SYMMETRIC BI-DERIVATION ON PRIME GAMMA RINGS

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Abstract. Maksan in [3] and [4], and Vukman in [8] worked the trace of symmetric bi-derivation. In [9], Yenigül and Arıcan extended a part of the working on [8] to the ideal of ring. In this paper, we extended all of the working on [8] to the ideal of prime $\Gamma$-ring.

1. Introduction

The notion of a gamma ring was introduced by Nobusawa in [5] as a generalized notion of a ring and it was defined in [1] as follows: Let $M$ and $\Gamma$ be additive abelian groups. $M$ is called a $\Gamma$-ring if the following conditions are satisfied. For any $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, there hold

1. $a \alpha b \in M$
2. $(a + b)\alpha c = a \alpha c + b \alpha c$,
   \[ a(\alpha + \beta)b = a\alpha b + a\beta b, \]
   \[ a\alpha (b + c) = a\alpha b + a\alpha c, \]
3. $(a\alpha b)\beta c = a\alpha (b\beta c)$.

Every ring is a $\Gamma$-ring and many notions on the ring theory are generalized to the $\Gamma$-ring. Let $M$ be a $\Gamma$-ring. A $\Gamma$-subring of $M$ is an additive subgroup $N$ such that $\Gamma N \subset N$ and the subset $Z = \{a \in M \mid a\alpha m = m a\alpha, \text{ for any } m \in M, a \in \Gamma\}$ is called the center of $M$. A right (resp. left) ideal of $M$ is an additive abelian group $I$ such that $\Gamma M \subset I$ (resp. $M \Gamma \subset I$). If $I$ is both a right and left ideal, then we say that $I$ is an ideal. $M$ is called a prime $\Gamma$-ring if $a\Gamma M \Gamma b = 0$ imply $a = 0$ or $b = 0$, $(a, b \in M)$. Semi-prime ring is defined similarly. Throughout, $M$ will be a $\Gamma$-ring. A map $D(\cdot, \cdot) : M \times M \to M$ is called symmetric bi-additive if it is additive in both arguments and $D(x, y) = D(y, x)$ for any $x, y \in M$. Then the map $d : M \to M$ defined by $d(x) = D(x, x)$ is called the trace of $D$. A symmetric bi-additive map is called symmetric bi-derivation if

\[ D(\alpha xy, z) = D(x, z)\alpha y + x\alpha D(y, z) \]

for all $x, y, z \in M$ and $\alpha \in \Gamma$.

Definition 1. Let $M$ be a $\Gamma$-ring. For a subset $I$ of $M$, $\text{Ann}_I = \{a \in M \mid a\Gamma I = 0\}$ is called the left annihilator of $I$. A right annihilator $\text{Ann}_r I$ can be defined similarly.

To make the paper self-containing we give the sketch proofs of the following Lemmas:

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LEMMA 1. [4, Lemma 3.4.5]. Let $M$ be a semi-prime $\Gamma$-ring and $I$ a non-zero ideal of $M$. Then $\text{Ann} I = \text{Ann}_\Gamma I$.

**Proof.** $\text{Ann}_\Gamma I = \{ a \in M \mid \Gamma a = 0 \}$ is a right ideal of $M$, that is, $(\text{Ann}_\Gamma I) \Gamma M \subseteq \text{Ann}_\Gamma I$. Similarly for $\text{Ann}_I$ we can write $M \Gamma (\text{Ann}_I) \subseteq \text{Ann}_I$. Since $M$ is a semi-prime $\Gamma$-ring, $(\text{Ann}_\Gamma I) \Gamma I = \{ 0 \}$, so $\text{Ann}_\Gamma I \subseteq \text{Ann}_I$. In the same manner $\Gamma (\text{Ann}_I) \Gamma (\text{Ann}_I) = \{ 0 \}$ gives us that $\Gamma (\text{Ann}_I) = \{ 0 \}$ as $M$ is a semi-prime $\Gamma$-ring. That is, $\text{Ann}_I \subseteq \text{Ann}_\Gamma I$. So $\text{Ann}_I = \text{Ann}_\Gamma I$.

Let $M$ be a semi-prime $\Gamma$-ring and $I$ non-zero ideal of $M$. Then we will denote $\text{Ann} I = \text{Ann}_\Gamma I = \text{Ann}_I$.

LEMMA 2. [4, Lemma 3.4.6]. Let $M$ be a semi-prime $\Gamma$-ring, $I$ a non-zero ideal of $M$. Then

(i) $\text{Ann} I$ is an ideal of $M$,

(ii) $(\text{Ann} I) \cap I = \{ 0 \}$.

**Proof.** (i) Let $a \in \text{Ann} I$. So by Lemma 1 $a \Gamma I = 0 = I \Gamma a$. If $a, b \in \text{Ann} I$, then $x a (a - b) = x a a - x a b = 0$ and $(a - b) x a = a x a - b x a = 0$ for all $x \in I$ and $\alpha \in \Gamma$. So we have $a - b \in \text{Ann} I$. For all $a \in \text{Ann} I$, $x \in I, m \in M$ and $\alpha, \beta \in \Gamma$, $(a m) \beta x = a (m \beta x) = 0$ and $x \beta (a m) = (x \beta a) m = 0$ so we get $(\text{Ann} I) \Gamma M \subseteq \text{Ann} I$. Similarly we get $M \Gamma (\text{Ann} I) \subseteq \text{Ann} I$.

(ii) Since $(\text{Ann} I) \cap I$ is an ideal of $M$ and $(\text{Ann} I) \cap I \cap I = \{ 0 \}$, we have $(\text{Ann} I) \cap I \cap (\text{Ann} I) \cap I = \{ 0 \}$ and since $M$ is a semi-prime $\Gamma$-ring we get $(\text{Ann} I) \cap I = \{ 0 \}$.

2. Results

LEMMA 3. Let $M$ be a 2-torsion free semi-prime $\Gamma$-ring, $I$ a non-zero ideal of $M$ and $a, b \in M$. Then the following are equivalent,

(i) $axb = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$

(ii) $baxa = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$

(iii) $axb + bxax = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$.

If one of the conditions is fulfilled and $\text{Ann}_\Gamma I = 0$ then $acb = 0 = bac$ for all $\alpha \in \Gamma$, moreover if $M$ is a prime $\Gamma$-ring then $a = 0$ or $b = 0$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $axb = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. Then $baxa = 0$ for all $x, y \in I$ and $\alpha, \beta, \gamma, \beta' \in \Gamma$. By writing $y' \gamma' m$ for $y$, we get $baxb \gamma' \gamma' \beta' baxa = 0$ where $m \in M$ and $\gamma' \in \Gamma$, hence

$$baxb \gamma' \gamma' \beta' baxa \gamma' y = 0$$

Now since $M$ is a semi-prime $\Gamma$-ring we have $baxb \gamma' \gamma' \beta' baxa \gamma' y = 0$ for all $x, y \in I$ and $\alpha, \beta, \gamma \in \Gamma$. That is $baxa \in \text{Ann}_\Gamma I$. Therefore $baxa \in (\text{Ann}_\Gamma I) \cap I = \{ 0 \}$ by Lemma 1 and Lemma 2.

(ii) $\Rightarrow$ (i). This can be done similarly.

(iii) $\Rightarrow$ (i). Suppose that $axb + bxax = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. In the above equation, writing $x' b \alpha' m \beta' \alpha x \beta$ for $x$, then

$$axb x' b \alpha' m \beta' \alpha x \beta b = - baxb x' b \alpha' m \beta' \alpha x \beta b$$

$$= - (baxb x' b \alpha' m \beta' \alpha x \beta b)$$

$$= a x' b \alpha' m \beta' (bax \beta)$$

$$= - (baxb x' b \alpha' m \beta' \alpha x \beta b)$$
then we have \(2\alpha(x\beta\alpha'dm'b\alpha x)\beta b = 0\). Since \(M\) is 2-torsion free, we get
\[
(a\alpha x\beta b)m'b\alpha(x\alpha x\beta b) = 0
\]
for all \(x \in I, \alpha, \beta, \alpha', \beta' \in \Gamma\) and \(m \in M\). Next, since \(M\) is semi-prime \(\Gamma\)-ring, then \(a\alpha x\beta b = 0\) for all \(x \in I, \alpha, \beta \in \Gamma\).

If \(a\Gamma I\Gamma b = 0\), then we also have \((b\Gamma a)\Gamma I\Gamma(b\Gamma a) = 0\) and \((a\Gamma b)\Gamma I\Gamma(a\Gamma b) = 0\). Hence \(b\alpha a\beta x = 0\) and \(a\alpha b\beta x = 0\) for all \(x \in I, \alpha, \beta \in \Gamma\), since \(M\) is a semi-prime \(\Gamma\)-ring and \(I\) is a non-zero ideal of \(M\). This says that \(a\alpha b, b\alpha a \in \text{Ann} I\). Since \(\text{Ann} I = (0)\), we have \(a\Gamma b = 0 = b\Gamma a\). Finally if \(a\Gamma I\Gamma b = 0\), then \(a = 0\) or \(b = 0\) as \(M\) is a prime \(\Gamma\)-ring by [3, Lemma 2(iii)].

**Lemma 4.** Let \(M\) be a 2, 3-torsion free semi-prime \(\Gamma\)-ring, \(I\) a non-zero ideal of \(M\). Let \(D_1(\ldots)\) and \(D_2(\ldots)\) be a symmetric bi-derivations of \(M\) with \(d_1\) and \(d_2\) respectively. Then,

(i) If \(d_1(I\Gamma I\Gamma d_2(I) = 0\) then \(d_1(M)\Gamma I\Gamma d_2(M) = 0\).

(ii) If \(\text{Ann} I = 0\) and \(d_1(M)\Gamma I\Gamma d_2(M) = 0\) then \(d_1(M)\Gamma M\Gamma d_2(M) = 0\).

**Proof.** (i) Suppose for all \(x, y, z \in I\) and \(\alpha, \beta \in \Gamma\)
\[
d_1(x)\alpha\beta d_2(x) = 0
\]
Linearizing (1) we get
\[
0 = d_1(x + y)\alpha\beta d_2(x + y)
\]
\[
= d_1(x)\alpha\beta d_2(x) + d_1(x)\alpha\beta d_2(y) + 2d_1(x)\alpha\beta d_2(x, y)
\]
\[
+ d_1(y)\alpha\beta d_2(x) + d_1(y)\alpha\beta d_2(y) + 2d_1(y)\alpha\beta d_2(x, y)
\]
\[
+ 2d_1(x, y)\alpha\beta d_2(x) + 2d_1(x, y)\alpha\beta d_2(y)
\]
\[
+ 4d_1(x, y)\alpha\beta d_2(x, y)
\]
and using (1) we have
\[
d_1(x)\alpha\beta d_2(y) + 2d_1(x)\alpha\beta D_2(x, y) + d_1(y)\alpha\beta d_2(x)
\]
\[
+ 2d_1(y)\alpha\beta D_2(x, y) + 2D_1(x, y)\alpha\beta d_2(x)
\]
\[
+ 2D_1(x, y)\alpha\beta d_2(y) + 4D_1(x, y)\alpha\beta D_2(x, y)
\]
\[
= 0
\]
Replacing \(x\) by \((-x)\) in (2) we write for all \(x, y, z \in I\) and \(\alpha, \beta \in \Gamma\)
\[
d_1(x)\alpha\beta d_2(y) - 2d_1(x)\alpha\beta D_2(x, y) + d_1(y)\alpha\beta d_2(x)
\]
\[
- 2d_1(y)\alpha\beta D_2(x, y) - 2D_1(x, y)\alpha\beta d_2(x)
\]
\[
- 2D_1(x, y)\alpha\beta d_2(y) + 4D_1(x, y)\alpha\beta D_2(x, y)
\]
\[
= 0
\]
Since \(M\) is 2-torsion free, from (2) and (3) we get for all \(x, y, z \in I\) and \(\alpha, \beta \in \Gamma\)
\[
d_1(y)\alpha\beta d_2(x) + d_1(x)\alpha\beta d_2(y) + 4D_1(x, y)\alpha\beta D_2(x, y) = 0
\]
Writing \(x + w\) for \(x\) in (4) we can write for all \(x, y, z \in I\) and \(\alpha, \beta \in \Gamma\)
\[
D_1(x, w)\alpha\beta d_2(x) + d_1(y)\alpha\beta D_2(x, w) + 2D_1(x, y)\alpha\beta D_2(y, w)
\]
\[
+ 2D_1(y, w)\alpha\beta D_2(x, y) = 0
\]
Writing $y$ for $w$ in (5) and since $M$ is 3-torsion free, we obtain that for all $x, y, z \in I, \alpha, \beta \in \Gamma$
\[ D_1(x, y) \alpha z \beta d_2(y) + d_1(y) \alpha z \beta D_2(x, y) = 0 \] (6)
Replacing $z$ by $\beta \delta(k(y) \alpha \gamma \delta D_1(x, y) \alpha z$ we get for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$
\[ D_1(x, y) \alpha \beta \delta d_2(y) \alpha \gamma \gamma \delta D_1(x, y) \alpha z \beta d_2(y) \]
\[ = -d_1(y) \alpha \beta \delta d_2(y) \alpha \gamma \gamma \delta D_1(x, y) \alpha z \beta d_2(x, y) \]

and from (1) we get for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$
\[ d_1(x, y) \alpha \beta \delta d_2(y) \alpha \gamma \gamma \delta D_1(x, y) \alpha z \beta d_2(y) = 0 \]
and Since $M$ is a semi-prime $\Gamma$-ring we get for all $x, y, z \in M, \alpha, \beta \in \Gamma$
\[ D_1(x, y) \alpha \beta \delta d_2(y) = 0 \] (7)
Now writing $m \gamma z$ for $z$ in (7) where $m \in M, \gamma \in \Gamma$, we get for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$
\[ D_1(x, y) \alpha m \beta \gamma z \beta d_2(y) = 0 \] (8)
Next, replacing $x$ by $x \gamma m$ in (7) and from (8), we have for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$
\[ x \gamma D_1(m, y) \alpha z \beta d_2(y) = 0 \] (9)
Now, we say that (9) implies
\[ D_1(m, y) \alpha z \beta d_2(y) \in Ann_I \]
and also $D_1(m, y) \alpha z \beta d_2(y) \in I$ and so $D_1(m, y) \alpha z \beta d_2(y) \in (Ann_I) \cap I = 0$ by Lemma 1
and Lemma 2. Thus, for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(y) = 0 \] (10)
Now replacing $y$ by $x + y$ in (10) and using (10) we get, for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(x + y) + 2D_1(m, y) \alpha z \beta D_2(x, y) + D_1(m, x) \alpha z \beta d_2(y) \]
\[ = 2D_1(m, x) \alpha z \beta D_2(x, y) = 0 \] (11)
Writing $-x$ for $x$ in (11), we get for all $x, y, z \in I, m \in M$ and $\alpha, \beta \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(x) - 2D_1(m, y) \alpha z \beta D_2(x, y) - D_1(m, x) \alpha z \beta d_2(y) \]
\[ + 2D_1(m, x) \alpha z \beta D_2(x, y) = 0 \] (12)
Since $M$ is 2-torsion free, then from (11) and (12) we write for all $x, y, z \in I, m \in M$ and $\alpha, \beta \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(x) + 2D_1(m, x) \alpha z \beta D_2(x, y) = 0 \] (13)
Writing $\beta \delta d_2(x) \alpha \gamma \gamma \delta D_1(m, y) \alpha z$ for $z$ in (13) and using (10) we get, for all $x, y, z \in I,$
and $m, m' \in M$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(x) \alpha \gamma \gamma \delta D_1(m, y) \alpha z \beta d_2(x) = 0 \] (14)
Since $M$ is semi-prime $\Gamma$-ring, (14) implies for all $x, y, z \in I, m \in M$ and $\alpha, \beta \in \Gamma$
\[ D_1(m, y) \alpha z \beta d_2(x) = 0 \] (15)
Replacing $z$ by $m \gamma z$ in (15) we get for all $x, y, z \in I, m \in M$ and $\alpha, \beta \in \Gamma$
\[ D_1(m, y) \alpha m \beta \gamma z \beta d_2(x) = 0 \] (16)
Next, replacing $y$ by $y \gamma m$ in (15) and from (16), we get for all $x, y, z \in I, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$
\[ y \gamma D_1(m) \alpha z \beta d_2(x) = 0 \] (17)
From (17), we can write that $d_1(m) \alpha z \beta d_2(x) \in Ann_I$ and
\[ d_1(m) \alpha z \beta d_2(x) \in I \]. So we get $d_1(m) \alpha z \beta d_2(x) \in (Ann_I) \cap I = (0)$ by Lemma 1 and
Lemma 2. Thus, for all $x, y, z \in I, \alpha, \beta \in \Gamma$
\[ d_1(m)\alpha \beta d_2(x) = 0 \]  
(18)

Writing \( x + y \) for \( x \) in (18), from (18) and using the fact that \( M \) is 2-torsion free we get, for all \( x, y, z \in I, m \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \gamma d_2(x, y) = 0 \]  
(19)

Replacing \( z \) by \( x \gamma n \) in (19) we get for all \( x, y, z \in I, m, n, \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \gamma n d_2(x, y) = 0 \]  
(20)

Replacing \( x \) by \( n \gamma x \) in (19) from (20) we get, for all \( x, y, z \in I, m, n, \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \gamma n d_2(x, y) = 0 \]  
(21)

Since \( d_1(m)\alpha \beta d_2(n, y) \in \text{Ann}_I \) from (21) and \( d_1(m)\alpha \beta D_2(n, y) \in I \), we get
\[ d_1(m)\alpha \beta D_2(n, y) \in (\text{Ann}_I) \cap I = (0) \]  
(22)

by Lemma 1 and Lemma 2. Thus, for all \( x, y, z \in I, m, n \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \beta D_2(n, y) = 0 \]  
(23)

Writing \( \gamma n \) for \( z \) in (22), we get for all \( y, z \in I, m, n \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \gamma n d_2(n, y) = 0 \]  
(24)

and replacing \( y \) by \( n \gamma y \) in (22) and using (23), we get for all \( y, z \in I, m, n \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \gamma n d_2(n, y) = 0 \]  
(25)

Now since \( d_1(m)\alpha \beta d_2(n) \in \text{Ann}_I \) and \( d_1(m)\alpha \beta d_2(n) \in I \), from (24) we get
\[ d_1(m)\alpha \beta d_2(n) \in (\text{Ann}_I) \cap I = (0) \]  
(26)

by Lemma 1 and Lemma 2. Hence, for all \( z \in I, m, n \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \beta d_2(n) = 0 \]  
(27)

\( \text{(ii)} \) Suppose that for all \( z \in I, m, n \in M \) and \( \alpha, \beta, \gamma \in \Gamma 
\]
\[ d_1(m)\alpha \beta d_2(n) = 0 \]  
(28)

Replacing \( z \) by \( m' \beta d_2(n) \gamma z \beta' n' \gamma' d_1(m)\alpha m' \) we get, for all \( z \in I, m, n, m', n' \in M \) and \( \alpha, \beta, \gamma, \beta', \gamma' \in \Gamma 
\]
\[ d_1(m)\alpha m' d_2(n) \gamma z \beta \beta' n' \gamma' d_1(m)\alpha m' \beta d_2(y) = 0 \]  
(29)

Since \( M \) is a semi-prime \( \Gamma \)-ring, we get for all \( \alpha, \beta, \gamma \in \Gamma \) and \( m, n, m' \in M 
\]
\[ d_1(m)\alpha m' d_2(n) \gamma z \beta \beta' n' \gamma' d_1(m)\alpha m' d_2(y) = 0 \]  
(30)

Finally, since \( d_1(m)\alpha m' d_2(n) \in \text{Ann}_I(I) = (0) \), then we get for all \( m, n, m' \in M 
\]
\[ d_1(m)\alpha m' d_2(n) = 0 \]  
(31)

**LEMMA 5.** Let \( M \) be a 2-torsion free \( \Gamma \)-ring and I a nonzer one-sided ideal of \( M \). Let \( D(\cdot, \cdot) \) be a symmetric bi-derivation with the trace \( d \). Consider the following conditions:

\( \text{(i)} \) \( d(x) = 0 \), for all \( x \in I \)

\( \text{(ii)} \) \( D(x, y) = 0 \), for all \( x, y \in I \)

\( \text{(iii)} \) \( D(m, y) = 0 \), for all \( y \in I \) and \( m \in M \)

\( \text{(iv)} \) \( D(m, n) = 0 \), for all \( m, n \in M \).

Then (i) and (ii) are equivalent. Moreover if \( M \) is a prime \( \Gamma \)-ring or \( \text{Ann}_I(I) = 0 \) (resp. \( \text{Ann}_I(I) = 0 \)), then the above conditions are equivalent.

**PROOF.** Let \( I \) be a right ideal of \( M, m, n \in M, x, y \in I \) and \( \alpha, \beta, \gamma \in \Gamma \) be arbitrary elements. Since
\[ d(x + y) = d(x) + d(y) + 2D(x, y) \]
and \( M \) is 2-torsion free, (i) and (ii) are equivalent. Replacing \( x \) by \( \alpha m \) in \( D(x, y) = 0 \), we have
\[ 0 = D(\alpha m, y) = D(x, y)\alpha m + x\alpha D(m, y) \]
If \( M \) is a prime \( \Gamma \)-ring, then by [3, Lemma 2 (ii)] and above condition show that (ii) and (iii) are equivalent. If \( \text{Ann}_I(I) = (0) \), then the above condition shows that (ii) and (iii) are equivalent. Moreover replacing \( y \) by \( \gamma n \) in \( D(m, y) \), (iii) and (iv) are equivalent. Similarly we can prove these results for a left ideal \( I \).
LEMMA 6. Let $M$ be a 2-torsion free $\Gamma$-ring and $I$ be a $\Gamma$-sub-ring of $M$. Let $D_1(\ldots)$ and $D_2(\ldots)$ be the symmetric bi-derivations of $M$ with $d_1$ and $d_2$ respectively. If $D_1(d_2(x), x) = 0$ for all $x \in I$ then
\[ d_1(x)\alpha x\beta d_2(x) + d_2(x)\alpha x\beta d_1(x) = 0 \quad (27) \]
for all $x, y \in I$ and $\alpha, \beta \in \Gamma$.

PROOF. By linearizing $D_1(d_2(x + y), x + y) = 0$ for any $x, y \in I$ and $\alpha, \beta \in \Gamma$ we have
\[ D_1(d_2(x), y) + D_1(d_2(y), x) + 2D_1(D_2(x, y), x) + 2D_1(D_2(x, y), y) = 0 \]
Replacing $x$ by $-x$ the above equation,
\[ D_1(d_2(x), y) - D_1(d_2(y), x) + 2D_1(D_2(x, y), x) - 2D_1(D_2(x, y), y) = 0 \]
Now adding the last two equations together and using the fact that $M$ is 2-torsion free, we get
\[ D_1(d_2(x), y) + 2D_1(D_2(x, y), x) = 0 \quad (28) \]
Now writing $x\alpha y$ for $y$ in (28) we obtain
\[ d_1(x)\alpha yD_2(x, y) + d_2(x)\alpha D_1(x, y) = 0 \quad (29) \]
Finally replacing $y$ by $y\beta x$ in (29) we get the result.

LEMMA 7. Let $M$ be a 2,3-torsion free $\Gamma$-ring and $I$ a non-zero $\Gamma$-sub-ring of $M$. Let $D_1(\ldots)$ and $D_2(\ldots)$ be symmetric bi-derivations of $M$ with the traces $d_1$ and $d_2$ respectively. Let $F(\ldots)$ be a symmetric bi-additive map of $M$ with the trace $f$. If $d_1(d_2(x)) = f(x)$ for all $x \in I$, then
\[ D_1(y, d_2(y))\alpha x\beta d_2(y) + d_2(y)\alpha x\beta D_1(y, d_2(y)) = 0 \quad (30) \]
for all $x, y \in I$ and $\alpha, \beta \in \Gamma$.

PROOF. For any $x, y \in I$, since $d_1(d_2(x + y)) + d_1(d_2(-x + y)) = f(x + y) + f(-x + y)$ and $M$ is a 2-torsion free, we get
\[ D_1(d_2(x), d_2(y)) + 2d_1(D_2(x, y)) = 0 \quad (31) \]
By considering $x = y$ in (31), we have $d_1(d_2(x)) = 0$. Moreover replacing $x$ by $x + y$ in (31) we get
\[ D_1(D_2(x, y), d_2(y)) = 0 \quad (32) \]
for all $x, y \in I$. Replacing $x$ by $x\alpha y$ in (32) we write for all $x, y \in I$ and $\alpha, \beta \in \Gamma$
\[ D_1(x, d_2(y))\alpha d_2(y) + D_2(x, y)\alpha D_1(y, d_2(y)) = 0 \quad (33) \]
Writing $y\beta x$ for $y$ in (33) then we have (30).

Now by Lemma 3, 6, and 3, 5 and 6, we have the following.

THEOREM 1. Let $M$ be a 2-torsion free prime $\Gamma$-ring, $I$ a non-zero ideal of $M$ and also assume that $d_1$, $d_2$ are the traces of the symmetric bi-derivations $D_1(\ldots)$ and $D_2(\ldots)$ of $M$ respectively.
If $D_1(d_2(x), x) = 0$ for all $x \in I$, then $D_1 = 0$ or $D_2 = 0$.

By Lemma 3, 5, 7 and Theorem 1, we see the following.

THEOREM 2. Let $M$ be a 2,3-torsion free prime $\Gamma$-ring, $I$ a non-zero ideal of $M$. Let $D_1(\ldots)$ and $D_2(\ldots)$ be the symmetric bi-derivation of $M$, $F(\ldots)$ a symmetric bi-additive map of $M$ and $d_1$, $d_2$ and $f$ the traces of $D_1(\ldots)$ and $D_2(\ldots)$ and $F(\ldots)$ respectively such that $d_2(I) \subset I$. If $d_1(d_2(x)) = f(x)$ for all $x \in I$, then $D_1 = 0$ or $D_2 = 0$.

For a semi-prime $\Gamma$-ring, we have

THEOREM 3. Let $M$ be a 2-torsion free semi-prime $\Gamma$-ring, $I$ a non-zero ideal of $M$, $D(\ldots)$ the symmetric bi-derivation of $M$ and $d$ the trace of $D(\ldots)$. If $D(d(x), x) = 0$ for all $x \in I$ and $\text{Ann}_I, I$, then $D = 0$

PROOF. Take $D_1 = D_2 = D$ in Lemma 6. By Lemma 4 and since $M$ is a 2-torsion free semi-prime $\Gamma$-ring, $d(x) = 0$, $x \in I$ and by Lemma 5, the result is easily seen.
THEOREM 4. Let $M$ be a $2,3$-torsion free semi-prime $\Gamma$-ring, $I$ a non-zero ideal of $M$. Let $D(\cdot \cdot)$ be the symmetric bi-derivation of $M$, $F(\cdot \cdot \cdot)$ a symmetric bi-additive map of $M$ and $d,f$ the traces of $D(\cdot \cdot \cdot)$, $F(\cdot \cdot \cdot)$ respectively such that $d(I) \subset I$. If $d(d(x)) = f(x)$ for all $x \in I$, then $D = 0$.

PROOF. Replacing $x$ by $x\alpha z$ in (32), we have for all $x,y,z \in I$ and $\alpha \in \Gamma$

$$D(x,d(y))\alpha D(z,y) + D(x,y)\alpha D(z,d(y)) = 0$$  \hfill (34)

By writing $z\beta x$ for $z$ in (34) and from (34), we get for all $x,y,z \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,d(y))\alpha z\beta D(x,y) + D(x,y)\alpha z\beta D(x,d(y)) = 0$$  \hfill (35)

So by Lemma 3, we have for all $x,y,z \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,d(y))\alpha z\beta D(x,y) = 0$$  \hfill (36)

Writing $y+w$ for $y$ in (36), we get, for all $x,y,z \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,d(y))\alpha z\beta D(x,w) + D(x,d(y))\alpha z\beta D(x,y)$$
$$+ 2D(x,d(w))\alpha z\beta D(x,y)$$
$$+ 2D(x,D(y,w))\alpha z\beta D(x,w) = 0$$  \hfill (37)

Replacing $w$ by $-w$ in (37), we have for all $x,y,z \in I$ and $\alpha, \beta \in \Gamma$

$$-D(x,d(y))\alpha z\beta D(x,w) + D(x,d(w))\alpha z\beta D(x,y)$$
$$- 2D(x,d(w))\alpha z\beta D(x,y)$$
$$+ 2D(x,D(y,w))\alpha z\beta D(x,w) = 0$$  \hfill (38)

Adding up (37) and (38) and using the fact that $M$ is 2-torsion free, then we have for all $x,y,z \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,d(w))\alpha z\beta D(x,y) = 0$$  \hfill (39)

Replacing $z$ by $z\beta D(x,y)\alpha m\beta D(x,d(y))\alpha z$ in (39) and using (36) and the fact that $M$ is a semi-prime $\Gamma$-ring, we get for all $x,y,z,w \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,d(w))\alpha z\beta D(x,y) = 0$$  \hfill (40)

Substituting $w+t$ for $w$ in (40), and from (40) and since $M$ is 2-torsion free, we have for all $x,y,z,w,t \in I$ and $\alpha, \beta \in \Gamma$

$$D(x,D(w,t))\alpha z\beta D(x,y) = 0$$  \hfill (41)

Writing $k\gamma z$ for $z$ in (41) we get for all $x,y,z,w,t \in I$ and $\alpha, \beta, \gamma \in \Gamma$

$$D(x,D(w,t))\alpha k\gamma z\beta D(x,y) = 0$$  \hfill (42)

In the similar manner, writing $w\gamma k$ for $w$ in (41), and from (41) and (42), we have for all $x,y,z,w,t,k \in I$ and $\alpha, \beta, \gamma \in \Gamma$

$$D(x,w)\gamma D(k,t)\alpha z\beta D(x,y) + D(w,\gamma \gamma D(x,k)\alpha z\beta D(x,y) = 0$$  \hfill (43)

Writing $d(p)$ for $k$ in (43) and by (40) we get for all $x,y,z,w,t,k,p \in I$ and $\alpha, \beta, \gamma \in \Gamma$

$$D(x,w)\gamma D(t,d(p))\alpha z\beta D(x,y) = 0$$  \hfill (44)

Writing $p+k$ for $p$ in (44) and by (44) again, since $M$ is 2-torsion free, we get for all $x,y,w,u,k,p \in I$ and $\alpha, \beta, \gamma \in \Gamma$

$$D(x,w)\gamma D(t,D(k,p))\alpha z\beta D(x,y) = 0$$  \hfill (45)

Replacing $z$ by $q\gamma z$ in (45) we get for all $x,y,z,w,t,k,p \in I$ and $\alpha, \beta, \gamma, \gamma' \in \Gamma$

$$D(x,w)\gamma D(t,D(k,p))\alpha q\gamma z\beta D(x,y) = 0$$  \hfill (46)

Writing $\alpha'$ and $\alpha$ for $\alpha$ and $\gamma$ respectively in (46) we have for all $x,y,z,w,t,k,p,q \in I$ and $\alpha, \beta, \gamma, \gamma' \in \Gamma$

$$D(x,w)\gamma D(t,D(k,p))\alpha q\alpha z\beta D(x,y) = 0$$  \hfill (47)

Replacing $t$ by $\alpha \gamma q$ in (45) again and using (47) we have for all $x,y,z,w,t,k,p,q \in I$ and $\alpha, \beta, \gamma, \alpha' \in \Gamma$
\[ D(x,w)\gamma\alpha D(q, D(k, p))\alpha z\beta D(x, y) = 0 \] (48)

Writing \( \alpha t k \) for \( k \) in (45) and from (47) and (48), we get for all \( x, y, z, w, t, k, p, q \in I \) and \( \alpha, \beta, \gamma, \alpha' \in \Gamma \)

\[ D(x,w)\gamma D(t,q)\alpha D(k,p)\alpha z\beta D(x, y) + D(x,w)\gamma D(q,p)\alpha' D(t,k)\alpha z\beta D(x, y) = 0 \] (49)

Replacing \( p \) by \( t \) in (49) and since \( M \) is 2-torsion free, we get for all \( x, y, z, w, t, k, p, q \in I \) and \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \Gamma \)

\[ D(x,w)\gamma D(t,q)\alpha D(k,t)\alpha z\beta D(x, y) = 0 \] (50)

Replacing \( z \) by \( z' m' D(x, w)\gamma D(t,q) \) in (50) we have, for all \( x, y, z, w, t, k, p, q \in I \) and \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \Gamma \)

\[ D(x,w)\gamma D(t,q)\alpha D(k,t)\alpha z' m' D(x, w)\gamma D(t,q) \beta D(x, y) = 0 \] (51)

Taking \( \beta \) for \( \alpha' \) and \( x, y \) for \( k, t \) respectively in the previous and using the fact that \( M \) is a semi-prime \( \Gamma \)-ring then we have, for all \( x, y, z, w, q \in I \) and \( \alpha, \beta, \gamma \in \Gamma \)

\[ D(x,w)\gamma D(y,q) \beta D(x, y) = 0 \] (52)

From (52), Since \( D(x,w)\gamma D(y,q) \beta D(x, y) \in \text{Ann}_I I = 0 \), we get for all \( x, y, w, t, q \in I \) and \( \beta, \gamma \in \Gamma \)

\[ D(x,w)\gamma D(y,q) \beta D(x, y) = 0 \] (53)

Writing \( w oz \) for \( w \) in (53), we get for all \( x, y, z, w, t, q \in I \) and \( \alpha, \beta, \gamma \in \Gamma \)

\[ D(x,w)\alpha z\gamma D(y,q) \beta D(x, y) = 0 \] (54)

Replacing \( z, t \) and \( q \) by \( D(x,y)\beta z' \gamma' m, x \) and \( w \) respectively, we get for all \( x, y, z, w \in I, m \in M \) and \( \alpha, \beta, \gamma, \beta', \gamma' \in \Gamma \)

\[ D(x,w)\alpha D(x,y)\beta z' \gamma' m D(x, w) \beta D(x, y) = 0 \] (55)

Taking \( \alpha \) for \( \beta \) in (55) and Since \( M \) is a semi-prime \( \Gamma \)-ring, we have for all \( x, y, z, w \in I \) and \( \alpha, \beta \in \Gamma \)

\[ D(x,w)\alpha D(x,y)\beta z = 0 \] (56)

From (56), since \( D(x,w)\alpha D(x,y) \in \text{Ann}_I I = 0 \), we get for all \( x, y, w \in I \) and \( \alpha \in \Gamma \)

\[ D(x,w)\alpha D(x,y) = 0 \] (57)

Writing \( y/\beta w \) for \( w \) in (57) and using (57) again, we have for all \( x, y, w \in I \) and \( \alpha, \beta \in \Gamma \)

\[ D(y,x)\beta y = 0 \] (58)

By writing \( x \) for \( y \) in (58), we get for all \( x, w \in I \) and \( \alpha, \beta \in \Gamma, d(x)\beta y = 0 \). Thus by Lemma 4 and since \( M \) is a semi-prime \( \Gamma \)-ring, we have \( d(m) = 0 \) for all \( m \in M \). So by Lemma 5 we get \( D = 0 \).

**THEOREM 5.** Let \( M \) be a 2,3-torsion free prime \( \Gamma \)-ring, \( I \) a non-zero ideal of \( M \) and \( d \) the trace of \( D(.,.) \), non-zero symmetric bi-derivation of \( M \). If \( d(I) \subset I \cap Z \), then \( M \) is commutative, where \( Z \) is the center of \( M \).

**PROOF.** By Lemma 5 \( Z \neq 0 \). Let \( x, y \in I, m \in M \) and \( \alpha \in \Gamma \) be arbitrary elements. We denote \( x y - y x \) by \( [x, y]_\alpha \). Since \( 0 = [d(x+y), m] = 2[D(x,y), m] \), we get

\[ [D(x,y), m]_\alpha = 0 \] (59)

Replacing \( x \) by \( d(y)\beta x, \beta \in \Gamma \) in (59), by using [3, Lemma 1.(ii)] we get

\[ D(d(y), \beta) x = 0 \] (60)

Thus by [3, Lemma 1.(vi)] we get for all \( x, y \in I, m \in M \) and \( \alpha, \beta \in \Gamma \)

\[ D(d(y), \beta) = 0 \] (61)

Let \( D(d(y), y) = 0 \) for all \( x, y \). Then by using Theorem 1, \( D = 0 \) which contradicts with the hypothesis. If \( I, M \) for all \( \alpha \in \Gamma \), then \( I \subset Z \) and so by [3, Lemma 2(i)], \( M \) is commutative.
REFERENCES


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