ON TOPOLOGICAL BCI-ALGEBRAS II

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Abstract. As a continuation of [7], we introduce the notions of topological subalgebras, topological ideals and topological homomorphisms in topological BCI-algebras and study some related properties. In the section 3, we investigate the compactness in a TBCI-algebra X and quotient TBCI-algebra X/I where I is a topological ideal of X. In the section 4, we introduce the notion of topological homomorphisms, study some properties for this notion and show that an open topological homomorphism f from TBCI-algebra X to TBCI-algebra Y gives rise to a one-to-one correspondence between the closed c-ideals of Y and the closed c-ideals of X which contains ker f.

1. Introduction

The notion of BCK-algebras was proposed by Y. Iami and K. Iseki in 1966. In the same year, K. Iseki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. R. A. Alo and E. Y. Deeba [1] attempted to study the topological aspects of the BCK-structures, and they initiated the study of various topologies on BCK-algebras analogous to that which has already been studied on lattices, but no attempts have been made to study the topological structures making the BCK-operation continuous. In [7], Y. B. Jun et al. initiated the study of topological BCI-algebras (briefly, TBCI-algebras) and some properties of this structure, and gave a characterization of a TBCI-algebra in terms of neighborhoods, and showed that a TBCI-algebra X is Hausdorff if and only if \{0\} is closed in X. They also gave a filter base \(\Omega\) generating a BCI-topology, and made a BCI-algebra X into a TBCI-algebra for which \(\Omega\) is a fundamental system of neighborhoods of 0. As a continuation of [7], we introduce the notions of topological subalgebras, topological ideals and topological homomorphisms in topological BCI-algebras and study some related properties. In the section 3, we get the following results: (i) If I is a topological ideal of a TBCI-algebra X, then the natural projection \(\Phi_I\) from X to \(X/I\) is open and continuous. (ii) Every topological subalgebra of a compact (locally compact) TBCI-algebra is compact (locally compact). (iii) Let I be a topological ideal of a compact TBCI-algebra X, then I and \(X/I\) are compact. (iv) Let I be a topological c-ideal of a locally compact TBCI-algebra X, then \(X/I\) is locally compact. (v) Let I be a compact topological ideal of a transitive open TBCI-algebra X, \(\beta = \{I_x \mid x \in X\}\) be a base for topological \(\mathcal{F}\) and \(I_x\) compact for each \(x \in X\). Then we have that if \(Q \subseteq X/I\) is compact, so is \(P = \Phi_I(Q)\); and if \(X/I\) is locally compact, then X is locally compact. In
the section 4, we introduce the notion of topological homomorphisms, study some properties
for this notion and show that an open topological homomorphism \( f \) from TBCI-algebra \( X \)
to TBCI-algebra \( Y \) gives rise to a one-to-one correspondence between the closed c-ideals of
\( Y \) and those closed c-ideals of \( X \) which contains \( \text{ker} f \).

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.
Recall that a \( BCI \)-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following axioms for every \( x, y, z \in X \),

\( (I) \) \( ((x * y) * (x * z)) * (z * y) = 0 \),

\( (II) \) \( (x * (x * y)) * y = 0 \),

\( (III) \) \( x * x = 0 \),

\( (IV) \) \( x * y = 0 \) and \( y * x = 0 \) imply \( x = y \).

A partial ordering \( \leq \) on \( X \) can be defined by \( x \leq y \) if and only if \( x * y = 0 \).
In a \( BCI \)-algebra \( X \), the following hold:

(1) \( x * 0 = x \).

(2) \( (x * y) * z = (x * z) * y \).

(3) \( 0 * (x * y) = (0 * x) * (0 * y) \).

(4) \( x * y = 0 \) implies \( (x * z) * (y * z) = 0 \) and \( (z * y) * (z * x) = 0 \).

(5) \( x * 0 = 0 \) implies \( x = 0 \).

A nonempty subset \( S \) of a \( BCI \)-algebra \( X \) is called a \( BCI \)-subalgebra of \( X \) if \( x * y \in S \)
whenever \( x, y \in S \).

A non-empty subset \( I \) of a \( BCI \)-algebra \( X \) is called a \( BCI \)-ideal of \( X \) if

(i) \( 0 \in I \),

(ii) \( x * y \in I \) and \( y \in I \) imply \( x \in I \).

A \( BCI \)-ideal \( I \) of a \( BCI \)-algebra \( X \) is said to be \text{closed} if \( 0 * x \in I \) whenever \( x \in I \). Here
we call this a \( c \)-\text{ideal} of \( X \).

3. Compactness in topological \( BCI \)-algebras

In this paper we shorten the statement “\( U \) is an open set containing \( x \)” to the phrase
“\( U \) is a neighborhood of \( x \)”.

\textbf{Definition 3.1.} (Jun et al. [7]) A topology \( \mathcal{T} \) on a \( BCI \)-algebra \( X \) is said to be a \( BCI \)-
topology, and \( X \), furnished with \( \mathcal{T} \), is called a \text{topological \( BCI \)-algebra} (briefly, \text{\( TBCI \)-algebra})
if \( \langle x, y \rangle \mapsto x * y \) is continuous from \( X \times X \), furnished with the cartesian product topology
defined by \( \mathcal{T} \), to \( X \). Moreover if \( X \) is a \( p \)-semisimple \( BCI \)-algebra, we say that \( X \), furnished
with \( \mathcal{T} \), is a \( p \)-\text{semisimple \( TBCI \)-algebra}.

\textbf{Example 3.2.} (Jun et al. [7]) (1) A \( BCI \)-algebra with discrete or indiscrete topology is a
\( TBCI \)-algebra.

(2) Consider a \( BCI \)-algebra \( X = \{0, a, b, c, d\} \) with Cayley table (Table 1) and Hasse
diagram (Figure 1):
Table 1

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
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<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1

Note that $\mathcal{T} := \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{0, a\}, \{b, c\}, \{b, d\}, \{c, d\}, \{0, a, b\}, \{0, a, c\}, \{0, a, d\}, \{b, c, d\}, \{0, a, b, c\}, \{0, a, b, d\}, \{0, a, c, d\}\}$ is a BCI-topology. By routine calculations we know that $X$ furnished with $\mathcal{T}$ is a TBCI-algebra.

We first give a characterization of a TBCI-algebra in terms of neighborhoods.

**Lemma 3.3.** (Jun et al. [7]) Let $\mathcal{T}$ be a BCI-topology for a BCI-algebra $X$. Then $X$, furnished with $\mathcal{T}$, is a TBCI-algebra if and only if for each $x$ and $y$ in $X$ and each neighborhood $W$ of $x \ast y$ there are neighborhoods $U$ of $x$ and $V$ of $y$ such that $U \ast V \subseteq W$.

**Lemma 3.4.** (Jun et al. [7]) Let $X$ be a TBCI-algebra. If $A$ is both an open subset in topological space and an ideal of BCI-algebra $X$, then it is a closed subset in topological space $X$.

**Lemma 3.5.** (Jun et al. [7]) Let $X$ be a TBCI-algebra. Then $\{0\}$ is closed subset if and only if $X$ is Hausdorff in topological space $X$.

**Definition 3.6.** A nonempty subset $S$ of a TBCI-algebra $X$ is called a topological subalgebra of $X$ if

(i) $S$ is a BCI-subalgebra of $X$ considered as an algebraic BCI-algebra,

(ii) $S$ is a closed set in the topological space $X$.

**Definition 3.7.** A nonempty subset $I$ of a TBCI-algebra $X$ is called a topological ideal of $X$ if

(i) $I$ is a BCI-ideal of $X$ considered as an algebraic BCI-algebra,

(ii) $I$ is open in the topological space $X$.

**Definition 3.8.** A nonempty subset $I$ of a TBCI-algebra $X$ is called a topological $c$-ideal of $X$ if

(i) $I$ is a $c$-ideal of $X$ considered as an algebraic BCI-algebra,

(ii) $I$ is open in the topological space $X$.

**Example 3.9.** Let $X = \{0, a, b, c, d\}$ be a TBCI-algebra as in Example 3.2(2). By routine calculations we know that $A = \{0, c\}$ is a topological subalgebra which is not a topological ideal, and $B = \{0, a\}$ is a topological ideal.

**Proposition 3.10.** Every topological ideal of a TBCI-algebra $X$ is a closed subset of $X$ considered as a topological space.

**Proof.** Follows from Lemma 3.4. □

**Proposition 3.11.** Let $X$ be a TBCI-algebra and $I$ a BCI-ideal of $X$ considered as an algebraic BCI-algebra. Define a map $\Phi_I : X \rightarrow X/I$ by $\Phi_I(x) = I_x$ for each $x \in X$ where $I_x$ is the equivalence class containing $x$. Then the set

$$\mathcal{T}_{X/I} := \{O \subseteq X/I | \Phi_I^{-1}(O) \text{ is open in } X\}$$
is a topology on $X/I$, which is called the quotient topology on $X/I$.

Proof. Straightforward. □

**Proposition 3.12.** If $I$ is a topological ideal of a TBCI-algebra $X$, then the mapping $\Phi_I$ as in Proposition 3.11 is open and continuous.

Proof. Clearly $\Phi_I$ is continuous. In order to prove $\Phi_I$ is open, let $W$ be an open set in $X$ and $W' = \Phi_I(W)$. We shall show that $W'$ is open in $X/I$ or equivalently $\Phi_I^{-1}(W')$ is open in $X$. Let $y \in \Phi_I^{-1}(W')$. Then $\Phi_I(y) \in W'$ and $\Phi_I(y) = \Phi_I(w)$ for some $w \in W$. It follows that $y \ast w \in I$ and $w \ast y \in I$. Note that $f : X \times X \to X$, $(x, y) \mapsto x \ast y$, is continuous. Then $B := \{(a, b) \in X \times X | a \ast b \in I\} = \{(a, b) \in X \times X | f(a, b) \in I\} = f^{-1}(I)$ is open in $X \times X$ since $I$ is open. Hence $B = \bigcup_{i \in \Lambda} (U_i \times V_i)$ for some open sets $U_i$ and $V_i$ in $X$ where $\Lambda$ is an index set and $i \in \Lambda$. It follows from $y \ast w \in I$ that there exists $i_0 \in \Lambda$ such that $y \in U_{i_0}$, $w \in V_{i_0}$ and $U_{i_0} \ast V_{i_0} \subseteq I$. Similarly $w \ast y \in I$ implies that there exists $j_0 \in \Lambda$ such that $w \in U_{j_0}$, $y \in V_{j_0}$ and $U_{j_0} \ast V_{j_0} \subseteq I$. Thus $y \in U_{i_0} \cap V_{j_0}$ and $w \in U_{j_0} \cap V_{i_0}$. Note that $(U_{i_0} \cap V_{j_0}) \ast (U_{j_0} \cap V_{i_0}) \subseteq U_{i_0} \ast V_{i_0} \subseteq I$ and $(U_{j_0} \cap V_{i_0}) \ast (U_{i_0} \cap V_{j_0}) \subseteq U_{j_0} \ast V_{j_0} \subseteq I$.

Let $x \in G$ where $G = U_{i_0} \cap V_{j_0}$. It follows that $x \ast w \in I$ and $w \ast x \in I$. Hence $I_x = I_w = \Phi_I(w) \in \Phi_I(W) = W'$, and so $G \subseteq \Phi_I^{-1}(W')$. Note that $y \in G \subseteq \Phi_I^{-1}(W')$ we have $\Phi_I^{-1}(W')$ is open in $X$. □

**Theorem 3.13.** If $I$ is a topological ideal of a TBCI-algebra $X$ with topology $\mathcal{T}_X$, then $X/I$ is a TBCI-algebra with topology $\mathcal{T}_{X/I}$.

Proof. It is sufficient to prove that $(I_x, I_y) \to I_x \ast I_y$ is continuous. Let $x, y \in X$ and let $W$ be a neighborhood of $I_x \ast I_y$. Then $\Phi_I^{-1}(W)$ is open in $X$ and $x \ast y \in \Phi_I^{-1}(W)$. Since $X$ is a TBCI-algebra, there exists neighborhoods $U'$ of $x$ and $V'$ of $y$ such that $U' \ast V' \subseteq \Phi_I^{-1}(W)$ by Lemma 3.3. Put $U := \Phi_I(U')$ and $V := \Phi_I(V')$. Then $U$ and $V$ are open in $X/I$ since $\Phi_I$ is open. Note that $I_x \in U$, $I_y \in V$ and

$$U \ast V = \{\Phi_I(u) \ast \Phi_I(v) | u \in U' \text{ and } v \in V'\}$$

$$= \{\Phi_I(u \ast v) | u \in U' \text{ and } v \in V'\}$$

$$= \Phi_I(U' \ast V') \subseteq W$$

so from Lemma 3.3 that $(I_x, I_y) \to I_x \ast I_y$ is continuous. □

**Theorem 3.14.** Let $X$ be a TBCI-algebra. Then

(i) If $X$ is compact (resp. locally compact), then every topological subalgebra $S$ of $X$ is compact (resp. locally compact).

(ii) If $X$ is compact and $I$ is a topological ideal, then $I$ and $X/I$ are compact.

(iii) If $X$ is locally compact, then every topological ideal $I$ is locally compact.

(iv) If $X$ is locally compact and $I$ is a topological $c$-ideal, then $X/I$ is locally compact.

Proof. (i) Note that $S$ is closed so that $S$ is compact if $X$ is compact. If $X$ is locally compact, then $\forall x \in S \exists$ a neighborhood $G$ of $x$ whose closure is compact in $X$. Then $U := G \cap S$ is a neighborhood of $x$ in $S$ and $\overline{U} \subseteq \overline{G} \cap S = \overline{G} \cap S$. Since $\overline{S} \cap S$ is compact in $S$, it follows that $\overline{U}$ is compact in $S$. Hence $S$ is locally compact.

(ii) Let $X$ be compact and $I$ a topological ideal. Then $I$ is closed by Proposition 3.10. Hence $I$ is compact. Note that $\Phi_I$ is a continuous mapping from $X$ onto $X/I$. Therefore $X/I$ is compact.

(iii) It is similar to the proof of (i).

(iv) Assume that $X$ is locally compact and $I$ is a topological $c$-ideal. Let $a \in X$, $A = \Phi_I(a)$ and let $U$ be a neighborhood of $a$ in $X$ such that $\overline{U}$ is compact. Since $I$ is
a c-ideal of X, we have \( \Phi_I(I) = \{I_0\} \) which is an open set in \( X/I \) because \( I \) is open in \( X \) and \( \Phi_I \) is open. Note that \( \{I_0\} \) is a BCI-ideal of \( X/I \) so that \( \{I_0\} \) is a closed set in \( X/I \) by Lemma 3.4. It follows from Lemma 3.5 that \( X/I \) is Hausdorff. Since \( \overline{\Phi_I(U)} \) is compact, the continuous image \( \Phi_I(U) \) is compact, and hence \( \overline{\Phi_I(U)} \) is closed in \( X/I \). From \( \Phi_I(U) \subseteq \Phi_I(U) \) it follows that \( \Phi_I(U) \subseteq \Phi_I(U) \) so that \( \Phi_I(U) \) is compact. On the other hand, put \( U^* = \{I_x \mid I_x \cap U \neq \emptyset\} \). Since \( a \in I_x \cap U = A \cap U \), we get \( A \in U^* \). Now we claim that \( U^* = \Phi_I(U) \) and then \( U^* \) is a neighborhood of \( A \). Let \( I_x \in U^* \). Then \( I_x \cap U \neq \emptyset \) and so \( \exists y \in I_x \cap U \). Hence \( I_x = I_y \in \Phi_I(U) \) and therefore \( U^* \subseteq \Phi_I(U) \). Conversely let \( I_x = \Phi_I(x) \in \Phi_I(U) \). Then \( x \in U \) and thus \( x \in I_x \cap U \), i.e., \( I_x \cap U \neq \emptyset \) which means that \( I_x \in U^* \). Consequently \( U^* = \Phi_I(U) \) and \( \overline{U^*} = \Phi_I(U) \) is a compact set in \( X/I \). Therefore \( X/I \) is locally compact. This completes the proof. \( \square \)

For a fixed element \( a \) of a TBCI-algebra \( X \), define a self-map \( f_a : X \to X \) by \( f_a(x) = a \ast x \) for all \( x \in X \).

**Definition 3.15.** A TBCI-algebra \( X \) is said to be transitive open if for each \( a \in X \) the self-map \( f_a \) is open and continuous.

**Lemma 3.16.** Let \( U \) be an open set in a transitive open TBCI-algebra \( X \) and let \( a \in X \). Then

(i) \( a \ast U \) is open.

(ii) \( f_a^{-1}(U) = \{x \in X \mid a \ast x \in U\} \) is open.

(iii) \( A \ast U \) is open for every subset \( A \) of \( X \).

**Proof.** Note that \( f_a(U) = a \ast U, f_a^{-1}(U) = V \) and \( A \ast U = \cup_{a \in A}(a \ast U) \). Hence we get the desired results. \( \square \)

**Theorem 3.17.** Every \( p \)-semisimple TBCI-algebra \( X \) is transitive open.

**Proof.** Let \( a \in X \). We shall prove that \( f_a \) is open and continuous. Let \( x \in X \) and let \( W \) be a neighborhood of \( f_a(x) = a \ast x \). Then there exist neighborhoods \( U \) and \( V \) of \( a \) and \( x \), respectively, such that \( U \ast V \subseteq W \) by Lemma 3.3. It follows that \( f_a(V) = a \ast V \subseteq U \ast V \subseteq W \), which means that \( f_a \) is continuous. Now let \( G \) be an open set in \( X \). We claim that \( f_a(G) \) is open. Let \( x \in f_a(G) \). Then \( x = f_a(u) = a \ast u \) for some \( u \in G \), and so \( u = a \ast (a \ast u) = a \ast x \) since \( X \) is \( p \)-semisimple. It follows that \( a \ast x \in G \) so that there exist neighborhoods \( P \) and \( Q \) of \( a \) and \( x \), respectively, such that \( P \ast Q \subseteq G \), which implies \( f_a(Q) = a \ast Q \subseteq P \ast Q \subseteq G \). Thus \( f_a(q) \in G \) for all \( q \in Q \), and so \( q = a \ast (a \ast q) = a \ast f_a(q) \in f_a(G) \), i.e., \( Q \subseteq f_a(G) \). Therefore \( f_a \) is an open map, ending the proof. \( \square \)

**Proposition 3.18.** Let \( \Delta_0 \) be a collection of closed subsets of a TBCI-algebra \( X \) having the finite intersection property. Then there is a maximal collection \( \Delta' \) of closed subsets of \( X \) having the finite intersection property such that \( \Delta_0 \subseteq \Delta' \).

**Proof.** Let \( \mathcal{F} = \{\Delta \mid \Delta \text{ is a collection of closed subsets of } X \text{ having the finite intersection property and } \Delta_0 \subseteq \Delta \} \). We shall prove that \( \mathcal{F} \) has a maximal element with respect to the partial order “\( \subseteq \)” on \( \mathcal{F} \). Let \( \Delta_1 \subseteq \Delta_2 \subseteq \cdots \subseteq \Delta_m \subseteq \cdots \) be a chain in \( \mathcal{F} \) and let \( \Delta = \bigcup_{i=1}^{\infty} \Delta_i \). Then \( \Delta_0 \subseteq \Delta \). We also can show that \( \Delta \) has the finite intersection property. In fact, let \( U_1, U_2, \ldots, U_n \in \Delta \), then there is an integer \( m \) such that \( U_1, U_2, \ldots, U_n \in \Delta_m \) and so \( \cap_{i=1}^{n} U_i \neq \emptyset \) since \( \Delta_m \in \mathcal{F} \). It follows that \( \Delta \in \mathcal{F} \) and hence it is an upper bound of the chain \( \{\Delta_i : i = 1, 2, \ldots, n, \ldots\} \). By Zorn’s Lemma, \( \mathcal{F} \) has a maximal element \( \Delta' \), it is the desired one.
Theorem 3.19. Let $X$ be a transfinite open TBCI-algebra and $I$ a compact topological ideal of $X$. Let $\beta := \{ I_x \mid x \in X \}$ be a base for topology $T$, where $I_x$ is compact for each $x \in X$. Then

(i) If $Q \subseteq X/I$ is compact, then so is $P = \Phi^{-1}_T(Q)$.

(ii) If $X/I$ is compact (resp. locally compact), then $X$ is compact (resp. locally compact).

Proof. (i) By Proposition 3.18, we only need to prove the theorem for a maximal collection $\Delta$ of the closed subsets in the subspace $P$ having the finite intersection property. We shall show that it also has non-empty intersection.

In the space $Q$, consider the collection $\Delta^*$ of all sets of the form $\Phi_I(F)$ such that $F \in \Delta$. Since $\Delta$ has the finite intersection property, so is the system $\Delta^*$. Let $\Delta^*$ denote the collection $\{ \Phi_I(F) \cap Q \mid F \in \Delta \}$. Then $\Delta^*$ is a collection of closed subsets in the space $Q$ having the finite intersection property. By hypothesis, $Q$ is compact, and so there exist a common point $A \in \Phi_I(F)$ for each $F \in \Delta$. Now let $U$ be an arbitrary neighborhood of zero in $X$ and $U^*$ a set of all equivalence classes contained in $A * U$, i.e., $U^* = \{ I_x \in X/I \mid I_x \subseteq A * U \}$. By Lemma 3.16, $A * U$ is open in $X$ and hence $\Phi_I(A * U)$ is open in $X/I$ since $\Phi_I$ is an open mapping. Clearly $U^* \subseteq \Phi_I(A * U)$. On the other hand, by hypothesis, $\{ I_x \mid x \in X \}$ is a base for topology $T$ and so $A * U = \bigcup_{i \in \Lambda} I_{i_0}$, for some index $i_0 \in \Lambda$. Thus for any $I_i = \Phi_I(a) \in \Phi_I(A * U)$, we have $a \in A * U = \bigcup_{i \in \Lambda} I_i$, and hence $a \in I_{i_0}$ for some $i_0 \in \Lambda$.

Therefore $I_{i_0} \subseteq A * U$ and so $I_{i_0} \subseteq U^*$, which shows that $\Phi_I(A * U) = U^*$. Hence $U^*$ is open in $X/I$. Since $A \subseteq A * U$, we get $A \in U^*$ and consequently $U^*$ meets every set of the system $\Delta^*$. We claim that $A * U$ meets every set of $\Delta$. Let $F \in \Delta$. Then $\Phi_I(F) \in \Delta^*$ and so $U^* \cap \Phi_I(F) \neq \emptyset$. Thus we can find $a \in F$ such that $I_{i_0} \in \Phi_I(F)$ and $I_{i_0} \in U^*$. Note that $I_{i_0} \subseteq U^*$ implies $I_{i_0} \subseteq A * U$ and hence $a \in A * U$. Therefore $(A * U) \cap F \neq \emptyset$. It follows that $A * U$ meets every set of $\Delta$. Put $FU^{-1}_1 := \{ x \in X \mid x * u \in F \text{ for some } u \in \bar{F} \}$. Since $(A * U) \cap F \neq \emptyset$, we have $FU^{-1}_1 \cap A \neq \emptyset$ for each $F \in \Delta$. Indeed let $a \in A * U \cap F$, then there is $x \in A$ and $a = x * u \in A * U$ and $a = x * u \in F$. Hence $x \in FU^{-1}_1 \cap A$. Now let $\Delta' = \{ FU^{-1}_1 \cap A \mid F \in \Delta, U \text{ is neighborhood of the zero in } X \}$. From the above fact we have that the system $\Delta'$ also has the finite intersection property. In fact, if $F_1U^{-1}_1 \cap A$ and $F_2U^{-1}_1 \cap A$ are in $\Delta'$, then taking $F = F_1 \cap F_2$ and $U = U_1 \cap U_2$ we have $FU^{-1}_1 \cap A \in \Delta'$ since $F \in \Delta$ by the maximal property of $\Delta$. Note that

$$\emptyset \neq FU^{-1}_1 \cap A \subseteq (F_1U^{-1}_1 \cap A) \cap (F_2U^{-1}_1 \cap A).$$

Therefore $\Delta'$ has the finite intersection property. Since $I_x$ is compact for each $x \in X$ by hypothesis, $A \subseteq I_x$ for some $x_0 \in X$ is compact in $X$ and so there exists $a \in \overline{FU^{-1}_1 \cap A}$ for each $FU^{-1}_1 \cap A \in \Delta'$. For any neighborhood $V$ of zero in $X$, we have that $a * V$ and $(a * V) \cap U$ are open by Lemma 3.16. Hence $FU^{-1}_1 \cap A$ and $a * V$ have nonempty intersection and consequently $FU^{-1}_1 \cap A$ and $a * V$ have nonempty intersection. It follows that $F$ meets the open set $(a * V) \cap U$. By Lemma 3.3, we have that for any neighborhood $W_a$ of $a$, there exist neighborhoods $V_a$ and $U_0$ of $a$ and $0$, respectively, such that $V_a * U_0 \subseteq W_a$. Put $\overline{V_0} = \{ x \in X \mid a * x \in V_a \}$, then $\overline{V_0}$ is an open set by Lemma 3.16 and thus $\overline{V_0}$ is a neighborhood of zero. Since $a * V_0 \subseteq \overline{V_0}$, we have $\overline{V_0} \subseteq V_a * U_0 \subseteq W_a$. By the above argument we have $((a * V_0) \cap F) \neq \emptyset$, and hence $W_a \cap F \neq \emptyset$. It follows that $a \in \overline{F}$ and so $a$ is common to all the sets of the system $\Delta$, which shows that $P$ is compact.

(ii) It follows from (i) that if $X/I$ is compact, so is $X$. Now let $X/I$ be locally compact and $a \in X$. Then $I_a \subseteq X/I$ and thus there is a neighborhood $U$ of $I_a$ such that it’s closure $\overline{U}$ is compact. Put $P = \Phi_T^{-1}(U)$. Then $P$ is a neighborhood of $x$. Noticing that $P \subseteq \Phi_T^{-1}(\overline{U})$,
we have $\mathcal{P} \subseteq \Phi_l^{-1}(\mathcal{U})$ since $\Phi_l^{-1}(\mathcal{U})$ is closed. Since $\Phi_l^{-1}(\mathcal{U})$ is compact by (i) and since $\mathcal{P}$ is a closed subset of $\Phi_l^{-1}(\mathcal{U})$, therefore $\mathcal{P}$ is compact. Hence $X$ is locally compact. \qed

**Corollary 3.20.** Let $X$ be a $p$-semisimple TBCI-algebra, $I$ a compact topological $c$-ideal and the system $\beta := \{ L_x \mid x \in X \}$ a base for the topology $\mathcal{T}$. If $Q \subseteq X/I$ is compact, then $P = \Phi_l^{-1}(Q)$ is compact in $X$. In particular, if $X/I$ is compact (locally compact), then $X$ is compact (locally compact).

**Proof.** By Theorem 3.17, $X$ is transitive open and then for each $a \in X$, $f_a$ is continuous and open. Since $X$ is a $p$-semisimple, we can see that $f_a$ is bijective. Indeed, if $f_a(x) = f_a(y)$, then $a \ast x = a \ast y$ and so $x = a \ast (a \ast x) = a \ast (a \ast y) = y$, which implies $f_a$ is injective. Moreover for each $x \in X$, $f_a(a \ast x) = a \ast (a \ast x) = x$ and so $f_a$ is surjective. Therefore $f_a$ is a homeomorphism. Now we claim that $I_a = f_a(I)$, for each $a \in X$. Let $x \in f_a(I)$. Then $x = f_a(y)$ for some $y \in I$. Since $I$ is $c$-ideal, $x \ast a = (a \ast y) \ast a = 0 \ast y \in I$. Note that $a \ast x = a \ast (a \ast y) = y \in I$. Then we have $x \in I_a$. Conversely if $x \in I_a$, then $x \ast a \in I$ and $a \ast x \in I$. Thus there exists $y \in I$ such that $y = a \ast x$. Hence $x = a \ast (a \ast x) = a \ast y = f_a(y) \in f_a(I)$. Therefore $I_a = f_a(I)$ and hence $I_a$ is compact. This shows that $X$ satisfies the hypothesis of Theorem 3.19 and so Corollary 3.20 holds. \qed

**4. Topological homomorphisms in topological BCI-algebras**

**Definition 4.1.** Let $X$ and $Y$ be two TBCI-algebras. A mapping $g : X \to Y$ is called a topological homomorphism if

(i) $g$ is a homomorphism from $X$ to $Y$ as BCI-algebras,

(ii) $g$ is a continuous mapping in the topological spaces.

A topological homomorphism $g$ from $X$ to $Y$ is said to be open if $g$ is an open mapping of the topological spaces.

**Definition 4.2.** Let $X$ and $Y$ be two TBCI-algebras. A mapping $f : X \to Y$ is called a topological isomorphism if

(i) $f$ is an isomorphism of the BCI-algebras,

(ii) $f$ is a homeomorphism of the topological spaces.

If $X = Y$, the topological isomorphism of $X$ into $Y$ is called a topological automorphism. Two TBCI-algebras are said to be topological isomorphic if there exists a topological isomorphism of $X$ into $Y$.

**Proposition 4.3.** Let $X$ and $Y$ be transitive open TBCI-algebras and $g$ be a homomorphism of a BCI-algebra $X$ into BCI-algebra $Y$.

(i) If for each neighborhood $U^*$ of $0^*$ in $Y$, there exists a neighborhood $U$ of $0$ in $X$ such that $g(U) \subseteq U^*$, then $g$ is continuous.

(ii) If for each neighborhood $V$ of $0$ in $X$, there exists a neighborhood $V^*$ of $0$ in $Y$ such that $g(V) \supseteq V^*$, then $g$ is open.

**Proof.** (i) Let $a \in X$ and $g(a) = a^* \in Y$. Assume that $W^*$ is an arbitrary neighborhood of $a^*$. Then $U^* = \{ y^* \in Y \mid a^* \ast y^* \in W^* \}$ is an open set by Lemma 3.16 and $0^* \in U^*$, i.e., $U^*$ is a neighborhood of $0^*$. Hence by hypothesis, there exists a neighborhood $U$ of $0$ in $X$ such that $g(U) \subseteq U^*$. Thus the open set $a \ast U$ contains $a$ and $g(a \ast U) = g(a) \ast g(U) = a^* \ast g(U) \subseteq a^* \ast U^* \subseteq W^*$. Therefore $g$ is continuous.

(ii) Let $a \in X$ and $g(a) = a^*$. Assume that $W$ is an arbitrary neighborhood of $a$. Then $V = \{ x \in X \mid a \ast x \in W \}$ is an open set containing $0$ by Lemma 3.16. By hypothesis there exists a neighborhood $V^*$ of $0$ in $Y$ such that $g(V) \supseteq V^*$. Note that $a^* \ast V^*$ is an open set
containing \( a^* \) and \( a^* \circ V^* \subseteq g(a) \circ g(V) = g(a \circ V) \subseteq g(W) \). It follows that \( g(W) \) is open and so \( g \) is an open mapping. \( \square \)

**Theorem 4.4.** Let \( X \) be a TBCI-algebra and \( I \) a topological \( e \)-ideal of \( X \). Then the natural projection \( \Phi_I \) is an open topological homomorphism of \( X \) onto \( X/I \) and \( \{I_0\} \) is an open set in \( X/I \).

**Proof.** Obviously \( \Phi_I \) is a homomorphism of BCI-algebras, and it is open and continuous by Proposition 3.12. Hence \( \Phi_I \) is an open topological homomorphism of \( X \) onto \( X/I \). Since \( I \) is a topological \( e \)-ideal, we have \( \Phi_I(I) = \{I_0\} \) and so \( \{I_0\} \) is open in \( X/I \). \( \square \)

The following theorem provides the converse of Theorem 4.4.

**Theorem 4.5.** Let \( X \) and \( Y \) be two TBCI-algebras. Assume \( g \) is an open topological homomorphism of \( X \) onto \( Y \) having \( \ker g = \{0\} \) and the zero element \( \{0\} \) of \( Y \) is an open set in \( Y \). Then we have

(i) \( I \) is a topological ideal of a TBCI-algebra \( X \).

(ii) if we define \( f : X/I \rightarrow Y \) by \( f(I_x) = g(x) \), then \( f \) is a topological isomorphism, i.e., \( X/I \) is topologically isomorphic to \( Y \).

**Proof.** (i) It is easy to see that \( I \) is a BCI-ideal of \( X \). Moreover since \( g \) is continuous and \( \{0\} \) is open in \( Y \), we have that \( I = \ker g = g^{-1}(\{0\}) \) is open in \( X \). Therefore \( I \) is a topological ideal of \( X \).

(ii) Note that \( f(I_x \circ I_y) = f(I_{x+y}) = g(x \circ y) = g(x) \circ g(y) = f(I_x) \circ f(I_y) \) for all \( I_x, I_y \in X/I \). Thus \( f \) is a homomorphism of the BCI-algebra \( X/I \) onto \( Y \). Next for each \( y \in Y \), since \( g \) is onto, there is \( x \in X \) such that \( g(x) = y \). Hence \( f(I_x) = g(x) = y \), which implies that \( f \) is onto. Finally if \( f(I_x) = f(I_y) \), then \( g(x) = g(y) \). It follows that \( g(x \circ y) = g(x) \circ g(y) = 0' \) and \( g(y \circ x) = g(y) \circ g(x) = 0' \) and hence \( x \circ y \in \ker g = I \) and \( y \circ x \in \ker g = I \). Hence \( I_x = I_y \). This shows that \( f \) is injective. Combining the above arguments we have that \( f \) is an isomorphism of the BCI-algebra \( X/I \) onto the BCI-algebra \( Y \). Finally we show that \( f \) is a homeomorphism of the topological space \( X/I \) into the topological space \( Y \). We shall verify that both \( f \) and \( f^{-1} \) are continuous. Let \( A = I_x \in X/I \) and \( f(A) = a' \). Assume that \( U^* \) is an arbitrary neighborhood of \( a' \). Then \( g(a) = f(I_o) = f(A) = a' \). Since \( g \) is continuous, there exists a neighborhood \( V \) of \( a \) such that \( g(V) \subseteq U^* \). Denote by \( V^* \) the neighborhood of \( A \) in \( X/I \) consisting of all \( I_x \) with \( x \in V \). Then \( f(V^*) = \{f(I_x) \mid I_x \in V^*\} = \{g(x) \mid x \in V\} = g(V) \subseteq U^* \), and hence \( f \) is continuous. Next let \( a' \in Y \) and \( f^{-1}(a') = A \). Denote by \( U^* \) an arbitrary neighborhood of \( A \) in the space \( X/I \). Then \( U := \Phi_I^{-1}(U^*) \) is an open set in \( X \). Let \( a \) be an element in \( U \) such that \( A = I_x \). Then \( g(a) = f(I_o) = f(A) = a' \). Since \( g \) is open, there exists a neighborhood \( V^* \) of \( a' \) such that \( V^* \subseteq g(U) \). Now we claim that \( f^{-1}(V^*) \subseteq U^* \). Indeed, for each \( f^{-1}(x) \in f^{-1}(V^*) \), we have \( x \in V^* \) and so \( x = g(u) \) for some \( u \in U \). Thus \( f^{-1}(x) = f^{-1}(g(u)) = f^{-1}(f(I_o)) = I_o \in U^* \) which shows that \( f^{-1}(V^*) \subseteq U^* \). Therefore \( f^{-1} \) is continuous, ending the proof. \( \square \)

From Theorem 4.5 we have the following corollary.

**Corollary 4.6.** Let \( X \) and \( Y \) be two TBCI-algebras with \( \{0\} \) being open set in \( Y \). If \( g \) is an open topological homomorphism of \( X \) onto \( Y \) having \( \ker g = \{0\} \), then \( g \) is a topological isomorphism and thus \( X \) and \( Y \) are topological isomorphic.

**Theorem 4.7.** Let \( X \) and \( Y \) be two TBCI-algebras and \( f \) an open topological homomorphism of \( X \) onto \( Y \). Denote by \( N^f \) the kernel of \( f \). Then \( f \) gives rise to a one-to-one correspondence between the closed \( e \)-ideals of \( Y \) and the closed \( e \)-ideals of \( X \) which contain \( N^f \) as follows:
(i) If $N^*$ is a closed c-ideal of $Y$, then the closed c-ideal $N$ of $X$ corresponding to it is just the inverse image $N = f^{-1}(N^*)$;
(ii) If $N$ is a closed c-ideal of $X$ containing $N'$, then the closed c-ideal $N^*$ of $Y$ corresponding to it is just the image $N^* = f(N)$.
(iii) The two correspondences thus defined are mutually inverse to one another. Moreover topological ideals correspond to one another. Finally if $N$ and $N^*$ are two topological ideals corresponding to one another in this fashion, then the quotient TBCI-algebras $X/N$ and $Y/N^*$ are topological isomorphic.

Proof. We consider first the correspondence of $N$ to $N^*$. Let $N^*$ be a closed c-ideal of $Y$. Then $N = f^{-1}(N^*)$ is also closed and contains $N'$ since $f$ is continuous. Moreover for each $y, x \in Y$, we have $f(y) = f(x) \cdot f(y) \in N^*$. Since $N^*$ is an ideal of $Y$, it follows that $f(x) \in N^*$ and hence $x \in f^{-1}(N^*) = N$. This shows that $N$ is an ideal of BCI-algebra $X$. Finally let $x \in N$, then $f(x) \in N^*$ and thus $f(0 \cdot x) = f(0) \cdot f(x) = 0'$ since $N^*$ is a c-ideal, where $0'$ is the zero element in $Y$. Therefore $0 \cdot x \in f^{-1}(N^*) = N$ and $N$ is a closed c-ideal of $X$. Now we show that the topological ideals correspond to one another and the quotient TBCI-algebras are topological isomorphic. If $N^*$ is a topological ideal of $Y$ and if $\Phi_{N^*}$ denotes the natural projection of $Y$ onto $Y/N^*$, then $h := \Phi_{N^*} \circ f$ is an open topological homomorphism of $X$ onto $Y/N^*$, with kernel $N$, and $\{N^*\} = \Phi_{N^*}(N)$ is an open set in $Y/N^*$. By Theorem 4.5, $N = \ker h$ is a topological ideal of $X$ and $X/N$ is topological isomorphic to $Y/N^*$.

Conversely let $N$ be a closed c-ideal of $X$ containing $N'$ and consider the correspondence of $N^*$ to $N$ where $N^* = f(N)$ and $N' \subseteq N$. We first show that the inverse image of $N^*$ under the mapping $f$ coincides with $N$. Indeed if $f(a) \in N^*$, then there exists $b \in N$ such that $f(a) = f(b)$. Thus $f(a \bullet b) = f(a) \bullet f(b) = 0'$, i.e., $a \bullet b \in N'$ so that $a \in N$ since $N$ is a BCI-ideal and $b \in N$. Hence $f^{-1}(N^*) \subseteq N$ and so $f^{-1}(N^*) = N$. This from fact it follows that $f(X \setminus N) = Y \setminus N^*$. Since $f$ is open and $X \setminus N$ is an open set, $Y \setminus N^*$ is also open and hence the set $N^*$ is closed in $Y$. Finally we show that $N^*$ is a c-ideal of $Y$. Let $y', x' \in N^*$, then there exist $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$. It follows that $f(0 \cdot x) = 0 \cdot f(x) = 0 \cdot x' = x'$ and hence $0 \cdot x = f(0) \cdot f(x) = f(0 \cdot x) \in f(N) = N^*$. It follows that $N^*$ is a c-ideal of BCI-algebra $Y$. Moreover for each $x' \in N^*$, there exists $x \in N$ such that $f(x) = x'$ and hence $0 \cdot x' = f(0) \cdot f(x) = f(0 \cdot x) \in f(N) = N^*$. Therefore $N^*$ is a closed c-ideal of $Y$, ending the proof. □

References

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