A CHARACTERIZATION OF POINTWISE MULTIPLIERS ON THE MORREY SPACES

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Dedicated to the memory of Professor Hiroshi Takehana

Abstract. Let $L_{p_i,\phi_i}$ $(i = 1, 2, 3)$ be Morrey spaces. A function $g$ is called a pointwise multiplier from $L_{p_1,\phi_1}$ to $L_{p_2,\phi_2}$ if the pointwise product $fg$ belongs to $L_{p_3,\phi_3}$ for each $f \in L_{p_1,\phi_1}$. We denote by $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2})$ the set of all pointwise multipliers from $L_{p_1,\phi_1}$ to $L_{p_2,\phi_2}$. A sufficient condition on $p_i$ and $\phi_i$ $(i = 1, 2, 3)$ for $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2})$ was given in [9]. In this paper, we give a necessary condition. In connection with these conditions, we also give sufficient conditions for $PWM(L_{p_1,\phi_1}, L_{p_2,\phi_2}) = \{0\}$.

1. Introduction

The theory of the Morrey spaces has been developed by many authors, Peetre [10], [11], Adams [1], Chiarenza and Frasca [3], Mizuhara [4], Arai and Mizuhara [2], etc. We investigate pointwise multipliers on the Morrey spaces.

Let $E$ and $F$ be spaces of real- or complex-valued functions defined on a set $X$. A function $g$ defined on $X$ is called a pointwise multiplier from $E$ to $F$, if the pointwise product $fg$ belongs to $F$ for each $f \in E$. We denote by $PWM(E, F)$ the set of all pointwise multipliers from $E$ to $F$.

$L^p$-spaces $(0 < p \leq \infty)$ on a measure space $X$ are complete quasi-normed linear spaces. If $1 \leq p \leq \infty$, then they are Banach spaces. If $X$ is $\sigma$-finite, and if $1/p_1 + 1/p_2 = 1/p_3$, then it is known that

$$PWM(L^{p_1}(X), L^{p_2}(X)) = L^{p_3}(X) \quad \text{and} \quad \|g\|_{L^{p_3}} = \|g\|_{L^{p_3}},$$

(1.1)

where $\|g\|_{L^{p_3}}$ is the operator norm of $g \in PWM(L^{p_1}(X), L^{p_2}(X))$, i.e.

$$\|g\|_{L^{p_3}} = \inf \{ \beta > 0 : \|fg\|_{L^{p_2}} \leq \beta \|f\|_{L^{p_1}} \text{ for all } f \in L^{p_1}(X) \}.$$ 

On some assumptions the equalities in (1.1) were generalized to the Morrey spaces $L_{p,\phi}(X)$ in [9], where $0 < p \leq \infty$, $\phi : X \times (0, +\infty) \to (0, +\infty)$ and $X$ is a space of homogeneous type. In this paper, we consider the necessity of the assumptions in the case of $\phi : (0, +\infty) \to (0, +\infty)$ and $X = \mathbb{R}^n$. Many authors studied in this case.

For $a \in \mathbb{R}^n$ and $r > 0$, let $B(a, r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$. For a measurable set $E \subset \mathbb{R}^n$, we denote by $|E|$ the Lebesgue measure of $E$. For $0 < p \leq \infty$ and $\phi : (0, +\infty) \to (0, +\infty)$, let

$$L_{p,\phi}(\mathbb{R}^n) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^p,\phi} < +\infty \},$$

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where
\[
||f||_{p, \phi} = \begin{cases} 
\sup_{B(a, r)} \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(x)|^p \, dx \right)^{1/p}, & 0 < p < \infty, \\
\sup_{B(a, r)} \frac{1}{\phi(r)} \inf_{x \in B(a, r)} |f(x)|, & p = \infty.
\end{cases}
\]

Then
\[
L_{\infty, \phi}(\mathbb{R}^n) = \begin{cases} 
\{0\}, & \inf_{0 < r < \infty} \phi(r) = 0, \\
L^\infty(\mathbb{R}^n), & \inf_{0 < r < \infty} \phi(r) > 0.
\end{cases}
\]

If \(\inf_{0 < r < \infty} \phi(r) > 0\), then \(||f||_{\infty, \phi} = ||f||_{L^\infty(\mathbb{R}^n)}/\inf_{0 < r < \infty} \phi(r)\).

For \(\phi(r) = r^{(\lambda-n)/p}\) (\(0 < p < \infty, 0 < \lambda < n\)), let \(L^{p, \lambda}(\mathbb{R}^n) = L_{p, \phi}(\mathbb{R}^n)\). Then \(L^{p, \lambda}(\mathbb{R}^n)\) is the classical Morrey spaces introduced in [5]. If \(\phi(r) = r^{-n/p}\), then \(L_{p, \phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)\).

If \(\phi(r) \equiv 1\), then \(L_{p, \phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)\).

The function space \(L_{p, \phi}(\mathbb{R}^n)\) is a complete quasi-normed linear space. If \(1 \leq p < \infty\), then it is a Banach space.

A function \(\theta : (0, +\infty) \to (0, +\infty)\) is said to be almost increasing (almost decreasing) if there exists a constant \(C > 0\) such that \(\theta(r) \leq C\theta(s)\) (\(\theta(r) \geq C\theta(s)\)) for \(r \leq s\).

If \(\inf_{1 \leq r} \phi(t) = 0\) for some \(r > 0\), then \(L_{p, \phi}(\mathbb{R}^n) = \{0\}\). Let \(\inf_{1 \leq r} \phi(t) > 0\) for every \(r > 0\) and \(\psi(r) = \inf_{1 \leq r} \phi(t)\). Then \(\psi\) is decreasing and \(L_{p, \psi}(\mathbb{R}^n) = L_{p, \phi}(\mathbb{R}^n)\) with equivalent norms.

If \(\inf_{r \geq r} \phi(t)^{n/p} = 0\) for some \(r > 0\), then \(L_{p, \phi}(\mathbb{R}^n) = \{0\}\). Let \(\inf_{r \geq r} \phi(t)^{n/p} > 0\) for every \(r > 0\) and \(\psi'(r) = r^{-n/p} \inf_{r \geq r} \phi(t)^{n/p}\). Then \(\psi'(r)^{n/p}\) is increasing and \(L_{p, \phi}(\mathbb{R}^n) = L_{p, \psi'}(\mathbb{R}^n)\) with equivalent norms.

To consider pointwise multipliers from \(L_{p_1, \phi_1}(\mathbb{R}^n)\) to \(L_{p_2, \phi_2}(\mathbb{R}^n)\), we may assume that \(\phi_i\) \((i = 1, 2)\) are almost decreasing and \(\phi_i(t)^{n/p_i}\) \((i = 1, 2)\) are almost increasing.

If \(\phi\) is almost decreasing and \(\phi(r)^{n/p}\) is almost increasing, then \(\phi\) satisfies
\[
\frac{1}{A} \leq \frac{\phi(s)}{\phi(r)} \leq A \quad \text{for} \quad \frac{s}{2} \leq \frac{r}{s} \leq 2,
\]
where \(A > 0\) is independent of \(r, s > 0\).

In the case of \(\phi : (0, +\infty) \to (0, +\infty)\) and \(X = \mathbb{R}^n\), Theorems 2.1, 2.2 and 2.3 in [9] can be stated as follows:

**Theorem 1.1.** Let \(0 < p_2 \leq p_1 \leq \infty\) and \(1/p_2 + 1/p_3 = 1/p_2\). Suppose that \(\phi_i\) \((i = 1, 2)\) are almost decreasing. Let \(\phi_2 = \phi_2/\phi_1\). If \(\phi_3(r)^{n/p_3}\) and \(\phi_2^{p_2/p_3}/\phi_1\) \((\phi_2/\phi_1\) when \(p_1 = p_2 = \infty, 1/\phi_1\) when \(p_2 < p_1 = \infty)\) are almost increasing, and if \(\liminf_{r \to 0} \phi_3(r) > 0\), then
\[
PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = L_{p_3, \phi_3}(\mathbb{R}^n).
\]

Moreover, the operator norm of \(g \in PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n))\) is comparable to \(\|g\|_{P_{p_3, \phi_3}}\). If \(\phi_2^{p_2/p_3}/\phi_1 = 1\), then the operator norm is equal to \(\|g\|_{P_{p_3, \phi_3}}\).

**Remark 1.1.** If \(\phi_1(r)^{n/p_1}\) and \(\phi_2^{p_2/p_1}/\phi_1\) are almost increasing, so are \(\phi_2(r)^{n/p_2}\) and \(\phi_3(r)^{n/p_3}\).

In this paper, we show that the almost increasingness of \(\phi_2^{p_2/p_1}/\phi_1\) is a necessary and sufficient condition for (1.3) when \(p_2 \leq p_1\), and that if \(p_2 > p_1\) or if \(\liminf_{r \to 0} \phi_3(r) = 0\) then
\[
PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\}.
\]

It turns out that Theorem 1.1 holds without the condition \(\liminf_{r \to 0} \phi_3(r) > 0\) in the case of \(\phi : (0, +\infty) \to (0, +\infty)\).
We state main results in the next section. Section 3 is for the preliminaries. In section 4 we give proofs of the results.

The letters $C$ will always denote a positive constant, not necessarily the same one. For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \leq \kappa(r) \leq C\theta(r), \quad r > 0.$$ 

2. Main results

**Theorem 2.1.** Let $0 < p_2 \leq p_1 < \infty$. Suppose that $\phi_i (i = 1, 2)$ are almost decreasing and that $\phi_i(r)r^{n/p_i} (i = 1, 2)$ are almost increasing. Then

$$PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = L_{p_2, \phi_2}(\mathbb{R}^n) ,$$

if and only if $\phi_2^{p_2/p_1}/\phi_1$ (or $\phi_2/\phi_1$ when $p_1 = p_2 = \infty$, $1/\phi_1$ when $p_2 < p_1 = \infty$) is almost increasing, where $1/p_3 = 1/p_2 - 1/p_1$ and $\phi_3 = \phi_2/\phi_1$. In this case, the operator norm of $g \in PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n))$ is comparable to $\|g\|_{p_1, \phi_1}.$

**Remark 2.1.** When $p_1 = p_2 = \infty$, $\phi_i(r) = \phi_i(r)r^{n/p_i}$ ($i = 1, 2$) are almost decreasing and almost increasing, i.e. $\phi_1 \sim \phi_2 \sim 1$ and $\phi_2/\phi_1$ is almost increasing. When $p_2 < p_1 = \infty$, $\phi_1(r) = \phi_1(r)r^{n/p_1}$ is almost decreasing and almost increasing, i.e. $\phi_1 \sim 1$ and $1/\phi_1$ is almost increasing.

**Remark 2.2.** In general, from Lemma 3.1 it follows that

$$PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) \supset L_{p_1, \phi_1}(\mathbb{R}^n).$$

**Theorem 2.2.** Suppose that $\phi_1$ is almost decreasing, that $\phi_1(r) \to +\infty$ as $r \to 0$ and that $\phi_1(r)r^{n/p_1}$ is almost increasing. If $0 < p_1 < p_2 < \infty$, then

$$PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\}.$$

**Lemma 2.3.** Let $0 < p_1, p_2 < \infty$. Suppose that $\phi_1$ is almost decreasing and that $\phi_1(r)r^{n/p_1}$ is almost increasing. If $\liminf_{r \to 0} \phi_2(r)/\phi_1(r) = 0$, then

$$PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\}.$$

**Remark 2.3.** If

$$PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\},$$

then there is a function $f \in L_{p_1, \phi_1}(\mathbb{R}^n)$ such that $f \notin L_{p_2, \phi_2}(\mathbb{R}^n)$, since a constant function 1 is not a pointwise multiplier.

For the classical Morrey spaces $L^{p, \lambda}(\mathbb{R}^n)$, we have the following.
Corollary 2.4. Let $0 < p_i < \infty$ and $0 < \lambda_i < n$ $(i = 1, 2)$. Then

$$PWM(L^{p_1,\lambda_1}(\mathbb{R}^n), L^{p_2,\lambda_2}(\mathbb{R}^n))$$

\[
\begin{cases}
\emptyset, & p_1 < p_2, \\
\emptyset, & p_1 = p_2 \text{ and } \lambda_1 < \lambda_2, \\
L^\infty(\mathbb{R}^n), & p_1 = p_2 \text{ and } \lambda_1 = \lambda_2, \\
\cup \{0\}, & p_1 = p_2 \text{ and } \lambda_1 > \lambda_2, \\
\emptyset, & p_1 > p_2 \text{ and } n + (\lambda_1 - n)p_2/p_1 < \lambda_2, \\
L^\infty(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_2 = n + (\lambda_1 - n)p_2/p_1, \\
L^{p_1,\lambda_1}(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_1 \leq \lambda_2 < n + (\lambda_1 - n)p_2/p_1, \\
\cup \{0\}, & p_1 > p_2 \text{ and } \lambda_1 p_2/p_1 < \lambda_2 < \lambda_1, \\
L^{p_2}(\mathbb{R}^n), & p_1 > p_2 \text{ and } \lambda_2 = \lambda_1 p_2/p_1, \\
\emptyset, & p_1 > p_2 \text{ and } \lambda_2 < \lambda_1 p_2/p_1.
\end{cases}
\]

where $p_3 = p_1 p_2/(p_1 - p_2)$ and $\lambda_3 = (p_1 \lambda_2 - p_2 \lambda_1)/(p_1 - p_2)$.

3. Preliminaries

We state lemmas to prove the theorems.

Lemma 3.1 ([9]). Let $0 < p_2 \leq p_1 \leq \infty$, $1/p_1 + 1/p_3 = 1/p_2$ and $\phi_1 \phi_3 = \phi_2$. Then

$$\|fg\|_{P_1,\phi_3} \leq \|f\|_{P_1,\phi_1} \|g\|_{P_1,\phi_2}.$$ 

Lemma 3.2 ([9]). Let $0 < p < \infty$. Suppose that $\phi$ satisfies (1.2) and $\phi(r)r^{n/p}$ is almost increasing. If $\text{supp } f \subset B(a,r)$ and if

$$\begin{aligned}
\sup_{B(b,s) \subset B(a,3r)} \frac{1}{\phi(s)} \left( \frac{1}{|B(b,s)|} \int_{B(b,s)} |f(x)|^p \, dx \right)^{1/p} &\leq M,
\end{aligned}$$

then

$$f \in L_{p,\phi}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{P,\phi} \leq CM,$$

where $C > 0$ independent of $f$, $B(a,r)$ and $M$.

Lemma 3.3. Let $0 < p < \infty$, $\phi$ be almost decreasing and $\phi(r)r^{n/p}$ be almost increasing. If $f$ is the characteristic function of the ball of radius $r > 0$, then

$$\|f\|_{P,\phi} \sim \frac{1}{\phi(r)}$$

Proof. For all balls $B(b,s) \subset B(a,3r)$, we have

$$\begin{aligned}
\frac{1}{\phi(s)} \left( \frac{1}{|B(b,s)|} \int_{B(b,s)} |f(x)|^p \, dx \right)^{1/p} &\leq \frac{1}{\phi(s)} \leq \frac{C}{\phi(r)}.
\end{aligned}$$

By Lemma 3.2 we obtain the desired result. \hfill \Box
Lemma 3.4. Let $0 < p < \infty$, $\phi$ be almost decreasing and $\phi(r)r^{n/p}$ be almost increasing. Let $\{B(a_j, r_j)\}_{j=0}^\infty$ be balls and $\{f_j\}_{j=1}^\infty$ be functions such that

$$
\begin{aligned}
&\begin{cases}
B(a_j, r_j) \subset B(a_0, r_0), \\
\text{supp } f_j \subset B(a_j, r_j/3), \\
\|f_j\|_{L^p} \leq c_1 \phi(r_0)r_j^{n/p}, \\
\|f_j\|_{L^p, \phi} \leq c_2,
\end{cases}
\quad \text{for } j = 1, 2, \ldots ,
\end{aligned}
$$

Then $f = \sum_{j=1}^\infty f_j$ is in $L_{p, \phi}(\mathbb{R}^n)$ and $\|f\|_{L^p, \phi} \leq C(c_1 + c_2)$.

Proof. For any ball $B(a, r) \subset B(a_0, 3r_0)$, let

$$
J_1 = \{j \in \mathbb{N} : B(a, r) \cap B(a_j, r_j/3) \neq \emptyset, r_j/3 \leq r\},
$$

$$
J_2 = \{j \in \mathbb{N} : B(a, r) \cap B(a_j, r_j/3) \neq \emptyset, r_j/3 > r\},
$$

and

$$
I_i = \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} \left| \sum_{j \in J_i} f_j(x) \right|^p \, dx \right)^{1/p}, \quad i = 1, 2.
$$

If $j \in J_1$, then $B(a_j, r_j/3) \subset B(a, 3r)$. It follows that

$$
\bigcup_{j \in J_1} B(a_j, r_j/3) \subset B(a, 3r) \quad \text{and} \quad \sum_{j \in J_1} (r_j/3)^n \leq (3r)^n.
$$

Hence

$$
I_1 \leq \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \sum_{j \in J_1} \int_{B(a_j, r_j/3)} |f_j(x)|^p \, dx \right)^{1/p} \leq \frac{c_1 \phi(r_0)}{\phi(r)} \left( \frac{1}{|B(a, r)|} \sum_{j \in J_1} r_j^n \right)^{1/p} \leq Cc_1.
$$

If $j \in J_2$, then $B(a, r) \subset B(a_j, r_j)$, i.e. $J_2$ has only one element. Hence

$$
I_2 \leq \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f_j(x)|^p \, dx \right)^{1/p} \leq \|f_j\|_{L^p, \phi} \leq c_2.
$$

By Lemma 3.2 we have $\|f\|_{L^p, \phi} \leq C(c_1 + c_2)$. \hfill \Box

4. PROOFS

First, we state a proof of Lemma 2.3.

Proof of Lemma 2.3. There are positive numbers $\{r_j\}_{j=1}^\infty$ such that

$$
r_j \to 0 \quad \text{and} \quad \phi_2(r_j)/\phi_1(r_j) \to 0 \quad \text{as } \quad j \to +\infty.
$$

If $g \in PWM(L_{p, \phi}(\mathbb{R}^n), L_{p, \phi}(\mathbb{R}^n))$, then the closed graph theorem shows that $g$ is a bounded operator. For any $a \in \mathbb{R}^n$, let $f$ be the characteristic function of the ball $B(a, r_j)$. 

By Lemma 3.3 we have
\[
\left( \frac{1}{|B(a,r)|} \int_{B(a,r)} |g(x)|^{p_2} \, dx \right)^{1/p_2} \leq \phi_2(r_j)\|f\|_{L^{p_2},\phi_2} \leq C \phi_2(r_j)\|g\|_{C^p} \leq \frac{C\phi_2(r_j)}{\phi_1(r_j)}\|g\|_{C^p} \to 0 \quad \text{as} \quad j \to +\infty.
\]
Therefore \( g(a) = 0 \) a.e. \( a \in \mathbb{R}^n. \)

Let \( I(a,r) \) be the cube \( \{x \in \mathbb{R}^n : |x_i - a_i| \leq r/2, i = 1, 2, \ldots, n\} \) whose edges have length \( r \) and are parallel to the coordinate axes.

**Proof of Theorem 2.1.** If \( \liminf_{r \to 0} \phi_3(r) > 0 \), then the sufficiency follows from Theorem 1.1. If \( \liminf_{r \to 0} \phi_3(r) = 0 \), then \( L_{p_3,\phi_3}(\mathbb{R}^n) = \{0\} \). From Lemma 2.3 it follows that
\[
PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) = L_{p_3,\phi_3}(\mathbb{R}^n).
\]
If \( \phi_2^{p_2/p_1}/\phi_1 \) is not almost increasing, then, for all \( k \in \mathbb{N} \), there are positive real numbers \( r_k \) and \( s_k \) such that
\[
s_k < r_k \quad \text{and} \quad \phi_2(s_k)^{p_2/p_1}/\phi_1(s_k) \geq \frac{4k^{p_2}}{p_2} \phi_2(r_k)^{p_2/p_1}/\phi_1(r_k).
\]
Let
\[
m_k = \left[ \frac{r_k \phi_2(r_k)^{p_2/n}}{s_k \phi_2(s_k)^{p_2/n}} \right] + 1,
\]
where \([\alpha]\) denotes the integer part of the positive real number \( \alpha \). By almost increasingness of \( \phi_2(r)^{p_2/n} \) and by almost decreasingness of \( \phi_2 \), we have
\[
m_k \sim \frac{r_k \phi_2(r_k)^{p_2/n}}{s_k \phi_2(s_k)^{p_2/n}}, \tag{4.1}
\]
\[
\frac{r_k}{m_k} > 6c_0s_k \quad \text{for some} \quad c_0 > 0. \tag{4.2}
\]
Let \( 0 \in \mathbb{R}^n \) be the origin. We divide the cube \( I(0,r_k) \) into \( m_k^n \) sub-cubes \( I(b_{k,j},r_k/m_k) \):
\( j = 1, 2, \ldots, m_k^n, \) i.e.
\[
I(0,r_k) = \bigcup_{j=1}^{m_k^n} I(b_{k,j},r_k/m_k),
\]
\[
I(b_{k,i},r_k/m_k)^{\circ} \cap I(b_{k,j},r_k/m_k)^{\circ} = \emptyset \quad \text{for} \quad i \neq j,
\]
where \( I(b,r)^{\circ} \) is the interior of \( I(b,r) \). Let
\[
g_{k,j}(x) = \begin{cases} 
\phi_3(c_0s_k), & x \in B(b_{k,j},c_0s_k), \\
0, & x \notin B(b_{k,j},c_0s_k),
\end{cases}
\]
\[
g_k = \sum_{j=1}^{m_k^n} g_{k,j}, \quad g = \left( \sum_{k=1}^{\infty} \frac{1}{2^k} (g_k)^{p_2} \right)^{1/p_2}.
\]
We show
\[
g \in PWM(L_{p_1,\phi_1}(\mathbb{R}^n), L_{p_2,\phi_2}(\mathbb{R}^n)) \setminus L_{p_3,\phi_3}(\mathbb{R}^n).
\]
We note that
\[
\text{supp } g_{k,j} \subset B(b_{k,j},r_k/(6m_k)),
\]
\[
B(b_{k,j},r_k/(2m_k)) \subset I(b_{k,j},r_k/m_k).
\]
For all \( f \in L_{p_1, \phi_1}(\mathbb{R}^n) \), by Hölder’s inequality, (1.2) and (4.1) we have

\[
\|fg_k\|_{L^{p_2}} \leq \left\{ \begin{array}{ll}
\left( \int_{B(b_k, \epsilon \alpha_s k)} |f(x)|^{p_1} \, dx \right)^{1/p_1} \left( \int_{B(b_k, \epsilon \alpha_s k)} |g_k, j(x)|^{p_2} \, dx \right)^{1/p_2} & \text{when } p_1 > p_2 \\
\phi_3(\epsilon \alpha_s k) \left( \int_{B(b_k, \epsilon \alpha_s k)} |f(x)|^{p_1} \, dx \right)^{1/p_1} & \text{when } p_1 = p_2
\end{array} \right.
\]

By Lemma 3.1 and Lemma 3.3 we have

\[
\|fg_k\|_{L^{p_2}} \leq \|f\|_{L^{p_1}} \|g_k\|_{L^{p_3, \phi_2}} \leq C \|f\|_{L^{p_1, \phi_1}}.
\]

By Lemma 3.4 we have \( f g_k \) is in \( L_{p_3, \phi_2} \) and \( \|f g_k\|_{L^{p_2, \phi_2}} \leq C_0 \|f\|_{L^{p_1, \phi_1}} \). This implies

\[
\int_{B(a, r)} |f(x)g_k(x)|^{p_2} \, dx \leq |B(a, r)| (\phi(r)C_0 \|f\|_{L^{p_1, \phi_1}})^{p_2} \text{ for each ball } B(a, r).
\]

From Beppo-Levi’s theorem it follows that

\[
\sum_{k=1}^{\infty} \frac{1}{2} |f(x)g_k(x)|^{p_2} = |f(x)|^{p_2} \sum_{k=1}^{\infty} \frac{1}{2} g_k(x)^{p_2}
\]

converges a.e. \( x \in B(a, r) \) and

\[
\int_{B(a, r)} |f(x)g(x)|^{p_2} \, dx \leq |B(a, r)| (\phi(r)C_0 \|f\|_{L^{p_1, \phi_1}})^{p_2} \text{ for each ball } B(a, r).
\]

Hence we have

\[
\|fg\|_{L^{p_2, \phi_2}} \leq C_0 \|f\|_{L^{p_1, \phi_1}},
\]

and

\[
g \in PWM(L_{p_1, \phi_1}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)).
\]

On the other hand, since \( \text{supp } g_k \subset B(O, \sqrt{\alpha_k}/2) \),

\[
\frac{1}{\phi_3(\sqrt{\alpha_k}/2)} \left( \frac{1}{|B(O, \sqrt{\alpha_k}/2)|} \int_{B(O, \sqrt{\alpha_k}/2)} |g(x)|^{p_3} \, dx \right)^{1/p_3} \geq \frac{1}{\phi_3(\sqrt{\alpha_k}/2)} \left( \frac{1}{|B(O, \sqrt{\alpha_k}/2)|} \int_{B(O, \sqrt{\alpha_k}/2)} \frac{1}{|2^{k/p_2} g_k(x)|} \, dx \right)^{1/p_3} \sim \frac{\phi_3(s_k)(m_k s_k)^{n/p_3}}{2^{k/p_2} \phi_3(r_k)(r_k)^{n/p_3}} = \left( \frac{m_k s_k \phi_2(s_k)^{p_3/2}}{r_k \phi_2(r_k)^{p_3/2}} \right)^{n/p_3} \frac{\phi_2(s_k)^{p_3/2}/\phi_1(s_k)}{2^{k/p_2} \phi_2(r_k)^{p_3/2}/\phi_1(r_k)} \geq 2^{k/p_2}
\]

for all \( k \in \mathbb{N} \) when \( p_1 > p_2 \),
and
\[
\frac{1}{\phi_2(\sqrt{m_{r_k}/2})} \text{ess sup} \{ g(x) : x \in B(O, \sqrt{m_{r_k}/2}) \} \\
\geq \frac{1}{\phi_2(\sqrt{m_{r_k}/2})} \text{ess sup} \left\{ \frac{1}{2^{k/p_2}} g_k(x) : x \in B(O, \sqrt{m_{r_k}/2}) \right\} \\
\sim \frac{\phi_2(s_k)}{2^{k/p_2} \phi_2(r_k)} = \frac{\phi_2(s_k)/\phi_2(r_k)}{2^{k/p_2}} \\
\geq 2^{k/p_2} \quad \text{for all } k \in \mathbb{N}, \quad \text{when } p_1 = p_2.
\]

This shows \( g \notin L_{p_3, \phi_3}(\mathbb{R}^n) \). \qed

Remark 4.1. If \( \{r_k\}_{k=1}^\infty \) in the above proof is bounded, then the support of \( g \) is compact.

Proof of Theorem 2.2. Let \( \{s_k\}_{k=1}^\infty \) be positive real numbers such that \( s_k > s_{k+1} \) \((k = 1, 2, \ldots)\) and \( s_k \to 0 \) as \( k \to +\infty \). Let
\[
l_k = \left[ \phi_1(s_k)^{p_1/m_{s_k}} \right]^{-1} + 1, \quad m_k = \left[ \phi_1(s_k)^{p_1/m_{s_k}} \right] + 1.
\]
Then, by the almost increasingness of \( \phi_1(r)^{p_1/m_{s_k}} \) and the almost decreasingness of \( \phi_1 \), we have
\[
l_k \sim \left( \phi_1(s_k)^{p_1/m_{s_k}} \right)^{-1}, \quad m_k \sim \phi_1(s_k)^{p_1/m_{s_k}}.
\]
Hence
\[
\phi_1(1/(l_km_k))^{p_1/m_{k^n}} \sim \phi_1(s_k)^{p_1/m_{s_k}}l_k^{n} \sim 1
\]
\[
\phi_1(1/(l_km_k))^{p_2/m_{k^n}} \sim \phi_1(s_k)^{p_2/m_{s_k}}l_k^{n} \sim 1
\]
\[
\text{as } k \to +\infty.
\]

For any fixed \( a_0 \in \mathbb{R}^n \), we divide the cube \( I(a_0, 1) \) into \( l_k^n \) sub-cubes \( I(b_{k,j}, 1/l_k) \): \( j = 1, 2, \ldots, l_k^n \), i.e.
\[
I(a_0, 1) = \bigcup_{j=1}^{l_k^n} I(b_{k,j}, 1/l_k),
\]
\[
I(b_{k,j}, 1/l_k) \cap I(b_{k,j'}, 1/l_k) = \emptyset \quad \text{for } j \neq j'.
\]
We divide the cube \( I(O, 1/l_k) \) into \( m_k^n \) sub-cubes \( I(e_{k,i}, 1/(l_km_k)) \): \( i = 1, 2, \ldots, m_k^n \), i.e.
\[
I(O, 1/l_k) = \bigcup_{i=1}^{m_k^n} I(e_{k,i}, 1/(l_km_k)),
\]
\[
I(e_{k,i}, 1/(l_km_k)) \cap I(e_{k,i'}, 1/(l_km_k)) = \emptyset \quad \text{for } i \neq i'.
\]

Then
\[
\begin{cases}
I(b_{k,j}, 1/l_k) = \bigcup_{i=1}^{m_k^n} I(b_{k,j} + e_{k,i}, 1/(l_km_k)), \\
I(b_{k,j} + e_{k,i}, 1/(l_km_k)) \cap I(b_{k,j} + e_{k,i'}, 1/(l_km_k)) = \emptyset \quad \text{for } i \neq i',
\end{cases}
\]
\[
j = 1, 2, \ldots, l_k^n.
\]
Let

\[ f_{k,j,i}(x) = \begin{cases} \phi_1(1/(l_k m_k)), & x \in I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \\ 0, & x \notin I(b_{k,j} + e_{k,i}, 1/(l_k m_k)) \end{cases} \]

\[ f_{k,i} = \sum_{j=1}^{l_k} f_{k,j,i}. \]

First we show

\[ \|f_{k,i}\|_{p_1, \phi_1} \leq C, \quad \text{for } i = 1, 2, \ldots, m_k^n, \text{ and for all } k \in \mathbb{N}. \]

We note that

\[ B(b_{k,j} + e_{k,i}, 1/(2l_k)) \subset I(b_{k,j} + e_{k,i}, 1/l_k) \subset I(a_0, 2) \subset B(a_0, \sqrt{n}), \]

\[ B(b_{k,j} + e_{k,i}, 1/(2l_k)) \cap B(b_{k,j'} + e_{k,i}, 1/(2l_k)) = \emptyset \quad \text{for } j \neq j'. \]

Since \( \phi_1(s_k) \to +\infty \) as \( k \to \infty \), we may assume \( m_k \geq 3\sqrt{n} \). Hence

\[ \text{supp } f_{k,j,i} = I(b_{k,j} + e_{k,i}, 1/(l_k m_k)) \subset B(b_{k,j} + e_{k,i}, \sqrt{n}/(2l_k m_k)) \subset B(b_{k,j} + e_{k,i}, 1/(6l_k)). \]

By (4.4) we have

\[ \|f_{k,j,i}\|_{L^{p_1}} \leq \phi_1(1/(l_k m_k))(1/(l_k m_k))^{n/p_1} \sim (1/l_k)^{n/p_1}. \]

By Lemma 3.3 we have

\[ \|f_{k,i}\|_{p_1, \phi_1} \leq C. \]

Hence, by Lemma 3.4 we have (4.6).

If \( g \in \text{PWM}(L_{p_1, \phi_0}(\mathbb{R}^n), L_{p_2, \phi_2}(\mathbb{R}^n)) \), then the closed graph theorem shows that \( g \) is a bounded operator. Hence

\[ \frac{\phi_1(1/(l_k m_k))}{\phi_2(\sqrt{n})} \left( \frac{1}{|B(a_0, \sqrt{n})|} \int_{\text{supp } f_{k,i}} |g(x)|^{p_2} \, dx \right)^{1/p_2} \]

\[ = \frac{1}{\phi_2(\sqrt{n})} \left( \frac{1}{|B(a_0, \sqrt{n})|} \int_{B(a_0, \sqrt{n})} |f_{k,i}(x)g(x)|^{p_2} \, dx \right)^{1/p_2} \]

\[ \leq \|f_{k,i}g\|_{p_2, \phi_2} \leq C\|g\|_{L^{p_2}}. \]

This is

\[ \int_{\text{supp } f_{k,i}} |g(x)|^{p_2} \, dx \leq C\phi_1(1/(l_k m_k))^{-p_2}\|g\|_{L^{p_2}}. \]

Since

\[ \text{supp } f_{k,i} = \bigcup_{j=1}^{l_k} I(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \]

\[ I(a_0, 1) = \bigcup_{i=1}^{m_k^n} \text{supp } f_{k,i}, \quad \text{for all } k \in \mathbb{N}, \]

\[ (\text{supp } f_{k,i})^c \cap (\text{supp } f_{k,j'})^c = \emptyset \quad \text{for } i \neq j', \]

we have by (4.5)

\[ \int_{I(a_0, 1)} |g(x)|^{p_2} \, dx \leq C m_k^n \phi_1(1/(l_k m_k))^{-p_2}\|g\|_{L^{p_2}} \to 0 \quad \text{as } k \to +\infty. \]

Therefore \( g = 0 \) a.e.
REFERENCES


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