SOME RESULTS ON HYPER$K$-ALGEBRAS

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ABSTRACT. In hyper$K$-algebras, the notion of a bounded hyper$K$-algebra and a homomorphism is introduced, and some properties related with a (weak) hyper$K$-ideal are investigated. The zero condition in a hyper$K$-algebra is considered, and then it is showed that every hyper$K$-algebra with the zero condition can be extended to a bounded hyper$K$-algebra.

1. Introduction The hyper algebraic structure theory was introduced in 1934 [7] by Marty at the 8th congress of Scandinavian Mathematicians. Since then many researchers have worked on this area. Imai and Iséki [3] introduced the notion of a BCK-algebra. Recently Jun et al. [6] applied the hyperstructures to BCK-algebras and introduced the concept of a hyperBCK-algebra which is a generalization of a BCK-algebra. Then Borzoei et al. [1] defined the notion of a hyper$K$-algebra. For background and notations we follow Borzoei et al. [1]. In this paper we introduced the notion of a bounded hyper$K$-algebra and a homomorphism of hyper$K$-algebras, and then we investigate some related results. We also consider the zero condition in hyper$K$-algebras. We show that every hyper$K$-algebra with the zero condition can be extended to a bounded hyper$K$-algebra.

2. Preliminaries

Definition 2.1 ([1], Definition 3.1). By a hyper$K$-algebra we mean a non-empty set $H$ endowed with a hyperoperation "$\circ$" and a constant $0$ satisfying the following conditions:

(HK1) \( x \circ z \circ (y \circ z) < x \circ y, \)
(HK2) \( (x \circ y) \circ z = (x \circ z) \circ y, \)
(HK3) \( x < x, \)
(HK4) \( x < y \) and \( y < x \) imply \( x = y, \)
(HK5) \( 0 < x, \)

for all \( x, y, z \in H \), where \( x < y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq H \), \( A < B \) is defined by \( \exists a \in A \) and \( \exists b \in B \) such that \( a < b. \)

Example 2.2 ([1], Example 3.2). (i) Define the hyper operation "$\circ$" on \( H = [0, +\infty) \) as follows:

\[
x \circ y := \begin{cases} 
[0, x] & \text{if } x \leq y \\
(0, y) & \text{if } x > y \neq 0 \\
\{x\} & \text{if } y = 0 
\end{cases}
\]

for all \( x, y \in H \). Then \( (H, \circ, 0) \) is a hyper$K$-algebra.

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(ii) Let \( H = \{0, a, b\} \). Consider the following table:

\[
\begin{array}{c|ccc}
\circ & 0 & a & b \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
a & \{a\} & \{0, a\} & \{0, a\} \\
b & \{b\} & \{a, b\} & \{0, a, b\}
\end{array}
\]

Then \( (H, \circ, 0) \) is a hyper \( K \)-algebra.

(iii) Let \( H = \{0, 1, 2\} \). Consider the following table:

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0, 1, 2\} & \{0, 1, 2\} \\
1 & \{1\} & \{0, 1, 2\} & \{0, 1, 2\} \\
2 & \{2\} & \{2\} & \{0, 1, 2\}
\end{array}
\]

Then \( (H, \circ, 0) \) is a hyper \( K \)-algebra.

**Theorem 2.3** ([1], Theorem 3.7). Let \( (H_1, \circ_1, 0) \) and \( (H_2, \circ_2, 0) \) be two hyper \( K \)-algebra such that \( H_1 \cap H_2 = \{0\} \) and \( H = H_1 \cup H_2 \). Then \( (H, \circ, 0) \) is a hyper \( K \)-algebra, where the hyper operation \( \circ \) on \( H \) is defined by:

\[
x \circ y = \begin{cases} 
  x \circ_1 y & \text{if } x, y \in H_1 \\
  x \circ_2 y & \text{if } x, y \in H_2 \\
  \{x\} & \text{otherwise,}
\end{cases}
\]

for all \( x, y \in H \), and we denote it by \( H_1 \oplus H_2 \).

**Theorem 2.4** ([1], Theorem 3.9). Let \( (H_1, \circ_1, 0_1) \) and \( (H_2, \circ_2, 0_2) \) be hyper \( K \)-algebras and \( H = H_1 \times H_2 \). We define a hyperoperation \( \circ \) on \( H \) as follows:

\[
(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ_1 a_2, b_1 \circ_2 b_2)
\]

for all \( (a_1, b_1), (a_2, b_2) \in H \), where for \( A \subseteq H_1 \) and \( B \subseteq H_2 \) by \( (A, B) \) we mean

\[
(A, B) = \{(a, b) : a \in A, b \in B\}, \quad 0 = (0_1, 0_2)
\]

and

\[
(a_1, b_1) < (a_2, b_2) \iff a_1 < a_2 \quad \text{and} \quad b_1 < b_2.
\]

Then \( (H, \circ, 0) \) is a hyper \( K \)-algebra, and it is called the hyper \( K \)-product of \( H_1 \) and \( H_2 \).

**Definition 2.5** ([1], Definition 4.1). Let \( I \) be a non-empty subset of a hyper \( K \)-algebra \( (H, \circ, 0) \). Then \( I \) is called a weak hyper \( K \)-ideal of \( H \) if

(Id1) \( 0 \in I \),

(Id2) \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in H \).

**Definition 2.6** ([1], Definition 4.4). Let \( I \) be a non-empty subset of a hyper \( K \)-algebra \( (H, \circ, 0) \). Then \( I \) is said to be a hyper \( K \)-ideal of \( H \) if

(Id1) \( 0 \in I \),

(Id2) \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \), for all \( x, y \in H \).

Note that every hyper \( K \)-ideal is a weak hyper \( K \)-ideal (see [1, Proposition 4.6]).

**Definition 2.7** ([1], Definition 4.11). Let \( (H, \circ, 0) \) be a hyper \( K \)-algebra and let \( S \) be a subset of \( H \) containing \( 0 \). If \( S \) is a hyper \( K \)-algebra with respect to the hyperoperation \( \circ \) on \( H \), we say that \( S \) is a hyper \( K \)-subalgebra of \( H \).

**Theorem 2.8** ([1], Theorem 4.12). Let \( S \) be a non-empty subset of a hyper \( K \)-algebra
\((H, \circ, 0)\). Then \(S\) is a hyper\(K\)-subalgebra of \(H\) if and only if \(x \circ y \subseteq S\) for all \(x, y \in S\).

3. Bounded hyper\(K\)-algebras

**Definition 3.1.** Let \((H, \circ, 0)\) be a hyper\(K\)-algebra. If there exists an element \(e \in H\) such that \(x < e\) for all \(x \in H\), then \(H\) is called a bounded hyper\(K\)-algebra and \(e\) is said to be the unit of \(H\).

Note that (HK4) implies that the unit of \(H\) is unique.

**Example 3.2.** (i) Let \((X, * , 0)\) be a bounded BCK-algebra. Define the hyper operation “\(\circ\)” on \(X\) as follows:

\[
x \circ y = \{x \ast y\}, \quad \forall x, y \in X.
\]

Then \((X, \circ, 0)\) is a bounded hyper\(K\)-algebra.

(ii) The hyper\(K\)-algebra \((H, \circ, 0)\) in Example 2.2(i) is not bounded, because if \(a \in H\) is unit, then \((a + 1) \circ a = (0, a)\). Thus \(0 \notin (a + 1) \circ a\), i.e., \(a + 1 \neq a\).

(iii) In Example 2.2(ii), \(H\) is bounded and \(b \in H\) is unit.

(iv) The hyper\(K\)-algebra \((H, \circ, 0)\) in Example 2.2(iii) is bounded and 2 \(\in H\) is unit.

**Proposition 3.3.** Let \(H_1\) and \(H_2\) be two bounded hyper\(K\)-algebras. Then the hyper\(K\)-product \(H_1 \times H_2\) of \(H_1\) and \(H_2\) is also bounded.

**Proof.** Let \(e_1 \in H_1\) and \(e_2 \in H_2\) be units and \((x, y) \in H_1 \times H_2\). Then \(x < e_1\) and \(y < e_2\) and so \((x, y) < (e_1, e_2)\). Therefore \(H_1 \times H_2\) is bounded and \((e_1, e_2)\) is its unit. □

The following example shows that if \(H_1\) and \(H_2\) are two bounded hyper\(K\)-algebras, then \(H_1 \oplus H_2\) may not be bounded. For the notation \(H_1 \oplus H_2\), we follow Borzoei [1].

**Example 3.4.** Let \(H_1\) and \(H_2\) be hyper\(K\)-algebras as in Examples 2.2(ii) and 2.2(iii) respectively. Then \(H_1\) and \(H_2\) are bounded, while \(H_1 \oplus H_2\) is not bounded.

**Definition 3.5.** Let \(H\) be a hyper\(K\)-algebra. If \(0 \circ x = \{0\}\) for all \(x \in H\), then we say that \(H\) satisfies the zero condition.

**Example 3.6.** Let \(H\) be a hyper\(K\)-algebra as in Example 2.2(i). Then \(H\) satisfies the zero condition.

**Theorem 3.7.** Let \((H_1, \circ_1, 0)\) be a hyper\(K\)-algebra, which satisfies the zero condition. Then \((H_1, \circ_1, 0)\) can be extended to a bounded hyper\(K\)-algebra.

**Proof.** Let \(e \notin H_1\) and \(H = H_1 \cup \{e\}\). Define the hyper operation “\(\circ\)” on \(H\) as follows:

\[
x \circ y = \begin{cases} 
\{e\} & \text{if } x = e, y \in H_1 \\
\{0\} & \text{if } x = e, y = e \\
\{0, x\} & \text{if } x \in H_1, y = e \\
x \circ_1 y & \text{if } x, y \in H_1,
\end{cases}
\]

for all \(x, y \in H\). We show that \((H, \circ, 0)\) is a bounded hyper\(K\)-algebra and \(e\) is its unit.

(HK1): If \(x, y, z \in H_1\), then by hypothesis (HK1) holds. Thus let at least one of \(x, y\) and \(z\) equal to \(e\). If \(x = e\) and \(y, z \in H_1\), then

\[
(e \circ z) \circ (y \circ z) = \{e\} \circ (y \circ z) = \{e\} < \{e\} = e \circ y.
\]

If \(z = e\) and \(x, y \in H_1\), then

\[
(x \circ e) \circ (y \circ e) = \{0, x\} \circ \{0, y\} = (0 \circ 0) \cup (0 \circ y) \cup (x \circ 0) \cup (x \circ y) < x \circ y.
\]
If $y = e$ and $x, z \in H_1$, then
\[(x \circ z) \circ (e \circ z) = (x \circ z) \circ \{e\} = \{0\} \cup (x \circ z) < \{0, x\} = x \circ e.\]

If $x = z = e$ and $y \in H_1$, then since $0 < e$ we have
\[(e \circ e) \circ (y \circ e) = \{0\} \circ \{0\} = (0 \circ 0) \cup (0 \circ y) < \{e\} = e \circ y.\]

If $y = z = e$ and $x \in H_1$, then
\[(x \circ e) \circ (e \circ e) = \{0, x\} \circ \{0\} = (0 \circ 0) \cup (x \circ 0) < \{0, x\} = x \circ e.\]

If $x = y = z = e$, then
\[(e \circ e) \circ (e \circ e) = \{0\} \circ \{0\} < \{0\} = e \circ e.\]

(HK2): If $x, y, z \in H_1$, then (HK2) holds. Thus we let at least one of $x, y, z$ equal to $e$.
If $x = e$ and $y, z \in H_1$, then
\[(e \circ y) \circ z = \{e\} \circ z = \{e\} = \{e\} \circ y = (e \circ z) \circ e.\]

If $y = e$ and $x, z \in H_1$, then since $H_1$ satisfies the zero condition we get that
\[(x \circ e) \circ z = \{0, x\} \circ z = (0 \circ z) \cup (x \circ z) = \{0\} \cup (x \circ z) = (x \circ z) \cup \{0\} = (x \circ z) \circ e.\]

If $x = y = e$ and $z \in H_1$, then since $H_1$ satisfies the zero condition we have
\[(e \circ e) \circ z = \{0\} \circ z = \{e\} = \{e\} \circ e = (e \circ z) \circ e.\]

(HK3) Since $e \circ e = \{0\}$, thus $0 < e \circ e$ and consequently $e < e$.
(HK4) and (HK5) are proved easily. Hence $(H, \circ, 0)$ is a hyper $K$-algebra. Moreover, since for any $x \neq e$, we have $x \circ e = \{0, x\}$, thus $x < e$. In other words $(H, \circ, 0)$ is bounded with unit $e$. □

4. Homomorphisms of hyper $K$-algebras

**Definition 4.1.** Let $H_1$ and $H_2$ be two hyper $K$-algebras. A mapping $f : H_1 \rightarrow H_2$ is said to be a **homomorphism** if

(i) $f(0) = 0$

(ii) $f(x \circ y) = f(x) \circ f(y)$, \( \forall x, y \in H_1 \).

If $f$ is 1-1 (or onto) we say that $f$ is a **monomorphism** (or **epimorphism**). And if $f$ is both 1-1 and onto, we say that $f$ is an **isomorphism**.

**Example 4.2.** Let $H$ be as in Example 2.2(i) and $t \in \mathbb{R}^+$ be constant. Define
\[f : H \rightarrow H, \quad f(x) = tx, \quad \forall x \in H.\]

Then $f$ is an isomorphism of hyper $K$-algebras. To do this, let $x, y \in H$ and $x \leq y$. Then $tx \leq ty$ and thus $f(x \circ y) = f([0, x]) = [0, tx] = tx \circ ty = f(x) \circ f(y)$. If $x > y \neq 0$, then $tx > ty$ and so
\[f(x \circ y) = f((0, y]) = (0, ty] = tx \circ ty = f(x) \circ f(y).\]

If $y = 0$, then
\[f(x \circ 0) = f([x]) = tx = tx \circ 0 = f(x) \circ f(0).\]

Also $f(0) = 0$, consequently $f$ is a homomorphism. Clearly $f$ is onto and 1-1. Thus $f$ is an isomorphism.

**Theorem 4.3.** Let $f : H_1 \rightarrow H_2$ be a homomorphism of hyper $K$-algebras. Then
(i) If $S$ is a hyper-$K$-subalgebra of $H_1$, then $f(S)$ is a hyper-$K$-subalgebra of $H_2$,
(ii) $f(H_1)$ is a hyper-$K$-subalgebra of $H_2$,
(iii) If $H_1$ satisfies the zero condition, then so is $f(H_1)$,
(iv) If $S$ is a hyper-$K$-subalgebra of $H_2$, then $f^{-1}(S)$ is a hyper-$K$-subalgebra of $H_1$,
(v) If $I$ is a (weak) hyper-$K$-ideal of $H_2$, then $f^{-1}(I)$ is a (weak) hyper-$K$-ideal of $H_1$,
(vi) $\ker f := \{x \in H_1 \mid f(x) = 0\}$ is a hyper-$K$-ideal and hence a weak hyper-$K$-ideal of $H_1$,
(vii) If $f$ is onto and $I$ is a hyper-$K$-ideal of $H_1$ which contains $\ker f$, then $f(I)$ is a hyper-$K$-ideal of $H_2$.

Proof. (i) Let $x, y \in f(S)$. Then there exist $a, b \in S$ such that $f(a) = x$ and $f(b) = y$. It follows from Theorem 2.8 that

$$x \circ y = f(a) \circ f(b) = f(a \circ b) \subseteq f(S)$$

so that $f(S)$ is a hyper-$K$-subalgebra of $H_2$.

(ii) It is straightforward by (i).

(iii) If $H_1$ satisfies the zero condition, then $0 \circ x = \{0\}$ for all $x \in H_1$. Let $y \in f(H_1)$. Then there exists $a \in H_1$ such that $f(a) = y$. It follows that

$$0 \circ y = f(0) \circ f(a) = f(0 \circ a) = f(\{0\}) = \{0\}$$

so that $f(H_1)$ satisfies the zero condition.

(iv) Since $0 \in S$, we have $f^{-1}(0) \subseteq f^{-1}(S)$. Since $f(0) = 0$, so $0 \in f^{-1}(0) \subseteq f^{-1}(S)$. Therefore $f^{-1}(S)$ is non-empty. Now let $x, y \in f^{-1}(S)$. Then $f(x), f(y) \in S$. Thus $f(x \circ y) = f(x) \circ f(y) \subseteq S$ and so $x \circ y \subseteq f^{-1}(S)$, which implies that $f^{-1}(S)$ is a hyper-$K$-subalgebra of $H_1$.

(v) Let $I$ be a weak hyper-$K$-ideal of $H_2$. Clearly $0 \in f^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \subseteq f^{-1}(I)$ and $y \in f^{-1}(I)$. Then $f(x \circ y) = f(x) \circ f(y) \subseteq I$ and $f(y) \in I$. Since $I$ is a weak hyper-$K$-ideal, it follows from (Id2) that $f(x) \in I$, i.e., $x \in f^{-1}(I)$. Hence $f^{-1}(S)$ is a weak hyper-$K$-ideal of $H_1$. Now let $I$ be a hyper-$K$-ideal of $H_2$. Obviously $0 \in f^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y < f^{-1}(I)$ and $y \in f^{-1}(I)$. Then there exist $t \in x \circ y$ and $z \in f^{-1}(I)$ such that $t < z$, i.e., $0 \in t \circ z$. Since $f(z) \in I$ and $0 \in t \circ z \subseteq (x \circ y) \circ z$, it follows that

$$0 = f(0) \in f((x \circ y) \circ z) = f(x \circ y) \circ f(z) \subseteq f(x \circ y) \circ I$$

so that $f(x) \circ f(y) = f(x \circ y) < I$. As $f(y) \in I$ and $I$ is hyper-$K$-ideal, by using (Id3) we have $f(x) \in I$, i.e., $x \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a hyper-$K$-ideal of $H_1$.

(vi) First we show that $\{0\} \subseteq H_2$ is a hyper-$K$-ideal. To do this, let $x, y \in H_2$ be such that $x \circ y < \{0\}$ and $y \in \{0\}$. Then $y = 0$ and so $x \circ 0 = x \circ y < \{0\}$. Therefore there exists $t \in x \circ 0$ such that $t < 0$. Thus $t = 0$, and consequently $0 \in x \circ 0$, i.e., $x < 0$, which implies that $x = 0$. This shows that $\{0\}$ is a hyper-$K$-ideal of $H_2$. Now by (v), $\ker f = f^{-1}(\{0\})$ is a hyper-$K$-ideal of $H_1$.

(vii) Since $0 \in I$, we have $0 = f(0) \in f(I)$. Let $x$ and $y$ be arbitrary elements in $H_2$ such that $x \circ y < f(I)$ and $y \in f(I)$. Since $y \in f(I)$ and $f$ is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = f(y_1)$ and $x = f(x_1)$. Thus

$$f(x_1 \circ y_1) = f(x_1) \circ f(y_1) = x \circ y < f(I).$$

Therefore there are $a \in x_1 \circ y_1$ and $b \in I$ such that $f(a) < f(b)$. So $0 \in f(a) \circ f(b) = f(a \circ b)$, which implies that $f(c) = 0$ for some $c \in a \circ b$. It follows that $c \in \ker f \subseteq I$ so that $a \circ b < I$. Now since $I$ is a hyper-$K$-ideal of $H_1$ and $b \in I$, we get $a \in I$. Thus $x_1 \circ y_1 < I$, which implies that $x_1 \in I$. Thereby $x = f(x_1) \in f(I)$, and so $f(I)$ is a hyper-$K$-ideal of $H_2$. □
The following theorem is straightforward, and we omit the proof.

**Theorem 4.4.** Let $f : H_1 \to H_2$ be an epimorphism of hyper $K$-algebras. Then there is a one to one correspondence between the set of all $K$-ideals of $H_1$ containing $Ker f$ and the set of all $K$-ideals of $H_2$.

**Lemma 4.5.** Let $f : H_1 \to H_2$ be a homomorphism of hyper $K$-algebras. If $x < y$ in $H_1$, then $f(x) < f(y)$ in $H_2$.

Proof. If $x < y$ in $H_1$, then $0 \in x \circ y$ and so

$$0 = f(0) \in f(x \circ y) = f(x) \circ f(y).$$

Therefore $f(x) < f(y)$.

**Theorem 4.6.** Let $f : H_1 \to H_2$ be an epimorphism of hyper $K$-algebras. If $H_1$ is bounded, then $H_2$ is also bounded.

Proof. Let $e$ be the unit of $H_1$ and $y \in H_2$ be an arbitrary element. Then there exists $x \in H_1$ such that $f(x) = y$. Since $x < e$, by Lemma 4.5 we have $y = f(x) < f(e)$. Thus $f(e)$ is the unit of $H_2$ and $H_2$ is bounded.

**Theorem 4.7.** Let $f : H_1 \to H_2$ and $g : H_1 \to H_3$ be two homomorphisms of hyper $K$-algebras such that $f$ is onto and $Ker f \subseteq Ker g$. Then there exists a homomorphism $h : H_2 \to H_3$ such that $h \circ f = g$.

Proof. Let $y \in H_2$ be arbitrary. Since $f$ is onto, there exists $x \in H_1$ such that $y = f(x)$. Define $h : H_2 \to H_3$ by $h(y) = g(x)$, $\forall y \in H_2$. Now we show that $h$ is well-defined. Let $y_1, y_2 \in H_2$ and $y_1 = y_2$. Since $f$ is onto, there are $x_1, x_2 \in H_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Therefore $f(x_1) = f(x_2)$ and thus $0 \in f(x_1) \circ f(x_2) = f(x_1 \circ x_2)$. It follows that there exists $t \in x_1 \circ x_2$ such that $f(t) = 0$. Thus $t \in Ker f \subseteq Ker g$ and so $g(t) = 0$. Since $t \in x_1 \circ x_2$ we conclude that

$$0 = g(t) \in g(x_1 \circ x_2) = g(x_1) \circ g(x_2),$$

which implies that $g(x_1) < g(x_2)$. On the other hand since $0 \in f(x_2) \circ f(x_1) = f(x_2 \circ x_1)$, similarly we can conclude that $0 \in g(x_2) \circ g(x_1)$, i.e., $g(x_2) < g(x_1)$. Thus $g(x_1) = g(x_2)$, which shows that $h$ is well-defined. Clearly $h \circ f = g$. Finally we show that $h$ is a homomorphism. Let $y_1, y_2 \in H_2$ be arbitrary. Since $f$ is onto there are $x_1, x_2 \in H_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then

$$h(y_1 \circ y_2) = h(f(x_1) \circ f(x_2)) = h(f(x_1 \circ x_2)) = (h \circ f)(x_1 \circ x_2) = g(x_1 \circ x_2) = g(x_1) \circ g(x_2) = (h \circ f)(x_1) \circ (h \circ f)(x_2) = h(f(x_1)) \circ h(f(x_2)) = h(y_1) \circ h(y_2).$$

Moreover since $f(0) = 0$ and $g(0) = 0$, we conclude that

$$h(0) = h(f(0)) = (h \circ f)(0) = g(0) = 0.$$

Thus $h$ is a homomorphism, ending the proof.

**Theorem 4.8.** Let $f : H_1 \to H_2$ be a monomorphism of hyper $K$-algebras. If $H_2$ is bounded with unit element $e$ and $e \in Im f$, then $H_1$ is also bounded and $f^{-1}(e)$ is its unit.

Proof. Let $x \in H_1$. Then $f(x) \in H_2$. Since $H_2$ is bounded we conclude that $f(x) < e$, and since $e \in Im f$, we get that $e = f(a)$ for some $a \in H_1$. Thus $f(x) < f(a)$. Therefore $0 \in f(x) \circ f(a) = f(x \circ a)$. It follows that there exists $b \in x \circ a$ such that $f(b) = 0$. Hence $b = 0$, because $f$ is 1-1. Thus $0 \in x \circ a$, i.e., $x < a$. Now since $a = f^{-1}(e)$, we conclude that
$x < f^{-1}(e)$, which shows $H_1$ is bounded with unit $f^{-1}(e)$. □

References