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A CHARACTERIZATION OF POINTWISE MULTIPLIERS ON THE MORREY SPACES

EIICHI NAKAI

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Dedicated to the memory of Professor Hiroaki Takehana

ABSTRACT. Let L_{p_i,ϕ_i} $(i=1,2,3)$ be Morrey spaces. A function g is called a pointwise multiplier from L_{p_1,ϕ_1} to L_{p_2,ϕ_2}, if the pointwise product fg belongs to L_{p_2,ϕ_2} for each $f \in L_{p_1,\phi_1}$. We denote by $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2})$ the set of all pointwise multipliers from L_{p_1,ϕ_1} to L_{p_2,ϕ_2}. A sufficient condition on p_i and ϕ_i $(i=1,2,3)$ for $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2}) = L_{p_3,\phi_3}$ was given in [9]. In this paper, we give a necessary condition. In connection with these conditions, we also give sufficient conditions for $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2}) = \{0\}$.

1. Introduction

The theory of the Morrey spaces has been developed by many authors, Peetre [10], [11], Adams [1], Chiarenza and Frasca [3], Mizuhara [4], Arai and Mizuhara [2], etc. We investigate pointwise multipliers on the Morrey spaces.

Let E and F be spaces of real- or complex-valued functions defined on a set X. A function g defined on X is called a pointwise multiplier from E to F, if the pointwise product fg belongs to F for each $f \in E$. We denote by $PWM(E,F)$ the set of all pointwise multipliers from E to F.

L^p-spaces $(0 < p \le \infty)$ on a measure space X are complete quasi-normed linear spaces. If $1 \le p \le \infty$, then they are Banach spaces. If X is σ-finite, and if $1/p_1 + 1/p_3 = 1/p_2$, then it is known that

$$PWM(L^{p_1}(X), L^{p_2}(X)) = L^{p_3}(X) \quad \text{and} \quad \|g\|_{\mathrm{Op}} = \|g\|_{L^{p_3}}, \tag{1.1}$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in PWM(L^{p_1}(X), L^{p_2}(X))$, i.e.

$$\|g\|_{\mathrm{Op}} = \inf\{\beta > 0 : \|fg\|_{L^{p_2}} \le \beta \|f\|_{L^{p_1}} \text{ for all } f \in L^{p_1}(X)\}.$$

On some assumptions the equalities in (1.1) were generalized to the Morrey spaces $L_{p,\phi}(X)$ in [9], where $0 < p \le \infty$, $\phi : X \times (0,+\infty) \to (0,+\infty)$ and X is a space of homogeneous type. In this paper, we consider the necessity of the assumptions in the case of $\phi : (0,+\infty) \to (0,+\infty)$ and $X = \mathbb{R}^n$. Many authors studied in this case.

For $a \in \mathbb{R}^n$ and $r > 0$, let $B(a,r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$. For a measurable set $E \subset \mathbb{R}^n$, we denote by $|E|$ the Lebesgue measure of E. For $0 < p \le \infty$ and $\phi : (0,+\infty) \to (0,+\infty)$, let

$$L_{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{p,\phi} < +\infty \right\},$$

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where

$$\|f\|_{p,\phi} = \begin{cases} \sup_{B(a,r)} \dfrac{1}{\phi(r)} \left(\dfrac{1}{|B(a,r)|} \displaystyle\int_{B(a,r)} |f(x)|^p \, dx \right)^{1/p}, & 0 < p < \infty, \\ \sup_{B(a,r)} \dfrac{1}{\phi(r)} \operatorname*{ess\,sup}_{x \in B(a,r)} |f(x)|, & p = \infty. \end{cases}$$

Then

$$L_{\infty,\phi}(\mathbb{R}^n) = \begin{cases} \{0\}, & \inf_{0<r<\infty} \phi(r) = 0, \\ L^\infty(\mathbb{R}^n), & \inf_{0<r<\infty} \phi(r) > 0. \end{cases}$$

If $\inf_{0<r<\infty} \phi(r) > 0$, then $\|f\|_{\infty,\phi} = \|f\|_{L^\infty}/(\inf_{0<r<\infty} \phi(r))$.

For $\phi(r) = r^{(\lambda-n)/p}$ $(0 < p < \infty,\ 0 < \lambda < n)$, let $L^{p,\lambda}(\mathbb{R}^n) = L_{p,\phi}(\mathbb{R}^n)$. Then $L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces introduced in [5]. If $\phi(r) = r^{-n/p}$, then $L_{p,\phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\phi(r) \equiv 1$, then $L_{p,\phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

The function space $L_{p,\phi}(\mathbb{R}^n)$ is a complete quasi-normed linear space. If $1 \le p \le \infty$, then it is a Banach space.

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that $\theta(r) \le C\theta(s)$ $(\theta(r) \ge C\theta(s))$ for $r \le s$.

If $\inf_{t \le r} \phi(t) = 0$ for some $r > 0$, then $L_{p,\phi}(\mathbb{R}^n) = \{0\}$. Let $\inf_{t \le r} \phi(t) > 0$ for every $r > 0$ and $\psi(r) = \inf_{t \le r} \phi(t)$. Then ψ is decreasing and $L_{p,\phi}(\mathbb{R}^n) = L_{p,\psi}(\mathbb{R}^n)$ with equivalent norms.

If $\inf_{t \ge r} \phi(t) t^{n/p} = 0$ for some $r > 0$, then $L_{p,\phi}(\mathbb{R}^n) = \{0\}$. Let $\inf_{t \ge r} \phi(t) t^{n/p} > 0$ for every $r > 0$ and $\psi(r) = r^{-n/p} \inf_{t \ge r} \phi(t) t^{n/p}$. Then $\psi(r) r^{n/p}$ is increasing and $L_{p,\phi}(\mathbb{R}^n) = L_{p,\psi}(\mathbb{R}^n)$ with equivalent norms.

To consider pointwise multipliers from $L_{p_1,\phi_1}(\mathbb{R}^n)$ to $L_{p_2,\phi_2}(\mathbb{R}^n)$, we may assume that ϕ_i $(i = 1, 2)$ are almost decreasing and $\phi_i(r) r^{n/p_i}$ $(i = 1, 2)$ are almost increasing.

If ϕ is almost decreasing and $\phi(r) r^{n/p}$ is almost increasing, then ϕ satisfies

$$\frac{1}{A} \le \frac{\phi(s)}{\phi(r)} \le A \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.2}$$

where $A > 0$ is independent of $r, s > 0$.

In the case of $\phi : (0, +\infty) \to (0, +\infty)$ and $X = \mathbb{R}^n$, Theorems 2.1, 2.2 and 2.3 in [9] can be stated as follows:

Theorem 1.1. *Let $0 < p_2 \le p_1 \le \infty$ and $1/p_1 + 1/p_3 = 1/p_2$. Suppose that ϕ_i $(i = 1, 2)$ are almost decreasing. Let $\phi_3 = \phi_2/\phi_1$. If $\phi_1(r) r^{n/p_1}$ and ${\phi_2}^{p_2/p_1}/\phi_1$ (ϕ_2/ϕ_1 when $p_1 = p_2 = \infty$, $1/\phi_1$ when $p_2 < p_1 = \infty$) are almost increasing, and if $\liminf_{r \to 0} \phi_3(r) > 0$, then*

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = L_{p_3,\phi_3}(\mathbb{R}^n). \tag{1.3}$$

Moreover, the operator norm of $g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n))$ is comparable to $\|g\|_{p_3,\phi_3}$. If ${\phi_2}^{p_2/p_1}/\phi_1 = 1$, then the operator norm is equal to $\|g\|_{p_3,\phi_3}$.

Remark 1.1. If $\phi_1(r) r^{n/p_1}$ and ${\phi_2}^{p_2/p_1}/\phi_1$ are almost increasing, so are $\phi_2(r) r^{n/p_2}$ and $\phi_3(r) r^{n/p_3}$.

In this paper, we show that the almost increasingness of ${\phi_2}^{p_2/p_1}/\phi_1$ is a necessary and sufficient condition for (1.3) when $p_2 \le p_1$, and that if $p_2 > p_1$ or if $\liminf_{r \to 0} \phi_3(r) = 0$ then

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = \{0\}.$$

It turns out that Theorem 1.1 holds without the condition $\liminf_{r \to 0} \phi_3(r) > 0$ in the case of $\phi : (0, +\infty) \to (0, +\infty)$.

We state main results in the next section. Section 3 is for the preliminaries. In section 4 we give proofs of the results.

The letters C will always denote a positive constant, not necessarily the same one.

For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r), \quad r > 0.$$

2. Main results

Theorem 2.1. *Let* $0 < p_2 \le p_1 \le \infty$. *Suppose that* ϕ_i $(i = 1, 2)$ *are almost decreasing and that* $\phi_i(r)r^{n/p_i}$ $(i = 1, 2)$ *are almost increasing. Then*

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = L_{p_3,\phi_3}(\mathbb{R}^n),$$

if and only if ${\phi_2}^{p_2/p_1}/\phi_1$ (ϕ_2/ϕ_1 *when* $p_1 = p_2 = \infty$, $1/\phi_1$ *when* $p_2 < p_1 = \infty$) *is almost increasing, where* $1/p_3 = 1/p_2 - 1/p_1$ *and* $\phi_3 = \phi_2/\phi_1$. *In this case, the operator norm of* $g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n))$ *is comparable to* $\|g\|_{p_3,\phi_3}$.

Remark 2.1. When $p_1 = p_2 = \infty$, $\phi_i(r) = \phi_i(r)r^{n/p_i}$ $(i = 1, 2)$ are almost decreasing and almost increasing, i.e. $\phi_1 \sim \phi_2 \sim 1$ and ϕ_2/ϕ_1 is almost increasing. When $p_2 < p_1 = \infty$, $\phi_1(r) = \phi_1(r)r^{n/p_1}$ is almost decreasing and almost increasing, i.e. $\phi_1 \sim 1$ and $1/\phi_1$ is almost increasing.

Remark 2.2. In general, from Lemma 3.1 it follows that

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) \supset L_{p_3,\phi_3}(\mathbb{R}^n).$$

Theorem 2.2. *Suppose that* ϕ_1 *is almost decreasing, that* $\phi_1(r) \to +\infty$ *as* $r \to 0$ *and that* $\phi_1(r)r^{n/p_1}$ *is almost increasing. If* $0 < p_1 < p_2 < \infty$, *then*

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = \{0\}.$$

Lemma 2.3. *Let* $0 < p_1, p_2 < \infty$. *Suppose that* ϕ_1 *is almost decreasing and that* $\phi_1(r)r^{n/p_1}$ *is almost increasing. If* $\liminf_{r\to 0} \phi_2(r)/\phi_1(r) = 0$, *then*

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = \{0\}.$$

Remark 2.3. If

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = \{0\},$$

then there is a function $f \in L_{p_1,\phi_1}(\mathbb{R}^n)$ such that $f \notin L_{p_2,\phi_2}(\mathbb{R}^n)$, since a constant function 1 is not a pointwise multiplier.

For the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, we have the following.

Corollary 2.4. *Let $0 < p_i < \infty$ and $0 < \lambda_i < n$ $(i = 1, 2)$. Then*

$$PWM(L^{p_1,\lambda_1}(\mathbb{R}^n), L^{p_2,\lambda_2}(\mathbb{R}^n)) \begin{cases} = \{0\}, & p_1 < p_2, \\ = \{0\}, & p_1 = p_2 \text{ and } \lambda_1 < \lambda_2, \\ = L^\infty(\mathbb{R}^n), & p_1 = p_2 \text{ and } \lambda_1 = \lambda_2, \\ \supsetneq \{0\}, & p_1 = p_2 \text{ and } \lambda_1 > \lambda_2, \\ = \{0\}, & p_1 > p_2 \text{ and } n + (\lambda_1 - n)p_2/p_1 < \lambda_2, \\ = L^\infty(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_2 = n + (\lambda_1 - n)p_2/p_1, \\ = L^{p_3,\lambda_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_1 \le \lambda_2 < n + (\lambda_1 - n)p_2/p_1, \\ \supsetneq L^{p_3,\lambda_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_1 p_2/p_1 < \lambda_2 < \lambda_1, \\ \supsetneq L^{p_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_2 = \lambda_1 p_2/p_1, \\ \supsetneq \{0\}, & p_1 > p_2 \text{ and } \lambda_2 < \lambda_1 p_2/p_1, \end{cases}$$

where $p_3 = p_1p_2/(p_1 - p_2)$ and $\lambda_3 = (p_1\lambda_2 - p_2\lambda_1)/(p_1 - p_2)$.

3. Preliminaries

We state lemmas to prove the theorems.

Lemma 3.1 ([9])**.** *Let $0 < p_2 \le p_1 \le \infty$, $1/p_1 + 1/p_3 = 1/p_2$ and $\phi_1\phi_3 = \phi_2$. Then*

$$\|fg\|_{p_2,\phi_2} \le \|f\|_{p_1,\phi_1}\|g\|_{p_3,\phi_3}.$$

Lemma 3.2 ([9])**.** *Let $0 < p < \infty$. Suppose that ϕ satisfies* (1.2) *and $\phi(r)r^{n/p}$ is almost increasing. If* $\operatorname{supp} f \subset B(a, r)$ *and if*

$$\sup_{B(b,s)\subset B(a,3r)} \frac{1}{\phi(s)} \left(\frac{1}{|B(b,s)|} \int_{B(b,s)} |f(x)|^p \, dx \right)^{1/p} \le M,$$

then

$$f \in L_{p,\phi}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{p,\phi} \le CM,$$

where $C > 0$ independent of f, $B(a, r)$ and M.

Lemma 3.3. *Let $0 < p < \infty$, ϕ be almost decreasing and $\phi(r)r^{n/p}$ be almost increasing. If f is the characteristic function of the ball of radius $r > 0$, then*

$$\|f\|_{p,\phi} \sim \frac{1}{\phi(r)}.$$

Proof. For all balls $B(b, s) \subset B(a, 3r)$, we have

$$\frac{1}{\phi(s)} \left(\frac{1}{|B(b,s)|} \int_{B(b,s)} |f(x)|^p \, dx \right)^{1/p} \le \frac{1}{\phi(s)} \le \frac{C}{\phi(r)}.$$

By Lemma 3.2 we obtain the desired result. □

Lemma 3.4. *Let $0 < p < \infty$, ϕ be almost decreasing and $\phi(r)r^{n/p}$ be almost increasing. Let $\{B(a_j, r_j)\}_{j=0}^{\infty}$ be balls and $\{f_j\}_{j=1}^{\infty}$ be functions such that*

$$\begin{cases} B(a_j, r_j) \subset B(a_0, r_0), \\ \operatorname{supp} f_j \subset B(a_j, r_j/3), \\ \|f_j\|_{L^p} \le c_1 \phi(r_0) {r_j}^{n/p}, \\ \|f_j\|_{p,\phi} \le c_2, \end{cases} \quad \text{for} \quad j = 1, 2, \cdots,$$

$$B(a_i, r_i) \cap B(a_j, r_j) = \emptyset \quad \text{for} \quad i, j = 1, 2, \cdots; i \ne j.$$

Then $f = \sum_{j=1}^{\infty} f_j$ is in $L_{p,\phi}(\mathbb{R}^n)$ and $\|f\|_{p,\phi} \le C(c_1 + c_2)$.

Proof. For any ball $B(a, r) \subset B(a_0, 3r_0)$, let

$$J_1 = \{j \in \mathbb{N} : B(a, r) \cap B(a_j, r_j/3) \ne \emptyset, r_j/3 \le r\},$$
$$J_2 = \{j \in \mathbb{N} : B(a, r) \cap B(a_j, r_j/3) \ne \emptyset, r_j/3 > r\},$$

and

$$I_i = \frac{1}{\phi(r)} \left(\frac{1}{|B(a, r)|} \int_{B(a,r)} \left| \sum_{j \in J_i} f_j(x) \right|^p dx \right)^{1/p}, \quad i = 1, 2.$$

If $j \in J_1$, then $B(a_j, r_j/3) \subset B(a, 3r)$. It follows that

$$\bigcup_{j \in J_1} B(a_j, r_j/3) \subset B(a, 3r) \quad \text{and} \quad \sum_{j \in J_1} (r_j/3)^n \le (3r)^n.$$

Hence

$$I_1 \le \frac{1}{\phi(r)} \left(\frac{1}{|B(a, r)|} \sum_{j \in J_1} \int_{B(a_j, r_j/3)} |f_j(x)|^p \, dx \right)^{1/p} \le \frac{c_1 \phi(r_0)}{\phi(r)} \left(\frac{1}{|B(a, r)|} \sum_{j \in J_1} {r_j}^n \right)^{1/p} \le C c_1.$$

If $j \in J_2$, then $B(a, r) \subset B(a_j, r_j)$, i.e. J_2 has only one element. Hence

$$I_2 \le \frac{1}{\phi(r)} \left(\frac{1}{|B(a, r)|} \int_{B(a,r)} |f_j(x)|^p \, dx \right)^{1/p} \le \|f_j\|_{p,\phi} \le c_2.$$

By Lemma 3.2 we have $\|f\|_{p,\phi} \le C(c_1 + c_2)$. □

4. Proofs

First, we state a proof of Lemma 2.3.

Proof of Lemma 2.3. There are positive numbers $\{r_j\}_{j=1}^{\infty}$ such that

$$r_j \to 0 \quad \text{and} \quad \phi_2(r_j)/\phi_1(r_j) \to 0 \quad \text{as} \quad j \to +\infty.$$

If $g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n))$, then the closed graph theorem shows that g is a bounded operator. For any $a \in \mathbb{R}^n$, let f be the characteristic function of the ball $B(a, r_j)$.

By Lemma 3.3 we have

$$\left(\frac{1}{|B(a,r_j)|}\int_{B(a,r_j)}|g(x)|^{p_2}\,dx\right)^{1/p_2} \le \phi_2(r_j)\|fg\|_{p_2,\phi_2}$$
$$\le \phi_2(r_j)\|f\|_{p_1,\phi_1}\|g\|_{\mathrm{Op}} \le C\frac{\phi_2(r_j)}{\phi_1(r_j)}\|g\|_{\mathrm{Op}} \to 0 \quad \text{as} \quad j\to+\infty.$$

Therefore $g(a)=0$ a.e. $a\in\mathbb{R}^n$. □

Let $I(a,r)$ be the cube $\{x\in\mathbb{R}^n : |x_i-a_i|\le r/2, i=1,2,\cdots,n\}$ whose edges have length r and are parallel to the coordinate axes.

Proof of Theorem 2.1. If $\liminf_{r\to 0}\phi_3(r) > 0$, then the sufficiency follows from Theorem 1.1. If $\liminf_{r\to 0}\phi_3(r) = 0$, then $L_{p_3,\phi_3}(\mathbb{R}^n)=\{0\}$. From Lemma 2.3 it follows that

$$PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = L_{p_3,\phi_3}(\mathbb{R}^n).$$

If ${\phi_2}^{p_2/p_1}/\phi_1$ is not almost increasing, then, for all $k\in\mathbb{N}$, there are positive real numbers r_k and s_k such that

$$s_k<r_k \quad \text{and} \quad \phi_2(s_k)^{p_2/p_1}/\phi_1(s_k) \ge 4^{k/p_2}\phi_2(r_k)^{p_2/p_1}/\phi_1(r_k).$$

Let

$$m_k=\left[\frac{r_k\phi_2(r_k)^{p_2/n}}{s_k\phi_2(s_k)^{p_2/n}}\right]+1,$$

where $[\alpha]$ denotes the integer part of the positive real number α. By almost increasingness of $\phi_2(r)r^{n/p_2}$ and by almost decreasingness of ϕ_2, we have

$$m_k \sim \frac{r_k\phi_2(r_k)^{p_2/n}}{s_k\phi_2(s_k)^{p_2/n}}, \tag{4.1}$$

$$\frac{r_k}{m_k} > 6c_0 s_k \quad \text{for some } c_0>0. \tag{4.2}$$

Let $\mathrm{O}\in\mathbb{R}^n$ be the origin. We divide the cube $I(\mathrm{O},r_k)$ into ${m_k}^n$ sub-cubes $I(b_{k,j},r_k/m_k)$: $j=1,2,\cdots,{m_k}^n$, i.e.

$$I(\mathrm{O},r_k)=\bigcup_{j=1}^{{m_k}^n} I(b_{k,j},r_k/m_k),$$
$$I(b_{k,i},r_k/m_k)^\circ \cap I(b_{k,j},r_k/m_k)^\circ=\emptyset \quad \text{for} \quad i\ne j,$$

where $I(b,r)^\circ$ is the interior of $I(b,r)$. Let

$$g_{k,j}(x)=\begin{cases}\phi_3(c_0s_k), & x\in B(b_{k,j},c_0s_k),\\ 0, & x\notin B(b_{k,j},c_0s_k),\end{cases}$$
$$g_k=\sum_{j=1}^{{m_k}^n} g_{k,j}, \quad g=\left(\sum_{k=1}^{\infty}\frac{1}{2^k}(g_k)^{p_2}\right)^{1/p_2}.$$

We show

$$g\in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n))\setminus L_{p_3,\phi_3}(\mathbb{R}^n).$$

We note that

$$\operatorname{supp} g_{k,j}\subset B(b_{k,j},r_k/(6m_k)),$$
$$B(b_{k,j},r_k/(2m_k))\subset I(b_{k,j},r_k/m_k).$$

For all $f \in L_{p_1,\phi_1}(\mathbb{R}^n)$, by Hölder's inequality, (1.2) and (4.1) we have

$$
\begin{aligned}
(4.3)\quad & \|fg_{k,j}\|_{L^{p_2}} \\
&\le \begin{cases} \left(\int_{B(b_{k,j},c_0s_k)} |f(x)|^{p_1}\,dx\right)^{1/p_1} \left(\int_{B(b_{k,j},c_0s_k)} |g_{k,j}(x)|^{p_3}\,dx\right)^{1/p_3} & \text{when } p_1 > p_2 \\ \left(\int_{B(b_{k,j},c_0s_k)} |f(x)|^{p_1}\,dx\right)^{1/p_1} \phi_3(c_0s_k) & \text{when } p_1 = p_2 \end{cases} \\
&\le \begin{cases} \phi_1(c_0s_k)|B(b_{k,j},c_0s_k)|^{1/p_1}\|f\|_{p_1,\phi_1}\phi_3(c_0s_k)|B(b_{k,j},c_0s_k)|^{1/p_3} & \text{when } p_1 > p_2 \\ \phi_1(c_0s_k)|B(b_{k,j},c_0s_k)|^{1/p_1}\|f\|_{p_1,\phi_1}\phi_3(c_0s_k) & \text{when } p_1 = p_2 \end{cases} \\
&= \phi_2(c_0s_k)|B(b_{k,j},c_0s_k)|^{1/p_2}\|f\|_{p_1,\phi_1} \\
&\sim \phi_2(s_k){s_k}^{n/p_2}\|f\|_{p_1,\phi_1} \sim \phi_2(r_k)(r_k/m_k)^{n/p_2}\|f\|_{p_1,\phi_1}.
\end{aligned}
$$

By Lemma 3.1 and Lemma 3.3 we have

$$\|fg_{k,j}\|_{p_2,\phi_2} \le \|f\|_{p_1,\phi_1}\|g_{k,j}\|_{p_3,\phi_3} \le C\|f\|_{p_1,\phi_1}.$$

By Lemma 3.4 we have fg_k is in L_{p_2,ϕ_2} and $\|fg_k\|_{p_2,\phi_2} \le C_0\|f\|_{p_1,\phi_1}$. This implies

$$\int_{B(a,r)} |f(x)g_k(x)|^{p_2}\,dx \le |B(a,r)|\left(\phi(r)C_0\|f\|_{p_1,\phi_1}\right)^{p_2} \quad \text{for each ball} \quad B(a,r).$$

From Beppo-Levi's theorem it follows that

$$\sum_{k=1}^{\infty} \frac{1}{2^k}|f(x)g_k(x)|^{p_2} = |f(x)|^{p_2}\sum_{k=1}^{\infty}\frac{1}{2^k}g_k(x)^{p_2}$$

converges a.e. $x \in B(a,r)$ and

$$\int_{B(a,r)} |f(x)g(x)|^{p_2}\,dx \le |B(a,r)|\left(\phi(r)C_0\|f\|_{p_1,\phi_1}\right)^{p_2} \quad \text{for each ball} \quad B(a,r).$$

Hence we have

$$\|fg\|_{p_2,\phi_2} \le C_0\|f\|_{p_1,\phi_1},$$

and

$$g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)).$$

On the other hand, since $\operatorname{supp} g_k \subset B(\mathrm{O}, \sqrt{n}r_k/2)$,

$$
\begin{aligned}
&\frac{1}{\phi_3(\sqrt{n}r_k/2)}\left(\frac{1}{|B(\mathrm{O},\sqrt{n}r_k/2)|}\int_{B(\mathrm{O},\sqrt{n}r_k/2)} |g(x)|^{p_3}\,dx\right)^{1/p_3} \\
&\quad \ge \frac{1}{\phi_3(\sqrt{n}r_k/2)}\left(\frac{1}{|B(\mathrm{O},\sqrt{n}r_k/2)|}\int_{B(\mathrm{O},\sqrt{n}r_k/2)} \left|\frac{1}{2^{k/p_2}}g_k(x)\right|^{p_3}\,dx\right)^{1/p_3} \\
&\quad \sim \frac{\phi_3(s_k)(m_ks_k)^{n/p_3}}{2^{k/p_2}\phi_3(r_k)(r_k)^{n/p_3}} = \left(\frac{m_ks_k\phi_2(s_k)^{p_2/n}}{r_k\phi_2(r_k)^{p_2/n}}\right)^{n/p_3}\frac{\phi_2(s_k)^{p_2/p_1}/\phi_1(s_k)}{2^{k/p_2}\phi_2(r_k)^{p_2/p_1}/\phi_1(r_k)} \\
&\quad \ge 2^{k/p_2} \quad \text{for all } k \in \mathbb{N}, \quad \text{when } p_1 > p_2,
\end{aligned}
$$

and

$$\frac{1}{\phi_3(\sqrt{n}r_k/2)} \operatorname{ess\,sup}\{g(x) : x \in B(\mathrm{O}, \sqrt{n}r_k/2)\}$$
$$\geq \frac{1}{\phi_3(\sqrt{n}r_k/2)} \operatorname{ess\,sup}\left\{\frac{1}{2^{k/p_2}} g_k(x) : x \in B(\mathrm{O}, \sqrt{n}r_k/2)\right\}$$
$$\sim \frac{\phi_3(s_k)}{2^{k/p_2}\phi_3(r_k)} = \frac{\phi_2(s_k)/\phi_1(s_k)}{2^{k/p_2}\phi_2(r_k)/\phi_1(r_k)}$$
$$\geq 2^{k/p_2} \quad \text{for all } k \in \mathbb{N}, \quad \text{when } p_1 = p_2.$$

This shows $g \notin L_{p_3,\phi_3}(\mathbb{R}^n)$. □

Remark 4.1. If $\{r_k\}_{k=1}^{\infty}$ in the above proof is bounded, then the support of g is compact.

Proof of Theorem 2.2. Let $\{s_k\}_{k=1}^{\infty}$ be positive real numbers such that $s_k > s_{k+1}$ ($k = 1, 2, \cdots$) and $s_k \to 0$ as $k \to +\infty$. Let

$$l_k = \left[\left(\phi_1(s_k)^{p_1/n} s_k\right)^{-1}\right] + 1, \quad m_k = \left[\phi_1(s_k)^{p_1/n}\right] + 1.$$

Then, by the almost increasingness of $\phi_1(r)r^{n/p_1}$ and the almost decreasingness of ϕ_1, we have

$$l_k \sim \left(\phi_1(s_k)^{p_1/n} s_k\right)^{-1}, \quad m_k \sim \phi_1(s_k)^{p_1/n}.$$

Hence

$$\phi_1(1/(l_k m_k))^{p_1}/{m_k}^n \sim \phi_1(s_k)^{p_1} {s_k}^n {l_k}^n \sim 1 \tag{4.4}$$

$$\begin{aligned}\phi_1(1/(l_k m_k))^{p_2}/{m_k}^n &\sim \phi_1(s_k)^{p_2} {s_k}^n {l_k}^n \\ &\sim \phi_1(s_k)^{p_2-p_1} \to +\infty \quad \text{as} \quad k \to +\infty.\end{aligned} \tag{4.5}$$

For any fixed $a_0 \in \mathbb{R}^n$, we divide the cube $I(a_0, 1)$ into ${l_k}^n$ sub-cubes $I(b_{k,j}, 1/l_k)$: $j = 1, 2, \cdots, {l_k}^n$, i.e.

$$I(a_0, 1) = \bigcup_{j=1}^{{l_k}^n} I(b_{k,j}, 1/l_k),$$
$$I(b_{k,j}, 1/l_k)^\circ \cap I(b_{k,j'}, 1/l_k)^\circ = \emptyset \quad \text{for} \quad j \neq j'.$$

We divide the cube $I(\mathrm{O}, 1/l_k)$ into ${m_k}^n$ sub-cubes $I(e_{k,i}, 1/(l_k m_k))$: $i = 1, 2, \cdots, {m_k}^n$, i.e.

$$I(\mathrm{O}, 1/l_k) = \bigcup_{i=1}^{{m_k}^n} I(e_{k,i}, 1/(l_k m_k)),$$
$$I(e_{k,i}, 1/(l_k m_k))^\circ \cap I(e_{k,i'}, 1/(l_k m_k))^\circ = \emptyset \quad \text{for} \quad i \neq i'.$$

Then

$$\begin{cases} I(b_{k,j}, 1/l_k) = \displaystyle\bigcup_{i=1}^{{m_k}^n} I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \\ I(b_{k,j} + e_{k,i}, 1/(l_k m_k))^\circ \cap I(b_{k,j} + e_{k,i'}, 1/(l_k m_k))^\circ = \emptyset \quad \text{for } i \neq i', \end{cases}$$
$$j = 1, 2, \cdots, {l_k}^n.$$

Let

$$f_{k,j,i}(x) = \begin{cases} \phi_1(1/(l_k m_k)), & x \in I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \\ 0, & x \notin I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \end{cases}$$

$$f_{k,i} = \sum_{j=1}^{{l_k}^n} f_{k,j,i}.$$

First we show

$$\|f_{k,i}\|_{p_1,\phi_1} \le C, \quad \text{for } i = 1, 2, \cdots, {m_k}^n, \text{ and for all } k \in \mathbb{N}. \tag{4.6}$$

We note that

$$B(b_{k,j} + e_{k,i}, 1/(2l_k)) \subset I(b_{k,j} + e_{k,i}, 1/l_k) \subset I(a_0, 2) \subset B(a_0, \sqrt{n}),$$
$$B(b_{k,j} + e_{k,i}, 1/(2l_k)) \cap B(b_{k,j'} + e_{k,i}, 1/(2l_k)) = \emptyset \quad \text{for } j \ne j'.$$

Since $\phi_1(s_k) \to +\infty$ as $k \to \infty$, we may assume $m_k \ge 3\sqrt{n}$. Hence

$$\begin{aligned} \operatorname{supp} f_{k,j,i} &= I(b_{k,j} + e_{k,i}, 1/(l_k m_k)) \\ &\subset B(b_{k,j} + e_{k,i}, \sqrt{n}/(2l_k m_k)) \subset B(b_{k,j} + e_{k,i}, 1/(6l_k)). \end{aligned}$$

By (4.4) we have

$$\|f_{k,j,i}\|_{L^{p_1}} \le \phi_1(1/(l_k m_k))(1/(l_k m_k))^{n/p_1} \sim (1/l_k)^{n/p_1}.$$

By Lemma 3.3 we have

$$\|f_{k,j,i}\|_{p_1,\phi_1} \le C.$$

Hence, by Lemma 3.4 we have (4.6).

If $g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n))$, then the closed graph theorem shows that g is a bounded operator. Hence

$$\begin{aligned} &\frac{\phi_1(1/(l_k m_k))}{\phi_2(\sqrt{n})} \left(\frac{1}{|B(a_0, \sqrt{n})|} \int_{\operatorname{supp} f_{k,i}} |g(x)|^{p_2}\, dx \right)^{1/p_2} \\ &\qquad = \frac{1}{\phi_2(\sqrt{n})} \left(\frac{1}{|B(a_0, \sqrt{n})|} \int_{B(a_0,\sqrt{n})} |f_{k,i}(x) g(x)|^{p_2}\, dx \right)^{1/p_2} \\ &\qquad \le \|f_{k,i} g\|_{p_2,\phi_2} \le C \|g\|_{\mathrm{Op}}. \end{aligned}$$

This is

$$\int_{\operatorname{supp} f_{k,i}} |g(x)|^{p_2}\, dx \le C \phi_1(1/(l_k m_k))^{-p_2} {\|g\|_{\mathrm{Op}}}^{p_2}.$$

Since

$$\begin{cases} \operatorname{supp} f_{k,i} = \displaystyle\bigcup_{j=1}^{{l_k}^n} I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \\ I(a_0, 1) = \displaystyle\bigcup_{i=1}^{{m_k}^n} \operatorname{supp} f_{k,i}, \\ (\operatorname{supp} f_{k,i})^\circ \cap (\operatorname{supp} f_{k,i'})^\circ = \emptyset \quad \text{for } i \ne i', \end{cases} \qquad \text{for all } k \in \mathbb{N},$$

we have by (4.5)

$$\int_{I(a_0,1)} |g(x)|^{p_2}\, dx \le C {m_k}^n \phi_1(1/(l_k m_k))^{-p_2} {\|g\|_{\mathrm{Op}}}^{p_2} \to 0 \quad \text{as} \quad k \to +\infty.$$

Therefore $g = 0$ a.e. □

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Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

E-mail address: enakai@cc.osaka-kyoiku.ac.jp