega}(X) \cap [\alpha_0, \alpha]$ is Lindelöf for every $\alpha < \omega_1 \cdot \omega_1$ with $\alpha_0 \leq \alpha$. Then X is paracompact.*

Proof. Note that, by Corollary 6 and 7, $X_{[0,\mu']}$ and $X^{[0,\nu']}$ are paracompact for each $\mu' < \omega_1 \cdot \omega_1$ and $\nu' < \omega_1 \cdot \omega$. Since $X_{[0,\alpha_0]}$ is paracompact, we may assume $\alpha_0 = 0$, that is, $H_{\omega_1\cdot\omega}(X) \cap [0, \alpha]$ is Lindelöf for every $\alpha < \omega_1 \cdot \omega_1$. As in the proof of Proposition 14, define a cub set $C \subset \omega_1$ with $X_{M(C)} \cap X^E = \emptyset$, where $E = \{\omega_1 \cdot n : n \leq \omega\}$. Let $\mathcal{W} = \{W(\gamma) : \gamma < \omega_1\}$ be a precise locally countable open refinement of the open cover $\mathcal{U} = \{X_{[0,M(\gamma))} : \gamma < \omega_1\}$. For each $\alpha \in H_{\omega_1\cdot\omega}(X)$, take $g(\alpha) < \omega$ and $f(\alpha) < \alpha$ such that

$H(\alpha) = X^{(\omega_1\cdot g(\alpha),\omega_1\cdot\omega]}_{(f(\alpha),\alpha]}$ meets at most countably many $W(\gamma)$'s and $(f(\alpha),\alpha]\cap M(C)=\emptyset$. For each $\gamma\in\mathrm{Succ}(C)$, since $H_{\omega_1\cdot\omega}(X)\cap[M(\mathrm{p}_C(\gamma)),M(\gamma)] = H_{\omega_1\cdot\omega}(X)\cap(M(\mathrm{p}_C(\gamma)),M(\gamma))$ is a clopen subspace of the Lindelöf space $H_{\omega_1\cdot\omega}(X)\cap[0,M(\gamma)]$, there is a countable subset $Z(\gamma)\subset H_{\omega_1\cdot\omega}(X)\cap(M(\mathrm{p}_C(\gamma)),M(\gamma))$ such that

$$X^{\{\omega_1\cdot\omega\}}_{(M(\mathrm{p}_C(\gamma)),M(\gamma))}\subset\bigcup_{\alpha\in Z(\gamma)}H(\alpha)\subset X_{(M(\mathrm{p}_C(\gamma)),M(\gamma))}.$$

Of course, $H=\bigcup_{\gamma\in\mathrm{Succ}(C)}(\bigcup_{\alpha\in Z(\gamma)}H(\alpha))$ covers $X^{\{\omega_1\cdot\omega\}}$. In a usual way, the following Claim shows that X is paracompact.

Claim. *$S=\{\delta\in C : X_{\{M(\gamma)\}}\cap\mathrm{Cl}\,H\neq\emptyset\}$ is not stationary in ω_1.*

Proof. Assume that S is stationary. For each $\delta\in S$, fix $h(\delta)\le\omega_1\cdot\omega$ and $\psi(\delta)>\delta$ such that $\langle M(\delta),h(\delta)\rangle\in\mathrm{Cl}\,H\cap W(\psi(\delta))$. Moreover since $\bigcup_{\alpha\in Z(\gamma)}H(\alpha)\subset X_{(M(\mathrm{p}_C(\gamma)),M(\gamma))}$ for each $\gamma\in\mathrm{Succ}(C)$, we can find $\varphi(\delta)\in\mathrm{Succ}(C)\cap\delta$ with $(\bigcup_{\alpha\in Z(\varphi(\delta))}H(\alpha))\cap W(\psi(\delta))\neq\emptyset$. Applying the PDL, we can find a stationary set $S'\subset S$ and $\delta_0\in\mathrm{Succ}(C)$ such that $\varphi(\delta)=\delta_0$ for each $\delta\in S'$ and the members of $\{\psi(\delta):\delta\in S'\}$ are all distinct. Then $\bigcup_{\alpha\in Z(\delta_0)}H(\alpha)$ meets uncountably many $W(\psi(\delta))$'s, $\delta\in S'$. Since $Z(\delta_0)$ is countable, we can find $\alpha_0\in Z(\delta_0)$ such that $H(\alpha_0)$ meets uncountably many $W(\psi(\delta))$'s, a contradiction. $\square$

Corollary 16. *Assume that there exists a paraLindelöf subspace X of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$ which is not paracompact. Then $I=\{\alpha\in\omega_1\cdot\omega_1\setminus H_{\omega_1\cdot\omega}(X):\mathrm{cf}\,\alpha=\omega_1\}$ is unbounded in $\omega_1\cdot\omega_1$.*

Proof. Assume that I is bounded by some $\alpha_0<\omega_1\cdot\omega_1$. The following general fact shows that $H_{\omega_1\cdot\omega}(X)\cap[\alpha_0,\alpha]$ is Lindelöf for every $\alpha<\omega_1\cdot\omega_1$ with $\alpha_0\le\alpha$.

Fact. *If Z is a subspace of $\rho+1$ for some ordinal ρ such that $\mathrm{cf}\,\beta\le\omega$ for every $\beta\in(\rho+1)\setminus Z$, then Z is Lindelöf.*

Proof. Assume that Z is not Lindelöf and let

$$\mu=\min\{\mu'\le\rho : Z\cap[0,\mu'] \text{ is not Lindelöf }\}.$$

Then we see that $\mu\notin Z$, μ is limit, $Z\cap[0,\mu']$ is Lindelöf for every $\mu'<\mu$ and $Z\cap[0,\mu]$ is not Lindelöf. Since $\mu\notin Z$, we have $\mathrm{cf}\,\mu=\omega$. Then $Z\cap[0,\mu]=\bigcup_{n\in\omega}(Z\cap[0,\mu(n)])$ can be represented as the countable union of Lindelöf subspaces, where $\{\mu(n):n\in\omega\}$ is a strictly increasing cofinal sequence in μ, so it is Lindelöf, a contradiction. $\square$

Taking account of these informations, the authors have tried to construct a paraLindelöf non-paracompact subspace of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$. But now we present the following by-product of these considerations. Here note that the locally finite union of clopen paracompact subspaces are also paracompact.

Example 17. There exists a non-paracompact subspace X of $(\omega_1\cdot\omega_1)\times(\omega_1\cdot\omega+1)$, which can be represented as the locally countable union of clopen paracompact subspaces, such that $X_{[0,\mu]}$ and $X^{[0,\nu]}$ are paracompact for each $\mu<\omega_1\cdot\omega_1$ and $\nu<\omega_1\cdot\omega$. In fact, unfortunately, this X is not paraLindelöf.

Our space is defined as follows:

$$X=\left[\bigcup_{\gamma,\beta\in\mathrm{Succ}}\{\omega_1\cdot(\gamma-1)+\beta\}\times\left(\{\omega_1\cdot\omega\}\cup\bigcup_{n\in\omega}\mathrm{Succ}((\omega_1\cdot n,\omega_1\cdot n+\beta))\right)\right]$$

$$\cup \left[\bigcup_{\delta \in \mathrm{Lim}} \{\omega_1 \cdot \delta\} \times \left(\bigcup_{n \in \omega} \{\omega_1 \cdot n + \delta + 1\} \right) \right],$$

where $\mathrm{Succ}((\omega_1 \cdot n, \omega_1 \cdot n + \beta))$ denotes the set of all successor ordinals in the open interval $(\omega_1 \cdot n, \omega_1 \cdot n + \beta)$. Observe that $X_{\{\omega_1 \cdot \gamma\}} = \emptyset$ for each $\gamma \in \mathrm{Succ}$ and $X^{\{\omega_1 \cdot n\}} = \emptyset$ for each $n \in \omega$.

Claim 1. *$X^{[0,\nu]}$ is paracompact for each $\nu < \omega_1 \cdot \omega$.*

Proof. Since $X^{(0,\omega_1]}$ is homeomorphic to $X^{(\omega_1 \cdot n, \omega_1 \cdot (n+1)]}$ for each $n \in \omega$, it suffices to show that $X^{[0,\omega_1]}$ is paracompact. Note that for each $\eta < \omega_1$, $H_\eta(X)$ has at most one limit ordinal, so $X^{\{\eta\}}$ is paracompact. Since $X^{\{\eta\}} = \emptyset$ for each $\eta \in \mathrm{Lim} \cup \{\omega_1\}$, $X^{[0,\omega_1]} = \bigoplus_{\eta \in \mathrm{Succ}} X^{\{\eta\}}$ is paracompact.

Claim 2. *$X_{[0,\mu]}$ is paracompact for each $\mu < \omega_1 \cdot \omega_1$.*

proof. Note that for each $\zeta < \omega_1 \cdot \omega_1$, $V_\zeta(X)$ has at most one limit ordinal ($\omega_1 \cdot \omega$ if it has), so $X_{\{\zeta\}}$ is paracompact. Assuming that $X_{[0,\mu]}$ is not paracompact for some $\mu < \omega_1 \cdot \omega_1$, let $\mu < \omega_1 \cdot \omega_1$ be the such minimal one. Then μ is limit and $\langle \mu, \omega_1 \cdot \omega \rangle \notin X_{[0,\mu]}$. If $\mathrm{cf}\, \mu = \omega$, then

$$X_{[0,\mu]} = \left(\bigoplus_{n \in \omega} X_{(M(n-1), M(n)]} \right) \cup \left(\bigoplus_{n \in \omega} X_{[0,\mu]}^{(\omega \cdot (n-1), \omega \cdot n]} \right)$$

can be represented as the countable union of paracompact clopen subspaces. Therefore $X_{[0,\mu]}$ is paracompact, a contradiction. If $\mathrm{cf}\, \mu = \omega_1$, then $\mu = \omega_1 \cdot \gamma$ for some $\gamma \in \mathrm{Succ}$. Since $X_{(\omega_1 \cdot (\gamma-1), \omega_1 \cdot \gamma]} = \bigoplus_{\beta \in \mathrm{Succ}} X_{\{\omega_1 \cdot (\gamma-1)+\beta\}}$ is paracompact, by the minimality of μ, $X_{[0,\mu]} = X_{[0,\omega_1 \cdot (\gamma-1)]} \bigoplus X_{(\omega_1 \cdot (\gamma-1), \omega_1 \cdot \gamma]}$ is paracompact, a contradiction.

Claim 3. *Let $V(\gamma) = X_{(\omega_1 \cdot (\gamma-1), \omega_1 \cdot \gamma)}$ for each $\gamma \in \mathrm{Succ}$ and $V(\delta, n) = X_{[0,\omega_1 \cdot \delta]}^{\{\omega_1 \cdot n + \delta + 1\}}$ for each $\delta \in \mathrm{Lim}$ and $n \in \omega$. Then $\mathcal{V} = \{V(\gamma) : \gamma \in \mathrm{Succ}\} \cup \{V(\delta, n) : \langle \delta, n \rangle \in \mathrm{Succ} \times \omega\}$ is a locally countable open refinement of the open cover $\mathcal{U} = \{X_{[0,\omega_1 \cdot \gamma]} : \gamma < \omega_1\}$.*

proof. Since other properties are not so hard, we only show that $\mathcal{V}$ is locally countable. Let $\langle \zeta, \eta \rangle \in X$.

First assume $\zeta \in (\omega_1 \cdot (\gamma - 1), \omega_1 \cdot \gamma)$ for some $\gamma \in \mathrm{Succ}$. Then there is $\beta \in \mathrm{Succ}$ with $\zeta = \omega_1 \cdot (\gamma - 1) + \beta$. If $\gamma' \in \mathrm{Succ}$ with $\gamma' \neq \gamma$, then $X_{\{\zeta\}} \cap V(\gamma') = \emptyset$. Moreover if $\delta \in \mathrm{Lim}$ with $\beta \leq \delta$, then $X_{\{\zeta\}} \cap V(\delta, n) = \emptyset$ for each $n \in \omega$. Therefore $X_{\{\zeta\}}$ is a neighborhood of $\langle \zeta, \eta \rangle$ which witnesses the local countability of $\mathcal{V}$ at $\langle \zeta, \eta \rangle$.

Next assume $\zeta = \omega_1 \cdot \delta$ for some $\delta \in \mathrm{Lim}$. Then by the construction of X, it is not difficult to show that $X_{[0,\zeta]}^{\{\eta\}}$ is a neighborhood of $\langle \zeta, \eta \rangle$ which witnesses the local countability of $\mathcal{V}$ at $\langle \zeta, \eta \rangle$.

Since $V(\gamma)$'s and $V(\delta, n)$'s are clopen in X, Claim 2 and 3 say that X can be represented as the locally countable union of clopen paracompact subspaces.

Claim 4. *X is not paraLindelöf.*

Proof. Let $V(\gamma, \beta) = X_{\{\omega_1 \cdot (\gamma-1)+\beta\}}$ for each $\gamma, \beta \in \mathrm{Succ}$, $V(\delta, n) = X_{[0,\omega_1 \cdot \delta]}^{\{\omega_1 \cdot n + \delta + 1\}}$ for each $\delta \in \mathrm{Lim}$ and $n \in \omega$. Assume that there is a precise locally countable open refinement $\mathcal{W} = \{W(\gamma, \beta) : \langle \gamma, \beta \rangle \in \mathrm{Succ} \times \mathrm{Succ}\} \cup \{W(\delta, n) : \langle \delta, n \rangle \in \mathrm{Lim} \times \omega\}$ of the open cover $\mathcal{V} = \{V(\gamma, \beta) : \langle \gamma, \beta \rangle \in \mathrm{Succ} \times \mathrm{Succ}\} \cup \{V(\delta, n) : \langle \delta, n \rangle \in \mathrm{Lim} \times \omega\}$. Let $\delta \in \mathrm{Lim}$ and $n \in \omega$. Since $\{W(\gamma, \beta) : \langle \gamma, \beta \rangle \in \mathrm{Succ} \times \mathrm{Succ}\}$ is locally countable, we can take $f(\delta, n) < \delta$ such that

$$I(\delta, n) = \{\langle \gamma, \beta \rangle \in \mathrm{Succ} \times \mathrm{Succ} : W(\gamma, \beta) \cap X_{(\omega_1 \cdot f(\delta,n), \omega_1 \cdot \delta]}^{\{\omega_1 \cdot n + \delta + 1\}} \neq \emptyset\}$$

is countable. Fix $n \in \omega$ and moving $\delta \in \mathrm{Lim}$, by the PDL, we can find a stationary set $S(n) \subset \mathrm{Lim}$ and $f(n) < \omega_1$ such that $f(\delta, n) = f(n)$ for each $\delta \in S(n)$. Let $\gamma_0 = \sup\{f(n) : n \in \omega\}$ and fix $\gamma \in \mathrm{Succ}$ with $\gamma_0 < \gamma$. Note $\gamma_0 \leq \gamma - 1$. For each $\beta \in \mathrm{Succ}$, since the unique member of $\mathcal{V}$ which contains the point $\langle \omega_1 \cdot (\gamma - 1) + \beta, \omega_1 \cdot \omega \rangle$ is $V(\gamma, \delta)$, $W(\gamma, \delta)$ also contains the point $\langle \omega_1 \cdot (\gamma - 1) + \beta, \omega_1 \cdot \omega \rangle$. So there is $n_0 \in \omega$ and an uncountable subset $K \subset \mathrm{Succ}$ such that $X^{(\omega_1 \cdot n_0, \omega_1 \cdot \omega]}_{\{\omega_1 \cdot (\gamma-1)+\beta\}} \subset W(\gamma, \beta)$ for each $\beta \in K$. Fix $\delta \in S(n_0)$ with $\gamma < \delta$. Since $I(\delta, n_0)$ is countable and K is uncountable, we can find $\beta \in K$ with $\delta + 1 < \beta$ and $\langle \gamma, \beta \rangle \notin I(\delta, n_0)$. By $\delta + 1 < \beta$, the point $\langle \omega_1 \cdot (\gamma - 1) + \beta, \omega_1 \cdot n_0 + \delta + 1 \rangle$ belongs to X. Moreover,

$$\langle \omega_1 \cdot (\gamma - 1) + \beta, \omega_1 \cdot n_0 + \delta + 1 \rangle \in X^{(\omega_1 \cdot n_0, \omega_1 \cdot \omega]}_{\{\omega_1 \cdot (\gamma-1)+\beta\}} \cap X^{\{\omega_1 \cdot n_0 + \delta + 1\}}_{(\omega_1 \cdot \gamma_0, \omega_1 \cdot \delta]}$$

$$\subset W(\gamma, \beta) \cap X^{\{\omega_1 \cdot n_0 + \delta + 1\}}_{(\omega_1 \cdot f(\delta, n_0), \omega_1 \cdot \delta]}.$$

Therefore $\langle \gamma, \beta \rangle \in I(\delta, n_0)$, a contradiction. □

In this connection, note that, as is well known, the space ω_1 is the locally countable union of closed paracompact subspaces $\{\alpha\}$'s, $\alpha < \omega_1$, but ω_1 is not paralindelöf. But:

Proposition 18. *Let $X \subset \rho + 1$ for some ordinal ρ. Assume that X is the locally countable union of clopen paracompact subspaces $X(\lambda)$'s, $\lambda \in \Lambda$. Then X is paracompact.*

Proof. Assume that X is not paracompact. Let

$$\mu = \min\{\mu' \leq \rho : X \cap [0, \mu'] \text{ is not paracompact }\}.$$

Then by the minimality of μ, μ is limit ordinal, $\mu \notin X$, $\mathrm{cf}\, \mu \geq \omega_1$ and $X \cap [0, \mu]$ is stationary in μ. By identifying $X = X \cap [0, \mu]$, we may assume that X is a stationary subset of μ and $X \cap [0, \mu']$ is paracompact for each $\mu' < \mu$.

Claim. *$X(\lambda)$ is bounded in μ for each $\lambda \in \Lambda$.*

Proof. Since $X(\lambda)$ is paracompact, it is not stationary in μ. So there is a cub set $C \subset \mathrm{Lim}(\mathrm{cf}\, \mu)$ such that $X(\lambda) \cap M(C) = \emptyset$, where M is a normal function for μ. For each $\gamma \in C \cap M^{-1}(X)$, fix $f(\gamma) < \gamma$ such that $X(\lambda) \cap (M(f(\gamma)), M(\gamma)] = \emptyset$. Then by the PDL, we find a stationary set $S \subset C \cap M^{-1}(X)$ and $\gamma_0 < \mathrm{cf}\, \mu$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Then $X(\lambda) \subset [0, M(\gamma_0)]$, and so $X(\lambda)$ is bounded.

Since $X(\lambda)$'s cover X and are open, for each $\gamma \in M^{-1}(X) \cap \mathrm{Lim}(\mathrm{cf}\, \mu)$, fix $f(\gamma) < \gamma$, $\lambda(\gamma) \in \Lambda$ and $g(\gamma) < \mathrm{cf}\, \mu$ such that $X \cap (M(f(\gamma)), M(\gamma)] \subset X(\lambda(\gamma)) \subset [0, M(g(\gamma))]$. By the PDL, we find a stationary set $S \subset M^{-1}(X) \cap \mathrm{Lim}(\mathrm{cf}\, \mu)$ and $\gamma_0 < \mathrm{cf}\, \mu$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Set $g(\gamma) = 0$ for each $\gamma \in \mathrm{cf}\, \mu \backslash (M^{-1}(X) \cap \mathrm{Lim}(\mathrm{cf}\, \mu))$ and $C = \{\gamma < \mathrm{cf}\, \mu : \forall \gamma' < \gamma (g(\gamma') < \gamma)\}$. Then members of $\{\lambda(\gamma) : \gamma \in S \cap C\}$ are distinct. Take $\alpha \in X$ with $M(\gamma_0) < \alpha$. Then $\alpha \in X(\lambda(\gamma))$ for each $\gamma \in S \cap C$ with $\alpha < M(\gamma)$. This contradicts the local countability of $\{X(\lambda) : \lambda \in \Lambda\}$. □

Burke [Bu] proved that submetacompact spaces in which every open cover has a σ-locally countable closed refinement are subparacompact. Using this we can see:

Proposition 19. *ParaLindelöf subspaces of products of two ordinals are subparacompact.*

Proof. Let X be a paraLindelöf subspace of products of two ordinals. Then by [KTY, Theorem 2.3], it is metacompact. By the regularity of X, every open cover has a σ-locally countable closed refinement. So by the result of [Bu], it is subparacompact. □

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