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$a\&\mathcal{I}$–IDEALS ON IS–ALGEBRAS

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Abstract. In this paper, we introduce the concept of an $a\&\mathcal{I}$-ideal in an **IS**-algebra and investigate their properties.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [3] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In 1993, Y. B. Jun et al. [5] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Y. B. Jun et al. [7] renamed the BCI-semigroup (resp, BCI-monoid and BCI-group) as the **IS**-algebra (resp. **IM**-algebra and **IG**-algebra) and studied further properties of these algebras (see [7] and [6]). In this paper, we introduce the concept of an $a\&\mathcal{I}$-ideal in an **IS**-algebra and investigate their properties.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra X satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK-algebra*. In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties for any $x, y, z \in X$:

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $(x * z) * (y * z) \leq x * y$.

A nonempty subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

For a BCI-algebra X, the set $X_+ := \{x \in X | 0 \leq x\}$ is called the *BCK-part* of X. If $X_+ = \{0\}$, then we say that X is a *p-semisimple* BCI-algebra. Note that a BCI-algebra X is p-semisimple if and only if $0 * (0 * x) = x$ for all $x \in X$.

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In [5], Y. B. Jun et al. introduced a new class of algebras related to BCI-algebras and semigroups, called a *BCI-semigroup*, and Jun et al. [7] renamed it as an **IS**-*algebra* for the convenience of study.

Definition 2.1 (Jun et al. [7]). An **IS**-*algebra* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms

(V) $I(X) := (X, *, 0)$ is a BCI-algebra,
(VI) $S(X) := (X, \cdot)$ is a semigroup,
(VII) the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

Especially, if $I(X) := (X, *, 0)$ is a p-semisimple BCI-algebra in Definition 2.1, we say that X is a **PS**-*algebra*. We shall write the multiplication $x \cdot y$ by xy, for convenience.

Example 2.2. Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "$\cdot$" by the following tables

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	c
c	0	0	c	b

Then, by routine calculations, we can see that X is an **IS**-algebra.

Remark. Every p-semisimple BCI-algebra gives an abelian group by defining $x + y := x * (0 * y)$, and hence a **PS**-algebra leads to the ring structure. On the while, every ring gives a BCI-algebra by defining $x * y := x - y$ and so we can construct an **IS**-algebra. This means that *the category of* **PS**-*algebras is equivalent to the category of rings.* In Example 2.2, we can see that $(a + b) + c = 0 \neq a = a + (b + c)$ if we define $x + y := x * (0 * y)$. Hence *the* **IS**-*algebra is a generalization of the ring.*

Lemma 2.3 (Jun et al. [5, Proposition 1]). *Let X be an* **IS**-*algebra. Then for any $x, y, z \in X$, we have*

(a) $0x = x0 = 0$,
(b) *$x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$.*

Definition 2.4 (Ahn et al. [1]). A non-empty subset A of a semigroup $S(X) := (X, \cdot)$ is said to be *left* (resp. *right*) *stable* if $xa \in A$ (resp. $ax \in A$) whenever $x \in S(X)$ and $a \in A$. Both left and right stable is *two-sided stable* or simply *stable*.

Definition 2.5 (Jun et al. [7]). A non-empty subset A of an **IS**-algebra X is called a *left* (resp. *right*) *$\mathcal{I}$-ideal* of X if

(iii) A is a left (resp. right) stable subset of $S(X)$,
(ii) for any $x, y \in I(X)$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

Both a left and a right $\mathcal{I}$-ideal is called a *two-sided $\mathcal{I}$-ideal* or simply an *$\mathcal{I}$-ideal*. Note that $\{0\}$ and X are $\mathcal{I}$-ideals. If A is a left (resp. right) $\mathcal{I}$-ideal of an **IS**-algebra X, then $0 \in A$. Thus A is an ideal of $I(X)$.

Definition 2.6. An **IS**-algebra X is called associative **IS**-algebra if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

3. Main Results

Definition 3.1. A non-empty subset A of an **IS**-algebra X is called a left(resp. right) $a\&\mathcal{I}$-ideal of X, if

(iii) A is a left (resp. right) stable subset of $S(X)$,

(iv) for any $x, y, z \in I(X)$, $(x * z) * (0 * y) \in A$ and $z \in A$ implies $y * x \in A$.

Both a left and a right $a\&\mathcal{I}$-ideal is called a *two-sided $a\&\mathcal{I}$-ideal* or simply an *$a\&\mathcal{I}$-ideal*. Clearly X is $a\&\mathcal{I}$-ideal.

Example 3.2. Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "$\cdot$" by the following tables:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then X is an **IS**-algebra([7]). We can easily check that {0, a} is an $a\&\mathcal{I}$-ideal of X.

Theorem 3.3. *If A is an $a\&\mathcal{I}$-ideal of* **IS**-*algebra X and $x \in A$, then $0 * x \in A$.*

Proof. Let $x \in A$. Then $(x * 0) * (0 * 0) = x \in A$. By Definition 3.1 (iv), we get $0 * x \in A$. The proof is finished. □

Theorem 3.4. *Any $a\&\mathcal{I}$-ideal is an $\mathcal{I}$-ideal, but the converse is not true.*

Proof. Assume that A is an $a\&\mathcal{I}$-ideal of X and $x * z \in A$ and $z \in A$. Then $(x * z) * (0 * 0) = x * z \in A$ and by (iV), we get $0 * x \in A$. So $0 * (0 * x) \in A$ by Theorem 3.3. Putting $x = z = 0$ in (iV), it follows that if $0 * (0 * y) \in A$, then $y \in A$. Now we obtain $x \in A$. Thus we prove that $x * z \in A$ and $z \in A$ implies $x \in A$, and so A is an $\mathcal{I}$-ideal of X.

To show the last part of theorem, we see Example 3.2. It is easy to verify that $A = \{0, b\}$ is an $\mathcal{I}$-ideal of X. But it is not an $a\&\mathcal{I}$-ideal of X as : $(b * b) * (0 * a) = 0 * 0 = 0 \in \{0, b\}$ and $b \in A$, but $a * b = c \notin A$. The proof is complete. □

Theorem 3.5. *Let A be an $\mathcal{I}$-ideal of X. Then the following are equivalent :*

(a) *A is an $a\&\mathcal{I}$-ideal of X,*

(b) *$(x * z) * (0 * y) \in A$ implies $y * (x * z) \in A$, for any $x, y, z \in I(X)$,*

(c) *$x * (0 * y) \in A$ implies $y * x \in A$, for any $x, y \in I(X)$.*

Proof. (a) $\Rightarrow$ (b) Supposed that $(x * z) * (0 * y) \in A$. For brevity we write $s = (x * z) * (0 * y)$, then $((x * z) * s) * (0 * y) = ((x * z) * (0 * y)) * s = 0 \in A$. By (iV), we have $y * (x * z) \in A$.

(b) $\Rightarrow$ (c) Letting $z = 0$ in (b), we get (c).

(c) $\Rightarrow$ (a) Let $(x * z) * (0 * y) \in A$ and $z \in A$. Since $(x * (0 * y)) * ((x * z) * (0 * y)) \leq x * (x * z) \leq z \in A$, we have $x * (0 * y) \in A$ as A is an $\mathcal{I}$-ideal. By (c), we have $y * x \in A$. Hence A is an $a\&\mathcal{I}$-ideal of X. □

Lemma 3.6. *Let A be an $\mathcal{I}$-ideal of an* **IS**-*algebra X. Then A is a $p\&\mathcal{I}$-ideal of X if and only if it satisfies $0 * (0 * x) \in A$ implies $x \in A$ for all $x \in A$.*

Theorem 3.7. *Any $a\&\mathcal{I}$-ideal is a $p\&\mathcal{I}$-ideal, but the converse is not true.*

Proof. Let A is an $a\&\mathcal{I}$-ideal of X. Then A is an $\mathcal{I}$-ideal by Theorem 3.4. Setting $x = z = 0$ in Theorem 3.5 (b), we get $0 * (0 * y) \in A$ implies $y \in A$. ¿From Lemma 3.6, A is a $p\&\mathcal{I}$-ideal.

To show the last part, we see the following Example 3.8.

Example 3.8. Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "$\cdot$" by the following tables:

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	c
c	0	0	c	b

Then X is an **IS**-algebra([9]). By routine calculations give that $A = \{0, a\}$ is a $p\&\mathcal{I}$-ideal of X. But it is not an $a\&\mathcal{I}$-ideal as : $(c * a) * (0 * b) = c * c = 0 \in A$ and $a \in A$, but $b * c = c \notin A$.

The following theorem provides a sufficient condition that the $\mathcal{I}$-ideal is $a\&\mathcal{I}$-ideal.

Theorem 3.9. *Let X be an associative* **IS***-algebra and A be an $\mathcal{I}$-ideal. Then A is an $a\&\mathcal{I}$-ideal*

Proof. Let $x * (0 * y) \in A$. By Theorem 3.5(c), if we show $y * x \in A$, then A is an $a\&\mathcal{I}$-ideal of X. Since $(y * x) * (x * (0 * y)) = (y * x) * ((x * 0) * y) = (y * x) * (x * y) = (y * (x * y)) * x = ((y * x) * y) * x = ((y * y) * x) * x = (0 * x) * x = 0 * (x * x) = 0 * 0 = 0$ and A is an $\mathcal{I}$-ideal of X, we get $y * x \in A$. Hence A is an $a\&\mathcal{I}$-ideal of X. $\square$

Theorem 3.10. *Let A and B be $\mathcal{I}$-ideals of X with $A \subset B$. If A is an $a\&\mathcal{I}$-ideal of X, then so is B.*

Proof. Supposed that A is an $a\&\mathcal{I}$-ideal and $x * (0 * y) \in B$. Put $s = x * (0 * y) \in B$, then $(x * s) * (0 * y) = (x * (0 * y)) * s = 0 \in A$. Since A is an $a\&\mathcal{I}$-ideal, $y * (x * s) \in A \subseteq B$. Now $((y * x) * s) * (y * (x * s)) = ((y * x) * (y * (x * s))) * s \leq (x * (x * s)) * s = (x * s) * (x * s) = 0 \in A \subseteq B$ Since B is an ideal of X and $y * (x * s) \in B$, we get $(y * x) * s \in B$. Therefore $y * x \in B$. Hence B is an $a\&\mathcal{I}$-ideal of X. $\square$

Corollary 3.11. *If zero $\mathcal{I}$-ideal of X is an $a\&\mathcal{I}$-ideal, then every ideal of X is an $a\&\mathcal{I}$-ideal.*

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