TOPOLOGY THEOREMS FORMULATED FROM HYPERBOLIC GEOMETRY CONSIDERATIONS

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Abstract. Hyperbolic geometry theorems are abstracted to formulate topology theorems for function spaces. These theorems generalize results which were established for complex spaces by Kiernan, Kiernan – Kobayashi, Kobayashi, and Noguchi. They also provide some new results in the framework of complex spaces including characterizations of hyperbolically imbedded complex subspaces modulo closed complex subspaces in terms of the family of holomorphic mappings into such spaces, and extension and convergence theorems for this family of mappings.

§**1. Introduction.** Since S. Kobayashi introduced the important notion of intrinsic pseudodistance in the theory of complex spaces, hyperbolic geometry has proved to be useful in several areas of study. In [5], [6], formulations of theorems in topology have been motivated by theorems from hyperbolic geometry. In this paper we present other such theorems. From these theorems some known results for complex spaces are established by purely topological methods and some new results are discovered for such spaces.

All spaces in this paper are assumed to be Hausdorff and if X, Y are spaces we represent the space of continuous functions equipped with the compact – open topology by $C(X,\ Y)$. Bagley and Yang established the following topological Ascoli – Arzelà Theorem in [3] (recall that a topological space is a $k-$ *space* if a subset C of the space is closed in the space whenever $C \cap Q$ is closed in Q for each compact Q of the space):

Theorem A. *If X is a $k-$space, Y a regular space, then $\Omega \subset C(X,\ Y)$ is compact iff*

(1) Ω is evenly continuous,

(2) $\Omega(x) = \{f(x) : f \in \Omega\}$ is a relatively compact subset of Y for each $x \in X$, and

(3) Ω is a closed subset of $C(X,\ Y)$.

If $\Omega \subset C(X,Y)$ we say that Ω is *evenly continuous from $p \in X$ to $q \in Y$* if for each U open in Y about q, there exist V, W open in X, Y about p, q respectively such that $\{f \in \Omega : f(p) \in W\} \subset \{f \in \Omega : f(V) \subset U\}$. If Ω is evenly continuous from each $p \in X$ to each $q \in Y$ we say that Ω is *evenly continuous* (from X to Y) [9]. The version of the Ascoli – Arzelà Theorem given below in Theorem B is readily derived from Theorem A since $\overline{\Omega(x)} = \overline{\Omega}(x)$ for each $x \in X$ and, under the hypothesis, Ω is evenly continuous iff $\overline{\Omega}$ is evenly continuous ($\overline{Q}$ represents the closure of the subset Q of a topological space).

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Theorem B. *Let X be a $k-$space and let Y be a regular space. Then $\Omega \subset C(X,Y)$ is a relatively compact subset of $C(X,Y)$ iff*

(1) Ω is evenly continuous, and

(2) $\Omega(x)$ is a relatively compact subset of Y for each $x \in X$.

In this paper we use Theorem B to establish several topological criteria for subsets of function spaces of continuous extensions to be relatively compact and apply these to spaces of holomorphic mappings from complex manifolds to complex spaces. If X_0, Y_0 are subspaces of the topological spaces X, Y respectively and $\mathcal{F} \subset \mathcal{C}(X_0, Y_0)$, $\mathcal{C}[X, Y; \mathcal{F}]$ will denote the collection of $g \in \mathcal{C}(X,Y)$ which are extensions of elements of $\mathcal{F}$. Here X_0 will be dense in X and, consequently, each such extension of $f \in \mathcal{C}(X_0, Y_0)$ will be unique and will be denoted by $\widetilde{f}$. It will be evident from the context which spaces X_0, Y_0, X, Y are under consideration. If X_0, Y_0 are complex subspaces of complex spaces X, Y respectively we will write $\mathcal{H}[X, Y^+; \mathcal{F}] = \mathcal{C}[X, Y^+; \mathcal{F}]$ if Y^+ is a complex space with Y as a complex subspace. Otherwise, $\mathcal{H}[X, Y^+; \mathcal{F}] = \mathcal{C}[X, Y^+; \mathcal{F}] \cap \mathcal{H}(X,Y)$. Definition 1 appears in [5].

Definition 1. If X and Y are topological spaces and $X_0 \subset X$ is dense, we say that $\Omega \subset C(X_0, Y)$ is *topologically uniformly normal with respect to* (X_0, X, Y) if for each $x \in X$, $y \in Y$ and net $\{(f_\alpha, x_\alpha, v_\alpha)\}$ in $\Omega \times X_0 \times X_0$ such that $x_\alpha \to x$, $v_\alpha \to x$ and $f_\alpha(x_\alpha) \to y$ we have $f_\alpha(v_\alpha) \to y$.

For our purposes we extend Definition 1 in the form of Definition 2.

Definition 2. If X and Y are topological spaces, $X_0 \subset X$ is dense, and Δ is a closed subset of Y we say that $\Omega \subset C(X_0, Y)$ is *topologically uniformly normal modulo Δ with respect to* (X_0, X, Y) if Ω is topologically uniformly normal with respect to $(X_0, X, Y - \Delta)$. We will use the abbreviation 'mod Δ' for ' modulo Δ'.

The motivation for this definition will become evident in the section on applications.

If Y is a topological space we let $Y^\infty = Y \cup \{\infty\}$ represent the one$-$point compactification of Y if Y is not compact, $Y^\infty = Y$ if Y is compact; if Γ satisfies $\Delta \subset \Gamma \subset \Delta \cup \{\infty\}$ we represent the quotient space obtained from Y^∞ by collasping a nonempty Γ to a point by Y_Γ^∞, and the induced canonical projection by $\mathbf{P}_\Gamma$ ($Y_\emptyset^\infty = Y^\infty$ and $\mathbf{P}_\emptyset(y) = y$). Topology function space theorems proved in §2 and applied in §3 include the following: Let X_0 be a dense subset of the k$-$space X, let Y be locally compact, let $\Delta \subset Y$, let $\Gamma = \Delta \cup \{\infty\}$ where Δ is a closed subset of Y, and suppose $\Omega \subset C(X_0,\ Y)$ is topologically uniformly normal mod Δ with respect to $(X_0,\ X,\ Y)$. Then

(1°) For each $f \in \overline{\Omega}$, $\mathbf{P}_\Gamma \circ f$ extends to $g \in C(X, Y_\Gamma^\infty)$.

(2°) $C[X,\ Y_\Gamma^\infty; \mathbf{P}_\Gamma \circ \Omega]$ is a relatively compact subset of $C(X, Y_\Gamma^\infty)$ (where $\mathbf{P}_\Gamma \circ \Omega = \{\mathbf{P}_\Gamma \circ f : f \in \Omega\}$).

(3°) $C[X,\ Y_\Gamma^\infty; \overline{\mathbf{P}_\Gamma \circ \Omega}]$ is a compact subset of $C(X, Y_\Gamma^\infty)$.

(4°) If $\{f_\alpha\}$ is a net in $\overline{\Omega}$ and $f_\alpha \to f$ then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

If, in addition, we assume that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$ then (1°) $-$ (4°) hold for the closed subsets Γ of Y^∞ satisfying $\Delta \subset \Gamma \subset \Delta \cup \{\infty\}$ ($\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, the unit disk in the complex plane, $\mathbf{D}^* = \mathbf{D} - \{0\}$, the punctured disk, satisfy this additional property and $\Sigma(x)$ represents the family of open sets having x as an element unless otherwise specified).

We now give definitions and background for the purpose of stating some of our applications of the results in §2 to the theory of complex spaces (see [13] for more information

on complex spaces). Let X be a complex space with a length function E (see [13]). We recall the definition of the *Kobayashi intrinsic pseudo-distance* k_X ; $k_X(p,q) = \inf_\sigma \ell(\sigma)$, where σ is a chain of holomorphic disks $\{\varphi_i\}_{i=1,...,m} \subset H(\mathbf{D}, X)$ such that $\varphi_1(0) = p$, $\varphi_i(z_i) = \varphi_{i+1}(0)$ for $i = 1, ..., m-1$ and $\varphi_m(z_m) = q$, and $\ell(\sigma) = \sum_{i=1}^m d_{\mathbf{D}}(0, z_i)$, where $d_{\mathbf{D}}$ is the Poincaré distance on $\mathbf{D}$.

The pseudo-distance k_X has an infinitesmal form given by

$$K_X(p,v) = \inf\{r > 0 : \varphi(0) = p,\ (d\varphi)_0(re) = v \text{ for some } \ \varphi \in H(\mathbf{D}, M)\}$$

where $p \in X$, $v \in T_p(X)$, the tangent space of X at p, $d\varphi$ is the *tangent map for* φ and e is the unit vector 1 at $0 \in \mathbf{D}$ (see [13][14]). If Y is a complex space and E is a length function on Y we denote by d_E the distance function generated on Y by E [13]. The *norm* $|df|_E$ *of the tangent map for* $f \in H(X,Y)$ *with respect to* E is defined by

$$|df|_E = \sup\{|(df)_p|_E : p \in X\} \qquad \text{where}$$
$$|(df)_p|_E = \sup\{E(f(p), (df)_p(v)) : K_X(p,v) = 1\}$$

(we use simply $|df|$ and $|(df)_p|$ when no confusion may arise). A complex space X is called *hyperbolic* if the pseudo – distance k_X is a distance function. Let X be a relatively compact open subset of a complex space Y and let Δ be a closed subspace of Y. In [12] X is said to be *hyperbolically imbedded in* Y *mod* Δ if for every pair of distinct points $p, q \epsilon \overline{X}$ not both elements of Δ, there exist neighborhoods U of p and V of q in Y such that $k_X(U \cap X, V \cap X) > 0$. If $\Delta = \emptyset$, X is said to be *hyperbolically imbedded* in Y. The notion of hyperbolic imbeddedness was introduced by Kobayashi in [13] and used to produce the generalization of the Big Picard Theorem given in Theorem C.

Theorem C. *Let X be a relatively compact hyperbolically imbedded complex subspace of a complex space Y. Then each $f \in H(\mathbf{D}^*, X)$ extends to $\tilde{f} \in H(\mathbf{D}, Y)$.*

The concept of hyperbolic inbeddedness modulo closed complex subspaces, due to Kiernan and Kobayashi [12], was utilized to restate results of Bloch [2] and Cartan [4] in a more general setting by using the Kobayashi intrinsic pseudo – distance defined on a complex space and the concept of tautness of Wu [17]. (The example which they had in mind is the one where Y is the complex projective space $\mathbf{P}_n(\mathbf{C})$, X is the complement of $n+2$ hyperplanes in general position in $\mathbf{P}_n(\mathbf{C})$ and Δ is the union of a certain collection of hyperplanes in $\mathbf{P}_n(\mathbf{C})$). Abate [1] showed that a complex space X is hyperbolic if and only if $H(\mathbf{D}, X)$ is a relatively compact subset of $C(\mathbf{D}, X^\infty)$. In [10], Kiernan established that a relatively compact complex subspace X of a complex space Y is hyperbolically imbedded in Y if and only if $H(\mathbf{D}, X)$ is a relatively compact subset of $H(\mathbf{D}, Y)$, and in [8] the authors have recently proved that a complex subset X of a complex space Y is hyperbolically imbedded in Y if and only if there is a distance function d on Y such that $d(f(x), f(y)) \leq k_{\mathbf{D}^*}(x,y)$ for all $x, y \in \mathbf{D}^*$ and $f \in H(\mathbf{D}^*, X)$. Kobayashi and Kiernan [12] proved that a relatively compact complex subspace X is hyperbolically imbedded in Y mod Δ, where Δ is a closed complex subspace of Y, if $H(\mathbf{D}, X)$ is relatively compact in $H(\mathbf{D}, Y)$ mod Δ. It is known [14] that $H(\mathbf{D}, X)$ *is a relatively compact subset of* $H(\mathbf{D}, Y)$ *mod* Δ *(locally relatively compact* in Lang [14]), if given any sequence F in $H(\mathbf{D}, X)$, there is a subsequence $\{f_n\}$ such that either it converges uniformly on compact subsets of $\mathbf{D}$ to an element of $H(\mathbf{D}, Y)$ or given a neighborhood U of Δ in Y and a compact subset K of $\mathbf{D}$, $f_n(K) \subset U$ ultimately. In §3 we generalize and unify these results , and offer characterizations of hyperbolically imbedded mod Δ complex subspaces in terms of function spaces, providing an answer to a question which was left open in [12] and [14]. We also extend the following theorems due

to Noguchi [15], [16] to non – relatively compact hyperbolically imbedded mod Δ spaces. The methods used in this paper should be compared with the complex space and measure – theoretic based methods of the aforementioned authors (see [1], [10]– [13], pp. 43 — 64 in [14], and [16]). A divisor A on a complex manifold M of dimension m has *normal crossings* (see [14]) if at each point in A there exists a system of coordin complexates $z_1, ..., z_m$ for M such that, locally,

$$M - A = (\mathbf{D}^*)^r \times \mathbf{D}^s \quad \text{with} \quad r + s = m.$$

Theorem D. *Let X be a relatively compact hyperbolically imbedded complex subspace of a complex space Y. Let M be a complex manifold and A be a divisor on M with normal crossings.*

(1) If $\{f_n\}$ is a sequence in $H(\mathbf{D}^*,\ X)$ and $f_n \to f \in H(\mathbf{D}^*,\ Y)$ then $\widetilde{f} \in H(\mathbf{D},\ Y)$ exists and $\widetilde{f_n} \to \widetilde{f}$.

(2) If $\{f_n\}$ is a sequence in $H(M - A,\ X)$ and $f_n \to f \in H(M - A,\ Y)$ then $\widetilde{f} \in H(M,\ Y)$ exists and $\widetilde{f_n} \to \widetilde{f}$.

§**2. Topological function space theorems.** Lemmas 1 and 2 will be used in conjunction with results from [6] to establish our function space theorems.

Lemma 1. *Let X_0 be a dense subset of the k – space X, let Y be a locally compact space, let $\Gamma = \Delta \cup \{\infty\}$ where Δ is a closed subset of Y and let $\Omega \subset C(X_0,\ Y)$. Then Ω is topologically uniformly normal mod Δ with respect to $(X_0,\ X,\ Y)$ iff $\mathbf{P}_\Gamma \circ \Omega$is topologically uniformly normal with respect to $(X_0,\ X,\ Y_\Gamma^\infty)$.*

Proof. **Necessity.** Suppose $(f_\alpha,\ x_\alpha,\ v_\alpha)$ is a net in $\Omega \times X_0 \times X_0$ and let $x \in X,\ y,\ z \in Y$ satisfy $\mathbf{P}_\Gamma \circ f_\alpha(x_\alpha) \to \mathbf{P}_\Gamma(y),\ \mathbf{P}_\Gamma \circ f_\alpha(v_\alpha) \to \mathbf{P}_\Gamma(z) \neq \mathbf{P}_\Gamma(y),\ x_\alpha \to x,\ v_\alpha \to x$. We may assume that $y \notin \Gamma$ and hence that $f_\alpha(x_\alpha) \to y;\ z = y$, a contradiction.

Sufficiency. Let $(x,\ y) \in X \times (Y - \Delta)$ and $(f_\alpha,\ x_\alpha,\ v_\alpha)$ be a net in $\Omega \times X_0 \times X_0$ satisfying $f_\alpha(x_\alpha) \to y,\ x_\alpha \to x,\ v_\alpha \to x$. There exist $W \in \Sigma(y)$ such that $W \cap (\Delta \cup \{\infty\}) = \emptyset$. Hence $\mathbf{P}_\Gamma \circ f_\alpha(x_\alpha) \to \mathbf{P}_\Gamma(y)$. Hence $\mathbf{P}_\Gamma \circ f_\alpha(v_\alpha) \to \mathbf{P}_\Gamma(y)$. and $f_\alpha(v_\alpha) \to y$. □

Lemma 2. *Let X_0 be a dense subset of the k – space X, let Y be a locally compact space, let Δ be a closed subset of Y; suppose for each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$and let $\Omega \subset C(X_0,\ Y)$. Then $\Omega$is topologically uniformly normal mod Δ with respect to $(X_0,\ X,\ Y)$. iff $\mathbf{P}_\Gamma \circ \Omega$is topologically uniformly normal with respect to $(X_0,\ X,\ Y_\Gamma^\infty)$, for each Γ closed in Y^∞ satisfying $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$.*

Proof. **Sufficiency.** This is obvious from Lemma 1.

Necessity. Let $x \in X$ and $(f_\alpha,\ x_\alpha,\ v_\alpha)$ be a net in $\Omega \times X_0 \times X_0$ satisfying $x_\alpha \to x,\ v_\alpha \to x,\ \mathbf{P}_\Gamma \circ f_\alpha(x_\alpha) \to \mathbf{P}_\Gamma(y),\ \mathbf{P}_\Gamma \circ f_\alpha(v_\alpha) \to \mathbf{P}_\Gamma(z)$ and $\mathbf{P}_\Gamma(y) \neq \mathbf{P}_\Gamma(z)$. It follows that $y \neq z$ and that $y,\ z \in \Delta \cup \{\infty\}$. If $\Gamma \neq \Delta$ then $\infty \in \Gamma$, so $y,\ z \in \Gamma$ and $\mathbf{P}_\Gamma(y) = \mathbf{P}_\Gamma(z)$. Hence $\Gamma = \Delta,\ \Gamma$ is compact and we may assume $y = \infty$. Let W be open in Y^∞ such that $\infty \in W$ and $W \cap \Gamma = \emptyset$. There is a subnet of $\{f_\alpha\}$, called again $\{f_\alpha\}$, a net $\{s_\alpha\}$, and a $q \in Y \cap \partial W$ such that $s_\alpha \to x$ and $f_\alpha(s_\alpha) \to q$. Thus $y = z$, a contradiction. □

Theorem E appears in [6] and when applied in concert with Lemmas 1 and 2 yields Theorems 1 and 2. In each instance below the space X is assumed to be a k – space and Y is assumed to be locally compact.

Theorem E. *Let X_0 be a dense subset of the space X. The following statements are equivalent for $\Omega \subset C(X_0, Y)$:*
(1) $\Omega$is topologically uniformly normal with respect to (X_0, X, Y).
(2) Ω satisfies the following two properties:
(a) Each $f \in \Omega$ extends to $\widetilde{f} \in C(X, Y^\infty)$, and
(b) $C[X, Y^\infty; \Omega]$ is a relatively compact subset of $C(X, Y^\infty)$.

Theorem 1. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y and let $\Gamma = \Delta \cup \{\infty\}$. The following statements are equivalent for $\Omega \subset C(X_0, Y)$:*
(1) $\Omega$is topologically uniformly normal mod Δ with respect to (X_0, X, Y).
(2) Ω satisfies the following two properties:
(a) For each $f \in \Omega$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in C(X, Y_\Gamma^\infty)$, and
(b) $C[X, Y_\Gamma^\infty; \mathbf{P}_\Gamma \circ \Omega]$ is a relatively compact subset of $C(X, Y_\Gamma^\infty)$.

Theorem 2. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y; suppose that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$ and let $\Omega \subset C(X_0, Y)$. The following are equivalent:*
(1) $\Omega$is topologically uniformly normal mod Δ with respect to (X_0, X, Y).
(2) Ω satisfies the following two properties for Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:
(a) For each $f \in \Omega$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in C(X, Y_\Gamma^\infty)$, and
(b) $C[X, Y_\Gamma^\infty; \mathbf{P}_\Gamma \circ \Omega]$ is a relatively compact subset of $C(X, Y_\Gamma^\infty)$.

Theorems 3, 4, 5 and 6 are derived from Theorems F and G which are proved in [6], and Lemmas 1 and 2.

Theorem F. *Let X_0 be a dense subset of the space X and suppose $\Omega \subset C(X_0, Y)$ is topologically uniformly normal with respect to (X_0, X, Y). Then*
(1) Each $f \in \overline{\Omega}$ extends to $\widetilde{f} \in C(X, Y^\infty)$.
(2) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{f}_\alpha \to \widetilde{f}$.

Theorem G. *Let X_0 be a dense subset of the space X. Then $\Omega \subset C(X_0, Y)$ is topologically uniformly normal with respect to (X_0, X, Y) iff the following two conditions hold:*
(1) Each $f \in \Omega$ extends to $\widetilde{f} \in C(X, Y^\infty)$, and
(2) $C[X, Y^\infty; \overline{\Omega}]$ is a compact subset of $C(X, Y^\infty)$.

Theorem 3. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y, let $\Gamma = \Delta \cup \{\infty\}$, and suppose $\Omega \subset C(X_0, Y)$is topologically uniformly normal mod Δ with respect to (X_0, X, Y). Then*
(1) For each $f \in \overline{\Omega}$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in C(X, Y_\Gamma^\infty)$, and
(2) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Proof. This follows from Lemma 1 and the fact that $f_\alpha \to f$ for a net $\{f_\alpha\}$ in Ω iff $\mathbf{P}_\Gamma \circ f_\alpha \to P_\Gamma \circ f$. □

Theorem 4. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y, let $\Gamma = \Delta \cup \{\infty\}$, and suppose $\Omega \subset C(X_0, Y)$ is topologically uniformly normal mod Δ with respect to (X_0, X, Y). The following are equivalent:*
(1) $\Omega$is topologically uniformly normal mod Δ with respect to (X_0, X, Y).
(2) Ω satisfies the following two properties:
(a) For each $f \in \Omega$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in C(X, Y_\Gamma^\infty)$, and
(b) $C[X, Y_\Gamma^\infty; \overline{\mathbf{P}_\Gamma \circ \Omega}]$ is a compact subset of $C(X, Y_\Gamma^\infty)$.

Theorem 5. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y; suppose that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$and let $\Omega \subset C(X_0,\ Y)$. Then*

(1) For each $f \in \overline{\Omega}$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in\ C(X, Y_\Gamma^\infty)$, and

(2) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$, for Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$.

Theorem 6. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y; suppose that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$and let $\Omega \subset C(X_0,\ Y)$. The following are equivalent:*

(1) $\Omega$is topologically uniformly normal mod Δ with respect to (X_0, X, Y).

(2) Ω satisfies the following two properties for Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:

(a) For each $f \in \Omega$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in\ C(X, Y_\Gamma^\infty)$, and

(b) $C[X, Y_\Gamma^\infty; \overline{\mathbf{P}_\Gamma \circ \Omega}]$ is a compact subset of $C(X, Y_\Gamma^\infty)$.

Theorems 7, 8, 9 and 10, our final results in this section, flow from Theorems H and I (proved in [6]) and Lemmas 1 and 2.

Theorem H. *Let X_0 be a dense subset of the space X and suppose that $\Omega \subset C(X_0, Y)$ is topologically uniformly normal with respect to (X_0, X, Y). Then*

(1) Ω is a relatively compact subset of $C(X_0, Y^\infty)$, and

(2) If $\{f_\alpha\}$ is a net in $\overline{\Omega}$ and $f_\alpha \to f$, then $\widetilde{f}_\alpha \to \widetilde{f}$.

Theorem I. *Let X_0 be a dense subset of the space X. Then $\Omega \subset C(X_0, Y)$ is topologically uniformly normal with respect to (X_0, X, Y) iff the following three conditions hold:*

(1) Ω is a relatively compact subset of $C(X_0, Y^\infty)$,

(2) Each $f \in \overline{\Omega}$ extends to $\widetilde{f} \in\ C(X, Y^\infty)$, and

(3) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{f}_\alpha \to \widetilde{f}$.

Theorem 7. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y, let $\Gamma = \Delta \cup \{\infty\}$, and suppose $\Omega \subset C(X_0,\ Y)$ is topologically uniformly normal mod Δ with respect to (X_0, X, Y). Then*

(1) $\mathbf{P}_\Gamma \circ \Omega$ is a relatively compact subset of $C(X_0, Y_\Gamma^\infty)$, and

(2) If $\{f_\alpha\}$ is a net in $\overline{\Omega}$ and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Theorem 8. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y; suppose that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$and let $\Omega \subset C(X_0,\ Y)$. Then, for each Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:*

(1) $\mathbf{P}_\Gamma \circ \Omega$ is a relatively compact subset of $C(X_0, Y_\Gamma^\infty)$, and

(2) If $\{f_\alpha\}$ is a net in $\overline{\Omega}$ and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Theorem 9. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y, let $\Gamma = \Delta \cup \{\infty\}$. Then $\Omega \subset C(X_0, Y)$ is topologically uniformly normal mod Δ with respect to (X_0, X, Y) iff the following three conditions hold:*

(1) $\mathbf{P}_\Gamma \circ \Omega$ is a relatively compact subset of $C(X_0, Y_\Gamma^\infty)$,

(2) For each $f \in \overline{\Omega}$, $\widetilde{\mathbf{P}_\Gamma \circ f}$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in\ C(X, Y_\Gamma^\infty)$, and

(3) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Theorem 10. *Let X_0 be a dense subset of the space X, let Δ be a closed subset of the space Y; suppose that at each $x \in X$ there is a base of neighborhoods $\Sigma(x)$ such that $V \cap X_0$ is connected for each $V \in \Sigma(x)$. Then $\Omega \subset C(X_0, Y)$ is topologically uniformly normal mod Δ with respect to (X_0, X, Y) iff the following three conditions hold for Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:*

(1) $\mathbf{P}_\Gamma \circ \Omega$ is a relatively compact subset of $C(X_0, Y_\Gamma^\infty)$,

(2) For each $f \in \overline{\Omega}$, $\mathbf{P}_\Gamma \circ f$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in C(X, Y_\Gamma^\infty)$, and

(3) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

§3. Applying the topological function space theorems to complex spaces. In this section we utilize the results in §2 to extend and generalize some theorems for complex spaces due to Abate [1], Kiernan [10], [11], Kiernan and Kobayashi [12], and Noguchi [15], [16]. Let X be a complex subspace (not necessarily relatively compact) of a complex space Y and let Δ be a closed complex subspace of Y. We say that X is *hyperbolically imbedded in Y mod Δ* if for every pair of distinct points $p, q \in \overline{X}$ not both elements of Δ, there exist neighborhoods U of p and V of q in Y such that $k_X(U \cap X, V \cap X) > 0$. To establish the results of this section we employ the notion of hyperbolic point for a complex subspace. Let X be a relatively compact complex subspace of the complex space Y. Kobayashi has called a point $p \in \overline{X}$ a *hyperbolic point for X* if for each $U \in \Sigma(p)$ some $V \in \Sigma(p)$ satisfies $\overline{V} \subset U$ and $k_X(\overline{V} \cap X,\ X - U) > 0$. We extend the notion of hyperbolic point in this paper by dropping the requirement of relative compactness on X. Lemma 3 characterizes hyperbolic point in terms of K_X.

Lemma 3. *Let X be a complex subspace of the complex space Y. Then $p \in \overline{X}$ is a hyperbolic point for X iff there is a $U \in \Sigma(p)$ and a $c > 0$ such that $K_X \geq cE$ on $U \cap X$.*

Proof. **Necessity.** If no such c exists there is a sequence $\{(f_n,\ v_n)\}$ such that $(f_n,\ v_n) \in H(\mathbf{D},\ X) \times T_{f_n(0)}(X)$, $K_X(f_n(0),\ v_n) = 1$ for each n, $f_n(0) \to p$, $|(df_n)_0| \to \infty$. From the hypothesis, there is a hyperbolic neighborhood U of p with $\overline{U}$ compact and a neighborhood N of 0 in $\mathbf{D}$ such that $f_n(N) \subset U$ ultimately. There is a subsequence of $\{f_n\}$, called again $\{f_n\}$, such that $f_n \to g \in H(N,\ Y)$, an impossibility.

Sufficiency. Let $U,\ V \in \Sigma(p)$ and $c > 0$ satisfy $\overline{V} \subset U$ and $K_X \geq cE$ on $U \cap X$. Then $k_X(\overline{V} \cap X,\ X - U) \geq cd_E(\partial V,\ \partial U) > 0$. □

Theorem 11. *The following statements are equivalent for a complex subspace X of a complex space Y, closed complex subspace Δ of Y and closed subset Γ of Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:*

(1) X is hyperbolically imbedded in Y mod Δ.

(2) $C[\mathbf{D},\ Y_\Gamma^\infty; \mathbf{P}_\Gamma \circ H(\mathbf{D}^*,\ X)]$ is a relatively compact subset of $C(\mathbf{D},\ Y_\Gamma^\infty)$.

(3) $C[\mathbf{D},\ Y_\Gamma^\infty; \overline{\mathbf{P}_\Gamma \circ H(\mathbf{D}^*,\ X)}]$ is a compact subset of $C(\mathbf{D},\ Y_\Gamma^\infty)$.

(4) $C[M,\ Y_\Gamma^\infty; \mathbf{P}_\Gamma \circ H(M - A,\ X)]$ is a relatively compact subset of $C(M,\ Y_\Gamma^\infty)$ for each complex manifold M and divisor A on M with normal crossings.

(5) $C[M,\ Y_\Gamma^\infty; \overline{\mathbf{P}_\Gamma \circ H(M - A,\ X)}]$ is a compact subset of $C(M,\ Y_\Gamma^\infty)$ for each complex manifold M and divisor A on M with normal crossings.

(6) $\mathbf{P}_\Gamma \circ H(M,\ X)$ is a relatively compact subset of $C(M,\ Y_\Gamma^\infty)$ for each complex manifold M.

(7) $\mathbf{P}_\Gamma \circ H(\mathbf{D},\ X)$ is a relatively compact subset of $C(\mathbf{D},\ Y_\Gamma^\infty)$.

Proof. The implications **(5)⇒(6)⇒(7)**, **(3)⇒(2)⇒(7)**, **(5)⇒(4)⇒(6)**, **(5)⇒(3)** are all obvious. To assist in establishing the remaining necessary implications we prove Lemma 4 which will be utilized in conjunction with results from §2.

Lemma 4. *Let X be a complex subspace of the complex space Y and let Δ be a closed complex subpace of Y. The following statements are equivalent:*

(i) X is hyperbolically imbedded in Y mod Δ.

(ii) Each point in $\overline{X} - \Delta$ is a hyperbolic point.

(iii) $H(\mathbf{D}^*,\ X)$ is topologically uniformly normal mod Δ with respect to $(\mathbf{D}^*,\ \mathbf{D},\ Y)$.

(iv) $H(M-A,\ X)$ is topologically uniformly normal mod Δ with respect to $(M-A,\ M,\ Y)$ for each complex manifold M and divisor A on M with normal crossings.

Proof. The equivalences **(ii)** $\Longleftrightarrow$ **(iii)** $\Longleftrightarrow$ **(iv)** are verified in [7](see theorems 1 and 7 in [7]).

(i)$\Rightarrow$**(ii)**. If $p \in \overline{X}$ is not a hyperbolic point then, by Lemma 3 and the homogeniety of $\mathbf{D}$, there are sequences $\{z_n\},\ \{f_n\}$ in $\mathbf{D},\ H(\mathbf{D},\ X)$ respectively and $U \in \Sigma(p)$ satisfying $z_n \to 0,\ f_n(0) \to p$, and $f_n(z_n) \to q \in \partial U$. It follows for $V \in \Sigma(p),\ W \in \Sigma(q)$ that

$$k_X(V \cap X,\ W \cap X) \leq k_X(f_n(0),\ f_n(z_n)) \leq d_{\mathbf{D}}(0,\ z_n) \text{ ultimately so } p \in \Delta.$$

(ii)$\Rightarrow$**(i)**. The argument used in establishing $\Leftarrow$ in Lemma 3 may be applied here. $\square$

(1)$\Rightarrow$**(5)**. Follows from Lemma 3 and Theorem 6.

(7)$\Rightarrow$**(1)**. Suppose $\mathbf{P}_\Gamma \circ H(\mathbf{D}, X)$ is relatively compact in $C(\mathbf{D},\ Y_\Gamma^\infty)$ and let $p \in \overline{X} - \Delta$ be a point in Y which is not a hyperbolic point for X. We may assume that there is a sequence $\{f_n\}$ in $H(\mathbf{D}, X)$ and $g \in C(\mathbf{D},\ Y_\Gamma^\infty)$ such that $\mathbf{P}_\Gamma \circ f_n \to g$, $|(df_n)_0| \to \infty$ and $f_n(0) \to p$. Let W be a hyperbolic neighborhood of p such that $\overline{W}$ is compact and $\overline{W} \cap \Gamma = \emptyset$. Then $\mathbf{P}_\Gamma(W)$ is a neighborhood of $g(0)$. There is a neighborhood V of 0 in $\mathbf{D}$ such that $\mathbf{P}_\Gamma \circ f_n(V) \subset \mathbf{P}_\Gamma(W)$ and hence $f_n(V) \subset W$ ultimately. This is a contradiction since $|(df_n)_0| \to \infty$. In view of Lemma 4 the proof of Theorem 11 is complete. $\square$

Results in §2, in [7] and Lemma 4 enable us to extend Theorem D (due to Noguchi [15], [16]) to hyperbolically imbedded mod Δ spaces in the form of Theorems 12, 13 and 14.

Theorem 12. *Let X be a complex subspace of the complex space Y and let Δ be a closed complex subpace of Y. X is hyperbolically imbedded in Y mod Δ iff the following three conditions hold for Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:*
(1) $\mathbf{P}_\Gamma \circ H(\mathbf{D}^*,\ X)$ is a relatively compact subset of $C(\mathbf{D}^*,\ Y_\Gamma^\infty)$,
(2) For each $f \in \overline{H(\mathbf{D}^*,\ X)}$, $\mathbf{P}_\Gamma \circ f$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in\ C(\mathbf{D},\ Y_\Gamma^\infty)$, and
(3) If $\{f_\alpha\}$ is a net in $H(\mathbf{D}^*,\ X)$ and $f_\alpha \to f \in C(\mathbf{D}^*,\ Y^\infty$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Proof. From Theorem 10 and Lemma 4. $\square$

Theorem 13. *Let X be a complex subspace of the complex space Y and let Δ be a closed complex subpace of Y. X is hyperbolically imbedded in Y mod Δ iff the following three conditions hold for any complex manifold M, divisor A on M with normal crossings, and Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:*
(1) $\mathbf{P}_\Gamma \circ H(M - A,\ X)$ is a relatively compact subset of $C(M,\ Y_\Gamma^\infty)$,
(2) For each $f \in \overline{H(M - A,\ X)}$, $\mathbf{P}_\Gamma \circ f$ extends to $\widetilde{\mathbf{P}_\Gamma \circ f} \in\ C(M,\ Y_\Gamma^\infty)$, and
(3) If $\{f_\alpha\}$ is a net in $H(M - A,\ X)$ and $f_\alpha \to f \in C(M - A,\ Y^\infty$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Proof. From Theorem 10 and Lemma 4. $\square$

Theorem 14. *Let X be a complex subspace of the complex space Y and let Δ be a closed complex subpace of Y such that X is hyperbolically imbedded in Y mod Δ. Then*

the following conditions hold for any complex manifold M, divisor A on M with normal crossings, and Γ closed in Y^∞ where $\Gamma = \Delta$ or $\Gamma = \Delta \cup \{\infty\}$:
(1) If $\{f_\alpha\}$ is a net in $\overline{H(\mathbf{D}^*,\ X)}$ and $f_\alpha \to f \in C(\mathbf{D}^*,\ Y^\infty)$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.
(2) If $\{f_\alpha\}$ is a net in $\overline{H(M - A,\ X)}$ and $f_\alpha \to f \in C(M - A,\ Y^\infty)$, then $\widetilde{\mathbf{P}_\Gamma \circ f_\alpha} \to \widetilde{\mathbf{P}_\Gamma \circ f}$.

Proof. From Theorem 8 and Lemma 4. □

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