NORMALITY ON DENSE COUNTABLE SUBSPACES

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Abstract. We prove that under CH there exists a countable dense subset X of $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is normal on X. This answers a question of A.V. Arhangel'skiĭ. Another result is that there exists a non-regular separable Hausdorff space Z which is normal on Y for every dense countable $Y \subset Z$. It is also established that there are Tychonoff separable spaces which are not normal on any countable dense subspace.

0. Introduction. The notion of normality on a subspace of a topological space was introduced by Arhangel'skiĭ in his survey on relative topological properties [Ar]. A space X is called *normal on a subspace* Y if any pair of disjoint closed sets F and G of X with $\overline{F \cap Y} = F$ and $\overline{G \cap Y} = G$ can be separated by open sets in X. Arhangel'skiĭ points out in [Ar, Proposition 22] that if X is normal on Y then Y is normal in X, i.e. for each pair A, B of closed disjoint subsets of X there are disjoint open subsets U, V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$. It is known [Ar] that a normal space is not necessarily normal in a bigger space. However, every Lindelöf (and hence every countable regular) space is strongly normal in any regular space [Ar,Corollary 10]. This makes it natural to ask whether every Tychonoff space is normal on its countable subspaces.

Recently, Arhangel'skiĭ proved that there is a dense countable subspace A of $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is not normal on A and asked whether the non-normality of a separable space is due to its non-normality on all dense countable subspaces. Is it true in particular, that $\mathbb{R}^{\mathfrak{c}}$ is non-normal on any countable dense subspace?

We show that the answer is negative under the Continuum Hypothesis: there exists a countable dense subset X in $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is normal on X.

Another reasonable hypothesis for characterizing normality could be normality on every countable subspace. It is easy to show that in general this is not the case, because there exist pseudocompact spaces in which all countable subsets are closed and C^*-embedded [Sh]. Clearly, such spaces are normal on every countable subspace without being normal. Therefore, if we want this hypothesis to hold, countable sets should be comparable (in some way) with the whole space as happens, for example, in separable spaces.

We prove that for Hausdorff separable spaces this hypothesis is not true: there exists a separable non-regular Hausdorff space which is not normal while being normal on any countable dense subspace.

Now, if a regular space is separable, then it has a dense normal subspace. Does this mean that X has to contain a "globally normal" (in some sense) subspace Y? This also turns out not to be true and we construct an example of a Tychonoff separable space X such that X is not normal on any countable dense subspace.

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1. Notation and terminology. The space $\mathbb{R}$ is the real line with its usual topology, $\mathbb{Q}$ is the set of rational numbers with the topology inherited from $\mathbb{R}$, and $I = [0,1] \subset \mathbb{R}$. The closure of a set A in a space X is denoted $\overline{A}$ when X is clearly defined and $\mathrm{cl}_X(A)$ if it might cause confusion. We identify any function with its graph and any ordinal with the set of preceeding ones. The symbol $\mathfrak{c}$ stands for the cardinality of continuum as well as for the respective ordinal. The abbreviation CH denotes the Continuum Hypothesis, i.e., the statement $\omega_1 = \mathfrak{c}$. The symbol $\Box$ stands for the end of a proof.

All other notions are standard and follow [En].

2. Normality and non-normality on dense countable subspaces. Let us show first that $\mathbb{R}^{\mathfrak{c}}$ can be normal on a countable dense subset.

2.1. Definition. *A space X is said to be normal on a subset $Y \subset X$ if every two disjoint closed subsets F, G of X satisfying $F = \overline{F \cap Y}$ and $G = \overline{G \cap Y}$ can be separated in X by disjoint open sets.*

2.2. Theorem. *If CH holds, then there is a countable dense subset X of $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is normal on X.*

Proof. Let $\{(A_\alpha, B_\alpha) : 1 \leq \alpha < \mathfrak{c}\}$ be an enumeration of all disjoint pairs of infinite subsets of ω.

Using transfinite recursion on ordinals α with $1 \leq \alpha < \mathfrak{c}$ we are going to construct subsets $X_\alpha = \{x_n^\alpha : n \in \omega\}$ and $Y_\alpha = \{y_\nu^\alpha : 1 \leq \nu \leq \alpha\}$ of the space $\mathbb{R}^\alpha$ in such a way that

(i) X_α is dense in $\mathbb{R}^\alpha$;

(ii) $x_n^\beta | \gamma = x_n^\gamma$ for all $n \in \omega$ and $1 \leq \gamma < \beta < \alpha$;

(iii) $y_\nu^\beta | \gamma = y_\nu^\gamma$ for all $1 \leq \nu \leq \gamma < \beta < \alpha$;

(iv) if $1 \leq \nu \leq \alpha$ and $\mathrm{cl}_{\mathbb{R}^\nu}(\delta_\nu(A_\nu)) \cap \mathrm{cl}_{\mathbb{R}^\nu}(\delta_\nu(B_\nu)) \neq \emptyset$ then $y_\nu^\alpha \in \mathrm{cl}_{\mathbb{R}^\alpha}(\delta_\alpha(A_\nu)) \cap \mathrm{cl}_{\mathbb{R}^\alpha}(\delta_\alpha(B_\nu))$.
Here $\delta_\alpha : \omega \to X_\alpha$ is defined by $\delta_\alpha(n) = x_n^\alpha$ for every $\alpha < \mathfrak{c}$ and $n \in \omega$.

Take an arbitrary countable dense subspace $X_1 = \{x_n^1 : n \in \omega\}$ of the space $\mathbb{R} = \mathbb{R}^1$. If $C_1 = \mathrm{cl}_{\mathbb{R}}(\delta_1(A_1)) \cap \mathrm{cl}_{\mathbb{R}}(\delta_1(B_1)) \neq \emptyset$ then choose any $y_1^1 \in C_1$. Otherwise let $y_1^1 = 0$. It is clear that the sets X_1 and $Y_1 = \{y_1^1\}$ satisfy (i)-(iv).

Suppose that for some $\alpha < \mathfrak{c}$ we have constructed the sets X_β, Y_β satisfying (i)–(iv) for all β with $1 \leq \beta < \alpha$. If α is a limit ordinal, let $x_n^\alpha = \cup\{x_n^\beta : \beta < \alpha\}$ for each $n \in \omega$ and $y_\nu^\alpha = \cup\{y_\nu^\beta : \nu \leq \beta < \alpha\}$ for each $\nu \in \alpha$.

Consider the set

$$(*) \qquad C_\alpha = \mathrm{cl}_{\mathbb{R}^\alpha}(\delta_\alpha(A_\alpha)) \cap \mathrm{cl}_{\mathbb{R}^\alpha}(\delta_\alpha(B_\alpha)).$$

If $C_\alpha \neq \emptyset$, let y_α^α be an arbitrary element of C_α. Otherwise put $y_\alpha^\alpha(\beta) = 0$ for all $\beta < \alpha$. It is clear that the sets $X_\alpha = \{x_n^\alpha : n \in \omega\}$ and $Y_\alpha = \{y_\nu^\alpha : \nu \leq \alpha\}$ satisfy the conditions (i)–(iv).

Suppose that $\alpha = \beta + 1$. There are two possibilities: $C_\alpha = \emptyset$ or $C_\alpha \neq \emptyset$ (see $(*)$ for the definition of C_α).

If $C_\alpha = \emptyset$, let $y_\alpha^\alpha(\gamma) = 0$ for every $\gamma < \alpha$. If $C_\alpha \neq \emptyset$, let y_α^α be any element of C_α. It will require more work to define y_ν^α for $1 \leq \nu < \alpha$.

Claim. There exist dense subsets $\{D_i : i \in \omega\}$ of the space X_β and sequences S_ν, T_ν for $1 \leq \nu \leq \beta$ such that

(1) the family $\mu = \{D_i : i \in \omega\} \bigcup \{S_\nu : 1 \leq \nu \leq \beta\} \bigcup \{T_\nu : 1 \leq \nu \leq \beta\}$ is disjoint and $\cup\mu = X_\beta$;

(2) $S_\nu \subset \delta_\beta(A_\nu)$ *and* $T_\nu \subset \delta_\beta(B_\nu)$ *for* $1 \le \nu \le \beta$*;*

(3) if $1 \le \nu \le \beta$ *and* $C_\nu \ne \emptyset$*, then* $S_\nu \to y^\beta_\nu$ *and* $T_\nu \to y^\beta_\nu$*.*

Proof of the claim. Take a countable base $\{U_n : n \in \omega\}$ in the space X_β. Since X_β is dense in itself, every U_n is infinite. For each ν with $1 \le \nu < \alpha$ choose non-trivial sequences $V_\nu \subset \delta_\beta(A_\nu)$ and $W_\nu \subset \delta_\beta(B_\nu)$ in such a way that $V_\nu \to y^\beta_\nu$ and $W_\nu \to y^\beta_\nu$ if $C_\nu \ne \emptyset$. This choice is possible because (iv) holds for X_β and Y_β. Now apply the following simple set-theoretic fact: if the sets E_n are infinite for every $n \in \omega$, then there exist infinite disjoint sets $\{G_n : n \in \omega\}$ such that $G_n \subset E_n$ for each $n \in \omega$.

By the above mentioned set-theoretic fact we can find infinite sets $S_\nu \subset V_\nu$, $T_\nu \subset W_\nu$ and $H_n \subset U_n$ for all $\nu \ge 1$, $\nu < \alpha$ and $n \in \omega$ in such a way that the family $\{S_\nu : 1 \le \nu < \alpha\} \bigcup \{T_\nu : 1 \le \nu < \alpha\} \bigcup \{H_n : n \in \omega\}$ is disjoint. The set $D = \bigcup\{H_n : n \in \omega\}$ is dense in X_β and disjoint from $E = \bigcup\{S_\nu \cup T_\nu : 1 \le \nu < \alpha\}$. Using again the countable base of X_β it is easy to split D into a disjoint union of dense subspaces $\{D'_i : i \in \omega\}$ of D and hence of X_β. Now add the set $X_\beta \setminus (D \cup E)$ to D'_0 and denote the resulting sets by D_i, $i \in \omega$ to conclude the proof of the claim.

Given a $\nu \ge 1$ with $\nu < \alpha$ put $y^\alpha_\nu(\gamma) = y^\beta_\nu(\gamma)$ for every $\gamma < \beta$ and $y^\alpha_\nu(\beta) = 0$. Let $x^\alpha_n(\gamma) = x^\beta_n(\gamma)$ for all $n \in \omega$ and $\gamma < \beta$. Put $x^\alpha_n(\beta) = 0$ if $x^\beta_n \in \bigcup\{S_\nu \cup T_\nu : 1 \le \nu \le \beta\}$ and $x^\alpha_n(\beta) = p_i$ if $x^\beta_n \in D_i$. Here $\{p_i : i \in \omega\}$ is some enumeration of $\mathbb{Q}$.

Let $X_\alpha = \{x^\alpha_n : n \in \omega\}$ and $Y_\alpha = \{y^\alpha_\nu : 1 \le \nu \le \alpha\}$. It is not difficult to verify that the families $\{X_\gamma : \gamma \le \alpha\}$ and $\{Y_\gamma : \gamma \le \alpha\}$ satisfy (i)–(iv).

To conclude the construction, put $x_n = \cup\{x^\beta_n : \beta < \mathfrak{c}\}$ for each $n \in \omega$ and $y_\nu = \cup\{y^\beta_\nu : \nu \le \beta < \mathfrak{c}\}$ for each $\nu \in \mathfrak{c}$. Let $X = \{x_n : n \in \omega\}$ and let $\delta : \omega \to X$ be defined by $\delta(n) = x_n$ for every $n \in \omega$. We claim that $\mathbb{R}^\mathfrak{c}$ is normal on X.

To see this, take two disjoint closed subsets $F, G \subset \mathbb{R}^\mathfrak{c}$ satisfying $F = \mathrm{cl}_{\mathbb{R}^\mathfrak{c}}(F \cap X)$ and $G = \mathrm{cl}_{\mathbb{R}^\mathfrak{c}}(G \cap X)$. If one of the sets $F \cap X$, $G \cap X$ is finite, then F and G can be separated because $\mathbb{R}^\mathfrak{c}$ is regular. If not, then the pair $(F \cap X, G \cap X)$ was listed in our numeration, i.e. there is $\beta \in \mathfrak{c}$ such that $F \cap X = \delta(A_\beta)$ and $G \cap X = \delta(B_\beta)$.

Assume that $\mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(A_\beta)) \cap \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(B_\beta)) = \emptyset$. Since $\mathbb{R}^\beta$ is normal there are disjoint open sets $U \supset \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(A_\beta))$ and $V \supset \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(B_\beta))$. Now $\pi^{-1}_\beta(U)$ and $\pi^{-1}_\beta(V)$ are disjoint open sets in $\mathbb{R}^\mathfrak{c}$ containing F and G respectively.

Suppose that $\mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(A_\beta)) \cap \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(B_\beta)) \ne \emptyset$. Then, by our construction, there exists a point $y^\beta_\beta \in \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(A_\beta)) \cap \mathrm{cl}_{\mathbb{R}^\beta}(\delta_\beta(B_\beta))$. It follows easily from (iii) and (iv) that $y_\beta \in \mathrm{cl}_{\mathbb{R}^\mathfrak{c}}(\delta(A_\beta)) \cap \mathrm{cl}_{\mathbb{R}^\mathfrak{c}}(\delta(B_\beta))$, i.e. $F \cap G \ne \emptyset$ which gives us a contradiction.

□

2.3. Proposition. *For any separable space* X *the following conditions are equivalent:*

(1) X *is normal on every dense countable subset;*

(2) *any two separable disjoint closed subspaces of* X *can be separated by disjoint open sets.*

Proof. It is immediate that (2)$\Longrightarrow$(1). Now suppose that (1) holds and let F and G be separable disjoint closed sets. Pick countable $A \subset F$ and $B \subset G$ such that $F = \overline{A}$ and $G = \overline{B}$. Since X is separable, there is a countable dense subspace $D \subset X$. Now apply the normality of X on $D \cup A \cup B$ to finish the proof.

□

2.4. Example. *There is a Hausdorff non-regular separable space* X *which is normal on each countable dense subspace.*

Proof. Let Y be a copy of $\omega_1 + 1$ in $\mathbb{R}^{\mathfrak{c}}$. Take a countable dense subspace D in $\mathbb{R}^{\mathfrak{c}}$ which is disjoint from Y. The space X will be the set $Y \cup D$ with the following topology:
(i) all points in X, except ω_1, have their basic neighbourhoods inherited from $\mathbb{R}^{\mathfrak{c}}$;
(ii) the basic neighbourhoods of ω_1 are the sets $(W \cap D) \cup \{\omega_1\}$ where W is open set in $\mathbb{R}^{\mathfrak{c}}$ containing the point ω_1.

It is easy to see that X is a non-regular space. It is Hausdorff because its topology is finer than the topology μ on $Y \cup D$ inherited from $\mathbb{R}^{\mathfrak{c}}$. We claim that the closure of each countable set A in X concides with the μ-closure of A in $Y \cup D$. Indeed, these closures can differ only in the point ω_1. Now if $\omega_1 \in \mathrm{cl}_{\mathbb{R}^{\mathfrak{c}}}(A)$, and $\omega_1 \notin A$, then $\omega_1 \in \mathrm{cl}_{\mathbb{R}^{\mathfrak{c}}}(A \setminus Y)$ which can take place if and only if $\omega_1 \in \mathrm{cl}_X(A)$.

According to Proposition 2.3 it suffices to prove that any separable disjoint closed sets F and G in X can be separated by open sets. The observation above yields that F and G are closed in $(Y \cup D, \mu)$. The space $(Y \cup D, \mu)$ is σ-compact, and hence normal. Consequently, the sets F and G can be separated in $(Y \cup D, \mu)$ and hence in X.

$\square$

2.5. Example. *There is a countable dense subset C of Niemytzki plane N such that N is not normal on C.*

Proof. Recall that $N = L \cup U$, where $L = \{(t, 0) : t \in \mathbb{R}\}$ and $U = \{(x, y) : x \in \mathbb{R} \text{ and } y > 0\}$. The topology at the points of U is Euclidean and the base at a $z = (t, 0) \in L$ is formed by the sets $\{z\} \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x - t)^2 + (y - \frac{1}{n})^2 < (\frac{1}{n})^2\}$ where n runs over positive integers.

Let $A = \{(x, y) \in U : x \textit{ and } y \textit{ are rational}\}$ and $B = \{(t, 0) : t \textit{ is rational}\}$. We will establish that N is not normal on $C = A \cup B$.

Let $\{t_n : n \in \omega\}$ be some enumeration of B. It is easy to construct open circles $\{W_n : n \in \omega\}$ such that their closures (in Euclidean topology) are disjoint, the radius of W_n is less than $\frac{1}{n}$, the circle W_n is contained in U and is tangent to the x-axis at the point t_n. Consider the set $P = A \setminus W$, where $W = \bigcup\{W_n : n \in \omega\}$. Then the sets $F = \overline{P}$ and $G = B = \overline{B}$ are closures of subsets of C. Let us prove that F and G are disjoint and can not be separated by open sets of N. Since rationals and irrationals in L can not be separated by open subsets of N, it suffices to verify that $F \cap L = L \setminus B$.

Since $W \cup B$ is a neighbourhood of B in N, no point of B belongs to F. Therefore F and G are disjoint. Let us prove that every $z = (t, 0) \in L \setminus B$ is in the closure of P.

Suppose not; then $z \notin \overline{P}$ for some $z = (t, 0) \in L \setminus B$. As a consequence $O \cap A \subset W$, where $O = \{z\} \cup \{(x, y) : (x - t)^2 + (y - \varepsilon)^2 < \varepsilon^2\}$ for some $\varepsilon > 0$. The set $O_1 = O \cap \{(x, y) \in U : y \geq \varepsilon\}$ intersects only finitely many W_n. Since $W \cap O_1$ is dense in O_1, only finitely many W_n's cover a dense subset of O_1. This means that $O_1 \subset \overline{W}_1 \cup \ldots \overline{W}_n$ for some n. The space O_1 is connected and the closed sets $\overline{W}_n$ are disjoint, which implies that $O_1 \subset \overline{W}_i$ for some $i \leq n$.

Now O and W_i are two circles tangent to L at distinct points. Therefore, there is a $\delta > 0$ such that $O_2 \cap W_i = \emptyset$ where $O_2 = \{(x, y) \in O : y < \delta\}$. Let $O_3 = \{(x, y) \in O : y > \frac{\delta}{2}\}$. The same reasoning as in the above paragraph shows that $O_3 \subset \overline{W}_j$ for some $j \in \omega$. Since $O_3 \setminus \overline{W}_i \neq \emptyset$ we have $i \neq j$ and $O_1 \subset \overline{W}_i \cap \overline{W}_j$ whence $\overline{W}_i \cap \overline{W}_j \neq \emptyset$ which is a contradiction. $\square$

Recently Arhangel'skiĭ proved that $C_p(\omega_1 + 1)$ is not normal on any dense subspace. However this space is not separable. Our next example shows that there exist separable spaces with a similar property.

2.6. Example. *There is a Tychonoff separable space which is not normal on any countable dense subspace.*

Proof. The underlying set of our space will be $X = L \cup Q \subset \mathbb{R}^2$, where $L = \{(t,0) : t \in \mathbb{R}\}$ and $Q = \{(p,q) : p,q \in \mathbb{Q} \textit{ and } q > 0\}$. All points of the set Q are isolated. The base at a point $z = (t,0) \in L$ is the family $\mathcal{B}_z = \cup\{\mathcal{B}_z^n : n \in \omega\}$ where $\mathcal{B}_z^n = \{\{z\} \cup U : U = V \cap Q\}$ and V is any open set in $\mathbb{R}^2$ which contains the set $T_n = \{(t,y) : 0 < y \leq \frac{1}{n}\}$.

It is clear that the space X is Hausdorff. The Tychonoff property of X will follow from the fact that X is zero-dimensional, i.e. has a base of clopen sets.

Indeed, let $z = (t,0) \in L$. Take any open $U \ni z$. There exists an $n \in \omega$ and a $W \in \mathcal{B}_z^n$ with the following properties:
(i) $z \in W \subset U$ and $W = V \cap Q$ where V is open in $\mathbb{R}^2$;
(ii) the boundary of $V \setminus \{z\}$ (in the Euclidean topology) does not intersect the set Q;
(iii) the closure of $V \setminus \{z\}$ (in the Euclidean topology) does not contain any point from L distinct from z.

It is immediate that the set W is clopen in X, and therefore X is zero-dimensional. Of course, X is separable because $\overline{Q} = X$. Any dense subspace of X has to contain Q, so it is sufficient to establish that X is not normal on Q.

Take any faithful enumeration $\{q_n : n \in \omega\}$ of the set $\mathbb{Q}$ of rational numbers. Denote by $\{V_n : n \in \omega\}$ any family of open circles in the upper half-plane which has the following properties:
(iv) V_n is tangent to L at the point $(q_n, 0)$;
(v) $\overline{V}_n \cap \overline{V}_m = \emptyset$ if $n \neq m$ (the closures are taken in $\mathbb{R}^2$);
(vi) the radii of V_n tend to zero.

Denote by V the set $\bigcup\{V_n : n \in \omega\}$. For each $n \in \omega$ take a sequence $P_n = \{(q_n, r_n^m) : m \in \omega\} \subset V_n \cap Q$ such that $r_n^m \to 0$ when $m \to \infty$. Let $A = \bigcup\{P_n : n \in \omega\}$ and $B = Q \setminus V$. We are going to establish that $\overline{A} \setminus A = \mathbb{Q} \times \{0\}$ and $\overline{B} \setminus B = (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ (from now until the end of the proof the closure is taken in X).

Since $W = V \cup (\mathbb{Q} \times \{0\})$ is open in X and $B \cap W = \emptyset$, we have $\overline{B} \cap (\mathbb{Q} \times \{0\}) = \emptyset$. Now $\mathbb{Q} \times \{0\} \subset \overline{A}$ implies that it suffices to show that $\overline{A} \cap ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\}) = \emptyset$ and $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\} \subset \overline{B}$.

Fix any point $z = (t,0)$ such that $t \in \mathbb{R} \setminus \mathbb{Q}$.

Claim. Suppose that $a, b > 0$ and $a < b$. Then there exists an $\varepsilon = \varepsilon(a,b)$ such that $((t - \varepsilon, t + \varepsilon) \times (a,b)) \cap A = \emptyset$.

Proof of the claim. There are only finitely many points $q_n \in \mathbb{Q}$ such that the radius of V_n is greater than or equal to $\frac{a}{2}$. Therefore there is an $\varepsilon > 0$ for which $(t - \varepsilon, t + \varepsilon)$ does not contain any of such q_n. Taking $\varepsilon(a,b) = \varepsilon$ we obtain what was needed.

Let $\varepsilon_n = \varepsilon(\frac{1}{n+2}, \frac{1}{n+1})$ for each $n \in \omega$. The set

$$U = \{z\} \cup \bigcup\{(t - \varepsilon_n, t + \varepsilon_n) \times (\tfrac{1}{n+2}, \tfrac{1}{n+1}) : n \in \omega\}$$

is an open neighbourhood of the point z and it follows from the claim that $U \cap A = \emptyset$. Therefore $\overline{A} \cap ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\}) = \emptyset$.

Let us establish that $z \in \overline{B}$. If it is not true, then there exists an $n \in \omega$ and a set G open in $\mathbb{R}^2$ such that $T_n = \{(t,y) : 0 < y \leq \frac{1}{n}\} \subset G$ and $G \cap Q \subset V$. It is easy to see that there exists a sequence $\{\varepsilon_m : m \in \omega\}$ of positive reals such that the set

$$H = \bigcup\{(t - \varepsilon_m, t + \varepsilon_m) \times (\tfrac{1}{n+m+1}, \tfrac{1}{n+m}) : m \in \omega\}$$

is contained in G.

There are only finitely many $k \in \omega$ such that $V_k \cap F \neq \emptyset$, where

$$F = (t - \varepsilon_0, t + \varepsilon_0) \times (\tfrac{1}{n+1}, \tfrac{1}{n}).$$

The closures of the sets V_k in $\mathbb{R}^2$ are disjoint and a finite subfamily of the closures of the elements of the family $\{V_k : k \in \omega\}$ cover the connected set F. This means $F \subset V_p$ for some $p \in \omega \setminus \{0\}$.

There exists $m \in \omega$ such that the set

$$F_m = (t - \varepsilon_m, t + \varepsilon_m) \times (\tfrac{1}{n+m+1}, \tfrac{1}{n+m})$$

intersects the closure of V_p but is not contained in it. Applying the same reasoning as in the previous paragraph to the set F_m, we can conclude that $F_m \subset V_q$ for some $q \in \omega$. Clearly, $q \neq p$ because V_q contains F_m and V_p does not. However, V_p meets F_m and hence V_q as well which contradicts the fact that the sets $\{V_k : k \in \omega\}$ are disjoint.

Now that we have the equalities $\overline{A} = A \cup (\mathbb{Q} \times \{0\})$ and $\bar{B} = B \cup ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\})$ it suffices to prove that the sets $\mathbb{Q} \times \{0\}$ and $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ can not be separated by open sets in X. The proof is essentially the same as for the Niemytzki plane: if there are open and disjoint U and V such that $\mathbb{Q} \times \{0\} \subset U$ and $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\} \subset V$, then we can assume that $V = \bigcup\{V_z : z \in (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}\}$, where $V_z \in \mathcal{B}_{n(z)}$ for each $z \in (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$. Since the space $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ has the Baire property in the topology of $\mathbb{R}$, there is an open interval $W \subset \mathbb{R}$ and an $n \in \omega$ such that $\{z \in W \times \{0\} : n(z) = n\}$ is dense in W. This implies that V is dense in the set $W \times (0, \frac{1}{n})$ in the Euclidean topology. If $y \in (\mathbb{Q} \times \{0\}) \cap W$, then any basic open set (in X) containing y has to intersect V. Therefore U and V are not disjoint. $\square$

We have already mentioned that Arhangel'skiĭ has constructed an example of a countable dense subspace of $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is not normal on it. However it is not clear without additional set-theoretic assumptions whether such a subspace exists in $\mathbb{R}^{\omega_1}$. The following group of results is some progress in that direction.

The construction below generalizes the one from [Tk, Example 1].

2.7. General construction. *Let X be any Tychonoff infinite separable space. Given non-empty disjoint regular open subsets U and V of the space X consider the following subspaces of X^{ω_1}:*
$P(U,V) = \{x \in X^{\omega_1} : x(\alpha) \notin U$ for any $\alpha < \omega_1$ and $x(\beta) \in V$ for some $\beta < \omega_1\}$ and
$Q(U,V) = \{x \in X^{\omega_1} : x(\alpha) \notin V$ for any $\alpha < \omega_1$ and $x(\beta) \in U$ for some $\beta < \omega_1\}$.
Then
(i) $\overline{P(U,V)} \cap Q(U,V) = \overline{Q(U,V)} \cap P(U,V) = \emptyset$;
(ii) the subspaces $P(U,V)$ and $Q(U,V)$ are separable;
(iii) $\overline{P(U,V)} \cap \overline{Q(U,V)} = (X \setminus (U \cup V))^{\omega_1}$.

Proof. *(i)* If $x \in P(U,V)$, then there is a $\beta < \omega_1$ such that $x(\beta) \in V$. The set $G_\beta = \{y \in X^{\omega_1} : y(\beta) \in V\}$ is open in X^{ω_1}, contains x and does not intersect $Q(U,V)$. Therefore $\overline{Q(U,V)} \cap P(U,V) = \emptyset$. The equality $\overline{P(U,V)} \cap Q(U,V) = \emptyset$ can be proved analogously.

(ii) Observe that $P(U,V) = F \cap G$, where $F = (X \setminus U)^{\omega_1}$ and $G = \bigcup\{G_\beta : \beta < \omega_1\}$ (see (i)). Since U is a regular open set, the space $X \setminus U$ is separable which implies the separability of F [En, Corollary 2.3.16]. The set G is open in X^{ω_1} so that $P(U,V)$ is an open set in a separable space F. Hence $P(U,V)$ is separable.

(iii) Since $P(U,V) \subset F$ (see (ii)), we have $\overline{P(U,V)} \subset F = (X \setminus U)^{\omega_1}$ because F is closed in X^{ω_1}. Analogously, $\overline{Q(U,V)} \subset (X \setminus V)^{\omega_1}$. As a consequence, we have $\overline{P(U,V)} \cap \overline{Q(U,V)} \subset (X \setminus (U \cup V))^{\omega_1}$.

On the other hand, given $x \in (X \setminus (U \cup V))^{\omega_1}$ and a finite $A \subset \omega_1$, take any $y \in X^{\omega_1}$ with $y(\alpha) = x(\alpha)$ for every $\alpha \in A$ and $y(\alpha) = y_0 \in V$ for each $\alpha \in \omega_1 \setminus A$. Then $y \in P(U,V)$ and $y|_A = x|_A$. This proves that $x \in \overline{P(U,V)}$. The proof of $x \in \overline{Q(U,V)}$ is analogous.

□

2.8. Examples. (1) *There is a dense open subspace X of $\mathbb{R}^{\omega_1}$ which is not normal on a countable dense set.*
(2) *There is a dense open (and hence locally compact) subspace Y of the Tychonoff cube I^{ω_1} which is not normal on a countable dense subspace.*

Proof. Take $U = (0, \frac{1}{3})$ and $V = (\frac{2}{3}, 1)$. The sets U and V are disjoint and regular open in I as well as in $\mathbb{R}$. This makes it possible to apply 2.7 to construct the sets $P(U,V)$ and $Q(U,V)$ in $\mathbb{R}^{\omega_1}$ or in I^{ω_1} respectively.

It follows from 2.7(iii) that the set $T = \overline{P(U,V)} \cap \overline{Q(U,V)}$ is nowhere dense in $\mathbb{R}^{\omega_1}$ or I^{ω_1} depending on which is considered. So let $X = \mathbb{R}^{\omega_1} \setminus T$ if we are proving (1) and $Y = I^{\omega_1} \setminus T$ if (2) is under consideration.

Let $A = \overline{P(U,V)} \cap X$ and $B = \overline{Q(U,V)} \cap X$ (the closures are taken in $\mathbb{R}^{\omega_1}$). The sets A and B are separable, closed and disjoint in X. By Proposition 2.3, to establish that X is not normal on some countable dense subspace it suffices to prove that A and B can not be separated by open sets in X. Since X is open in $\mathbb{R}^{\omega_1}$ our task reduces to proving that $P(U,V)$ and $Q(U,V)$ can not be separated by open disjoint subsets of $\mathbb{R}^{\omega_1}$.

Suppose that there exist open and disjoint $G, H \subset \mathbb{R}^{\omega_1}$ such that $P(U,V) \subset G$ and $Q(U,V) \subset H$. Without loss of generality we can assume that G and H are regular open. But then each one of them depends on countably many coordinates i.e. there is a countable $D \subset \omega_1$ such that $G = \pi_D^{-1}\pi_D(G)$ and $H = \pi_D^{-1}\pi_D(H)$ [RS]. In particular, the open sets $\pi_D(G)$ and $\pi_D(H)$ are disjoint. Since $\pi_D(P(U,V)) \subset \pi_D(G)$ and $\pi_D(Q(U,V)) \subset \pi_D(H)$ we have $\overline{\pi_D(P(U,V))} \cap \pi_D(Q(U,V)) = \emptyset$ which gives a contradiction because the point $x \in \mathbb{R}^D$ with $x(\alpha) = 0$ for all $\alpha \in D$ belongs to $\pi_D(P(U,V)) \cap \pi_D(Q(U,V))$.

The proof that X is not normal on a countable dense subset is complete. The same reasoning applied to I^{ω_1} establishes the same about Y.

□

3. Unsolved problems. All results obtained here were inspired by Arhangel'skiĭ's questions and investigations on the topic. We formulate here some problems of Arhangel'skiĭ which were posed at his seminar held in Moscow State University.

3.1. Problem. *Does there exist in ZFC a countable dense subspace X of $\mathbb{R}^{\mathfrak{c}}$ such that $\mathbb{R}^{\mathfrak{c}}$ is normal on X?*

3.2. Problem. *Does there exist in ZFC a countable dense subspace X of $\mathbb{R}^{\omega_1}$ such that $\mathbb{R}^{\omega_1}$ is not normal on X?*

3.3. Problem. *Does there exist a dense separable subspace X of $\mathbb{R}^{\mathfrak{c}}$ such that X is normal on any countable dense subspace?*

3.4. Problem. *Is it true that the Niemytzki plane is not normal on any countable dense subspace?*

3.5. Problem. *Is $\mathbb{R}^{[0,1]}$ normal on the set of rational polinomials on $[0,1]$?*

References

[Ar1] A.V. Arhangel'skiĭ, *Relative topological properties and relative topological spaces,* Topology and Its Applications, 1996, 70, N 1, 87–99.

[Ar2] A.V. Arhangel'skiĭ, *Normality and dense subspaces,* preprint.

[En] R. Engelking, *General Topology,* PWN, Warszawa, 1977.

[RS] K.A. Ross, A.H. Stone, *Products of separable spaces*, American Mathematical Monthly, 1964, 71, 398–403.

[Sh] D.B. Shakhmatov, *A pseudocompact Tychonoff space all countable subsets of which are closed and C^*-embedded,* Topology and Its Applications, 1986, 22, N 2, 139–144.

[Tk] M.G. Tkačenko, *Some results on inverse spectra I,* Comment. Math. Univ. Carolinae, 1981, 22, N 3, 621–633.

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