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CONVERGENCE RATE OF CONDITIONAL EXPECTATIONS

XIKUI WANG

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Abstract. As opposed to the traditional probabilistic approach, the convergence rate of conditional expectations is examined from the analytic point of view. This new approach provides simple and clear proofs for a two-sided uniform inequality for conditional expectations and related results. A necessary and sufficient condition for the convergence rate of conditional expectations is derived. Moreover, the existing lower bound for the convergence rate based on the probabilistic approach is sharpened by the new analytic approach.

1. Introduction. Let (Ω, Σ, P) be a complete probability space and $\mathcal{S}(\Sigma)$ be the set of all sub-σ-algebras of Σ. A metric d^* on $\mathcal{S}(\Sigma)$ is introduced in [1], [6], and [7] from probabilistic point of view to study the convergence rate of conditional expectations. The convergence rate is studied in [1] when $\Sigma_n, n = 1, 2, \cdots$, increases or decreases to Σ_∞ and $d^*(\Sigma_n, \Sigma_\infty) \to 0$, and is generalized in [7] to the case where $\Sigma_n, n = 1, 2, \cdots$, is not nested.

We investigate the convergence rate from the analytic point of view using a metric d on $\mathcal{S}(\Sigma)$ introduced in [8] and [9]. This new approach allows for direct and simple proofs of the results. A two-sided uniform inequality for the convergence of conditional expectations is derived whose lower bound sharpens that in [7]. Moreover, the conditional expectation $E(f|\Sigma_n)$ converges to $E(f|\Sigma_\infty)$ uniformly in $f \in L^\infty$ if and only if $d(\Sigma_n, \Sigma_\infty) \to 0$. This does not require that $\Sigma_n, n = 1, 2, \cdots$, be nested.

The new metric is introduced in section 2 and is compared with the metric based on the probabilistic approach. Main results are derived in section 3. We conclude with an example which shows that $\Sigma_n, n = 1, 2, \cdots$, increases or decreases to Σ_∞ does not necessarily imply $d(\Sigma_n, \Sigma_\infty) \to 0$.

2. The Metrics. Let Σ_1 be a sub-σ-algebra of Σ. The space $L^\infty(\Sigma_1) = L^\infty(\Omega, \Sigma_1, P)$ is a closed subspace of $L^\infty(\Omega, \Sigma, P)$ in the natural way, and is a subspace of $L^2(\Omega, \Sigma, P)$ since (Ω, Σ, P) is a probability space. Moreover, $L_1^\infty(\Sigma_1) = \{f \in L^\infty(\Sigma_1) : \|f\|_\infty \le 1\}$ is a unit ball in $L^\infty(\Sigma_1)$ and is closed in $L^2(\Omega, \Sigma, P)$.

Define a metric d on $\mathcal{S}(\Sigma)$, which is basically the Hausdorff metric on unit balls in the L^2-norm, as follows:

$$d(\Sigma_1, \Sigma_2) = \max\{\sup_{f \in L_1^\infty(\Sigma_1)} \inf_{g \in L_1^\infty(\Sigma_2)} \|f - g\|_2, \sup_{g \in L_1^\infty(\Sigma_2)} \inf_{f \in L_1^\infty(\Sigma_1)} \|f - g\|_2\}.$$

[9] shows that $(\mathcal{S}(\Sigma), d)$ is a complete metric space.

This metric d is modeled on a metric introduced in [2] on the set of all von Nuemann subalgebras of a Type II_1 factor, which has been useful in the study of index in Type II_1 factors, see [3] and [5].

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For any Σ_1 and Σ_2 in $\mathcal{S}(\Sigma)$, the metric d^* used in [7] is defined as

$$d^*(\Sigma_1, \Sigma_2) = \max\{\sup_{A\in\Sigma_1} \inf_{B\in\Sigma_2} P(A\Delta B), \sup_{B\in\Sigma_2} \inf_{A\in\Sigma_1} P(A\Delta B)\}.$$

The metric d^* in [1] has $+$ instead of max, but is essentially the same. The relationship between the two metrics d and d^* is studied in [8].

3. Main Results.

Let $e(f|\Sigma_1)$ be the orthogonal projection of $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \Sigma_1, P)$, and $E(f|\Sigma_1)$ be the restriction of $e(f|\Sigma_1)$ to $L^\infty(\Omega, \Sigma, P)$. Both of them are conditional expectations given Σ_1, but results are stated with respect to $E(f|\Sigma_1)$: $L^\infty(\Omega, \Sigma, P) \to L^\infty(\Omega, \Sigma_1, P)$ only.

Lemma 3.1. *For any Σ_1 and Σ_2 in $\mathcal{S}(\Sigma)$,*

$$\sup_{f\in L_1^\infty(\Sigma_2)} \|E(f|\Sigma_1) - f\|_2 \leq 2d(\Sigma_1, \Sigma_2).$$

Proof. For any $\epsilon > 0$ and any $f \in L_1^\infty(\Sigma_2)$, there exists a $g \in L_1^\infty(\Sigma_1)$ such that

$$\|f - g\|_2 < d(\Sigma_1, \Sigma_2) + \epsilon/2.$$

So,

$$\|E(f|\Sigma_1) - g\|_2 = \|E(f - g|\Sigma_1)\|_2 \leq \|f - g\|_2 < d(\Sigma_1, \Sigma_2) + \epsilon/2.$$

Hence,

$$\|E(f|\Sigma_1) - f\|_2 \leq \|E(f|\Sigma_1) - g\|_2 + \|f - g\|_2 < 2d(\Sigma_1, \Sigma_2) + \epsilon.$$

Therefore,

$$\sup_{f\in L_1^\infty(\Sigma_2)} \|E(f|\Sigma_1) - f\|_2 \leq 2d(\Sigma_1, \Sigma_2). \quad \square$$

If $\Sigma_1 \subset \Sigma_2$, then $E(f|\Sigma_1) = E(E(f|\Sigma_1)|\Sigma_2) = E(E(f|\Sigma_2)|\Sigma_1)$. So,

Corollary 3.2. *For any Σ_1 and Σ_2 in $\mathcal{S}(\Sigma)$ such that $\Sigma_1 \subset \Sigma_2$,*

$$\sup_{f\in L_1^\infty(\Sigma)} \|E(f|\Sigma_1) - E(f|\Sigma_2)\|_2 \leq 2d(\Sigma_1, \Sigma_2).$$

This corollary gives the uniform inequality for nested sub-σ-algebras. In general,

Theorem 3.3. *For any Σ_1 and Σ_2 in $\mathcal{S}(\Sigma)$, we have*

$$d(\Sigma_1, \Sigma_2) \leq \sup_{f\in L_1^\infty(\Sigma)} \|E(f|\Sigma_1) - E(f|\Sigma_2)\|_2 \leq 2[d(\Sigma_1, \Sigma_2)]^{1/2}.$$

Proof. For any $f \in L_1^\infty(\Sigma_1)$, $E(f|\Sigma_1) = f$ and $E(f|\Sigma_2)$ is the (L^2-norm) closest element of $L_1^\infty(\Sigma_2)$ to f. So,

$$\begin{aligned} \sup_{f\in L_1^\infty(\Sigma_1)} \inf_{g\in L_1^\infty(\Sigma_2)} \|f - g\|_2 &= \sup_{f\in L_1^\infty(\Sigma_1)} \|f - E(f|\Sigma_2)\|_2 \\ &\leq \sup_{f\in L_1^\infty(\Sigma)} \|E(f|\Sigma_1) - E(f|\Sigma_2)\|_2. \end{aligned}$$

Similarly,

$$\sup_{g\in L_1^\infty(\Sigma_2)} \inf_{f\in L_1^\infty(\Sigma_1)} \|f-g\|_2 \le \sup_{g\in L_1^\infty(\Sigma)} \|E(g|\Sigma_1)-E(g|\Sigma_2)\|_2.$$

Hence,

$$d(\Sigma_1,\Sigma_2) \le \sup_{f\in L_1^\infty(\Sigma)} \|E(f|\Sigma_1)-E(f|\Sigma_2)\|_2.$$

On the other hand, for any $f\in L_1^\infty(\Sigma)$, from Lemma 3.1, the Hölder's inequality, and the fact that $\|f-E(f|\Sigma_i)\|_2\le 1, i=1,2$, we have

$$\begin{aligned}
\|E(f|\Sigma_1)-E(f|\Sigma_2)\|_2^2 &= \int_\Omega E(f|\Sigma_1)[f-E(f|\Sigma_2)]dP + \int_\Omega E(f|\Sigma_2)[f-E(f|\Sigma_1)]dP \\
&= \int_\Omega [E(f|\Sigma_1)-E(E(f|\Sigma_1)|\Sigma_2)][f-E(f|\Sigma_2)]dP \\
&\quad + \int_\Omega [E(f|\Sigma_2)-E(E(f|\Sigma_2)|\Sigma_1)][f-E(f|\Sigma_1)]dP \\
&\le \|E(f|\Sigma_1)-E(E(f|\Sigma_1)|\Sigma_2)\|_2\|f-E(f|\Sigma_2)\|_2 \\
&\quad + \|E(f|\Sigma_2)-E(E(f|\Sigma_2)|\Sigma_1)\|_2\|f-E(f|\Sigma_1)\|_2 \\
&\le 4d(\Sigma_1,\Sigma_2).
\end{aligned}$$

Therefore,

$$\sup_{f\in L_1^\infty(\Sigma)} \|E(f|\Sigma_1)-E(f|\Sigma_2)\|_2 \le 2[d(\Sigma_1,\Sigma_2)]^{1/2}. \quad \square$$

Corollary 3.4. *Let $\Sigma_n, n=1,2,\cdots,\infty$, be an arbitrary sequence in $\mathcal{S}(\Sigma)$. Then,*

$$\lim_{n\to\infty} \|E(f|\Sigma_n)-E(f|\Sigma_\infty)\|_2 = 0$$

uniformly in $f\in L_1^\infty(\Sigma)$ if and only if

$$\lim_{n\to\infty} d(\Sigma_n,\Sigma_\infty)=0.$$

Theorem 3.5. *If $\lim_{n\to\infty} d(\Sigma_n,\Sigma_\infty)=0$, then for any $A\in\Sigma_\infty$ and $f\in L^\infty(\Sigma)$, we have*

$$\lim_{n\to\infty} |\int_A E(f|\Sigma_n)dP - \int_A E(f|\Sigma_\infty)dP| = 0.$$

Proof. Let $A_n=\{E(\chi_A|\Sigma_n)>1/2\}\in\Sigma_n$. Since $\frac{1}{2}=\|\frac{1}{2}-\chi_A\|_1$, by Lemma 2.1 in [4],

$$\begin{aligned}
|\int_A E(f|\Sigma_n)dP - \int_A E(f|\Sigma_\infty)dP| &\le |\int_A E(f|\Sigma_n)dP - \int_{A_n} E(f|\Sigma_n)dP| \\
&\quad + |\int_{A_n} E(f|\Sigma_n)dP - \int_A E(f|\Sigma_\infty)dP| \\
&\le \|f\|_\infty P(A_n\Delta A) \\
&= \|f\|_\infty[\|\frac{1}{2}-\chi_A\|_1 - \|\frac{1}{2}-E(\chi_A|\Sigma_n)\|_1] \\
&\le \|f\|_\infty\|\chi_A-E(\chi_A|\Sigma_n)\|_1 \\
&\le \|f\|_\infty\|\chi_A-E(\chi_A|\Sigma_n)\|_2 \\
&\le 2\|f\|_\infty d(\Sigma_n,\Sigma_\infty). \quad \square
\end{aligned}$$

4. An Example. It is shown in [7] that $d^*(\Sigma_1, \Sigma_2) \leq \sup_{f \in L_1^\infty(\Sigma)} \|E(f|\Sigma_1) - E(f|\Sigma_2)\|_2$. We show in our example that $\Sigma_n, n = 1, 2, \cdots$, increases to Σ_∞ does not necessarily imply that $d(\Sigma_n, \Sigma_\infty) \to 0$. Moreover, $d^*(\Sigma_1, \Sigma_2) < d(\Sigma_1, \Sigma_2)$. Therefore, the above lower bound in [7] is sharpened by our Theorem 3.3.

Let $\Omega = [0, 1)$ and P be the Lebesgue measure on Ω. Define $\Sigma_n, n = 1, 2, \cdots$, to be the σ-algebra generated by $\{[\frac{k-1}{2^n}, \frac{k}{2^n}), k = 1, 2, \cdots, 2^n\}$. Then $\Sigma_n$increases to Σ_∞, the σ-algebra of Borel sets of Ω. Since $L_1^\infty(\Sigma_{n+1}) \subset L_1^\infty(\Sigma_\infty)$, then $d(\Sigma_n, \Sigma_{n+1}) \leq d(\Sigma_n, \Sigma_\infty)$ for every n.

Now, $[\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}})$ and $[\frac{2k-1}{2^{n+1}}, \frac{k}{2^n})$ are in Σ_{n+1}, and $[\frac{k-1}{2^n}, \frac{k}{2^n})$ is in Σ_n. Define

$$g = \begin{cases} 1 & \text{on } \cup_{k=1}^{2^n} [\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}}). \\ -1 & \text{on } \cup_{k=1}^{2^n} [\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}). \end{cases}$$

Then

$$g \in L^2(\Sigma_{n+1})$$

and

$$\inf_{f \in L_1^\infty(\Sigma_n)} \|f - g\|_2 = \|g - E(g|\Sigma_n)\|_2 = \|g\|_2 = 1.$$

Therefore,

$$\sup_{g \in L_1^\infty(\Sigma_{n+1})} \inf_{f \in L_1^\infty(\Sigma_n)} \|f - g\|_2 \geq 1,$$

so

$$d(\Sigma_n, \Sigma_{n+1}) \geq 1.$$

This implies that $d(\Sigma_n, \Sigma_\infty)$ is at least 1 and hence does not go to 0. It is easy to see that

$$d^*(\Sigma_n, \Sigma_{n+1}) = 1/2 < d(\Sigma_n, \Sigma_{n+1}).$$

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Division of Comminity Health, Faculty of Medicine, Memorial University of Newfoundland, st. John's, nf, Canada a1b 3v6

E-mail: xikui@mun.ca

Current address: Department of Statistics,The University of Manitoba, Winnipeg,Manitoba,Canada r3t 2n2

E-mail: xikui_wang@umanitoba.ca