A FURTHER GENERALIZATION OF PARANORMAL OPERATORS

TAKEAKI YAMAZAKI AND MASAHIRO YANAGIDA

ABSTRACT. As a further generalization of paranormal operators, we shall introduce a new class “absolute-\((p, r)\)-paranormal” operators for \(p > 0\) and \(r > 0\) such that 
\[
\| T^* |T|^r x \| \geq \| |T|^p x \|^{p+r} \quad \text{for every unit vector } x.
\]
And we shall show several properties on absolute-\((p, r)\)-paranormal operators as generalizations of the results on absolute-\(k\)-paranormal and \(p\)-paranormal operators introduced in [10] and [6], respectively.

1. Introduction

In this paper, an operator means a bounded linear operator on a complex Hilbert space \(H\). An operator \(T\) is said to be positive (denoted by \(T \geq 0\)) if \((Tx, x) \geq 0\) for all \(x \in H\), and also \(T\) is said to be strictly positive (denoted by \(T > 0\)) if \(T\) is positive and invertible.

An operator \(T\) is said to be hyponormal if \(T^* T \geq TT^*\). As extensions of it, \(p\)-hyponormal and log-hyponormal operators are defined. An operator \(T\) is said to be \(p\)-hyponormal for \(p > 0\) if \((T^* T)^p \geq (TT^*)^p\), and \(T\) is said to be log-hyponormal if \(T\) is invertible and \(\log T^* T \geq \log TT^*\). It is easily seen that every \(p\)-hyponormal operator is \(q\)-hyponormal for \(p \geq q > 0\) by the celebrated L"owner-Heinz theorem “\(A \geq B \geq 0\) ensures \(A^\alpha \geq B^\alpha\) for any \(\alpha \in [0, 1]\)” and every invertible \(p\)-hyponormal operator for \(p > 0\) is log-hyponormal since \(\log t\) is an operator monotone function.

On the other hand, \(T\) is said to be paranormal if
\[
\| T^2 x \| \geq \| Tx \|^2 \quad \text{for every unit vector } x.
\]
Paranormal operators have been studied by many researchers, for example, [4], [8] and [11]. Particularly, Ando [4] showed that every log-hyponormal operator is paranormal. Afterward, in [10], we gave another simplified proof of this result by introducing class \(A\) as a new class of operators given by an operator inequality. In fact, \(T\) belongs to class \(A\) if
\[
\| T^2 \| \geq \| T \|^2,
\]
where \(|T| = (T^* T)^{1/2}\), and we showed that every log-hyponormal operator belongs to class \(A\) and every class \(A\) operator is paranormal.

We introduced class \(A(k)\) and absolute-\(k\)-paranormal operators for \(k > 0\) in [10] as generalizations of class \(A\) and paranormal operators, respectively. \(T\) belongs to class \(A(k)\) if
\[
(T^* |T|^2kT)^{1/p+r} \geq |T|^2,
\]
and \(T\) is said to be absolute-\(k\)-paranormal if
\[
\| |T|^kTx \| \geq \| Tx \|^{k+1} \quad \text{for every unit vector } x.
\]

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It is clear that class $A(1)$ equals class $A$ and absolute-1-paranormality equals paranormality since $||S|y| = \|Sy\|$ for any $S \in B(H)$ and $y \in H$. $T$ is said to be normaloid if $||T|| = r(T)$. We showed inclusion relations among these classes in [10]. Class $A$ and class $A(k)$ operators have been studied in [12], [13] and [15].

On the other hand, Fujii, Izumino and Nakamoto [6] introduced $p$-paranormal operators for $p > 0$ as another generalization of paranormal operators. $T$ is said to be $p$-paranormal if
\[ ||T|^p U|T|^p x|| \geq ||T|^p x||^2 \quad \text{for every unit vector} \quad x, \]
where the polar decomposition of $T$ is $T = U|T|$. It is clear that 1-paranormality equals paranormality. $p$-Paranormality is based on the following fact [5]: $T = U|T|$ is $p$-hyponormal if and only if $S = U|T|^p$ is hyponormal for $p > 0$. Actually, it was shown in [6] that $T = U|T|$ is $p$-paranormal if and only if $S = U|T|^p$ is paranormal for $p > 0$.

Fujii, Jung, S.H.Lee, M.Y.Lee and Nakamoto [7] introduced class $A(p, r)$ as a further generalization of class $A(k)$. $T$ belongs to class $A(p, r)$ for $p > 0$ and $r > 0$ if
\[ (||T^*|^p|T^*|^2|T^*|^r||x||^2) \geq ||T^*|^r x||^{p+r}, \]
and class $AI(p, r)$ is the class of all invertible operators which belong to class $A(p, r)$. It was pointed out in [15] that class $A(k, 1)$ equals class $A(k)$ for each $k > 0$.

In this paper, we shall introduce a further generalization of the classes of both absolute-$k$-paranormal and $p$-paranormal operators as a parallel concept to class $A(p, r)$. Then we shall generalize the results on absolute-$k$-paranormal and $p$-paranormal operators for this new class.

2. Definition and Properties of Absolute-$(p, r)$-Paranormal Operators

We introduce the following new class of operators.

**Definition.** For positive real numbers $p > 0$ and $r > 0$, $T$ is absolute-$(p, r)$-paranormal if
\[ ||T|^p |T^*|^r x||^r \geq ||T^*|^r x||^{p+r} \quad \text{for every unit vector} \quad x, \]
or equivalently,
\[ ||T|^p |T^*|^r x||^r \|x||^p \geq ||T^*|^r x||^{p+r} \quad \text{for all} \quad x \in H. \]

We remark that the definition of absolute-$(p, r)$-paranormal operators (2.1) and (2.2) are expressed in terms of only $T$ and $T^*$, without $U$ which appears in the polar decomposition of $T = U|T|$.

To consider the relations to absolute-$k$-paranormality and $p$-paranormality, we show another expression of absolute-$(p, r)$-paranormality as follows.

**Proposition 1.** For each $p > 0$ and $r > 0$, $T$ is absolute-$(p, r)$-paranormal if and only if
\[ ||T|^p U|T|^r x||^r \geq ||T|^r x||^{p+r} \quad \text{for every unit vector} \quad x, \]
where the polar decomposition of $T$ is $T = U|T|$.

The following result is easily obtained as a corollary of Proposition 1.

**Corollary 2.**
(i) For each $k > 0$, $T$ is absolute-$k$-paranormal iff $T$ is absolute-$(k, 1)$-paranormal.
(ii) For each $p > 0$, $T$ is $p$-paranormal iff $T$ is absolute-$(p, p)$-paranormal.
(iii) $T$ is paranormal iff $T$ is absolute-$1$-paranormal.

It turns out by Corollary 2 that absolute-$(p, r)$-paranormality is a further generalization of paranormality than both absolute-$k$-paranormality and $p$-paranormality.
Proof of Proposition 1. It is well known that $|T^*|^r = U|T|^r U^*$ for $r > 0$, so that (2.2) is equivalent to the following (2.4):

$$(2.4) \quad \| |T|^p U |T|^r U^* x \|^p \geq \| U |T|^r U^* x \|^p + r \quad \text{for all } x \in H.$$ 

It is also well known that $N(S^*) = N(S)$ for any $S \geq 0$ and $r > 0$. By using this fact, we have $R(|T|^r) \subseteq R(|T|^r) = N(|T|^r)^\perp = N(U)^\perp$, so that $\| |T|^r U^* x \| = \| |T|^r U^* x \|$ for all $x \in H$. Hence (2.4) is equivalent to the following (2.5):

$$(2.5) \quad \| |T|^p U |T|^r U^* x \|^p \geq \| |T|^r U^* x \|^p + r \quad \text{for all } x \in H.$$ 

Put $x = Uy$ in (2.5), then we have the following (2.6) since $|T|^r U^* U = |T|^r$:

$$(2.6) \quad \| |T|^p U |T|^r y \|^p \geq \| |T|^r y \|^p + r \quad \text{for all } y \in H.$$ 

(2.6) yields the following (2.7) since $\| y \| \geq \| Uy \|$ for all $y \in H$:

$$(2.7) \quad \| |T|^p U |T|^r y \|^p \geq \| |T|^r y \|^p + r \quad \text{for all } y \in H.$$ 

Hence (2.5) implies (2.7). Here we show that (2.7) implies (2.5) conversely. Put $y = U^* x$ in (2.7), then we have

$$(2.8) \quad \| |T|^p U |T|^r U^* x \|^p \geq \| |T|^r U^* x \|^p + r \quad \text{for all } x \in H.$$ 

(2.8) yields (2.5) since $\| x \| \geq \| U^* x \|$ for all $x \in H$. Hence (2.7) implies (2.5), so that (2.5) is equivalent to (2.7). Consequently, the proof of Proposition 1 is complete since (2.7) is equivalent to (2.3). \qed

Proof of Corollary 2. We remark that $\| |S| y \| = \| Sy \|$ holds for any $S \in B(H)$ and $y \in H$.

(i) Put $p = k > 0$ and $r = 1$ in (2.3), then we have (1.2).

(ii) Put $r = p > 0$ in (2.3), then we have (1.3).

(iii) Put $r = p = 1$ in (2.3), then we have (1.1). \qed

Ando [4] gave a characterization of paranormal operators via an operator inequality as follows: $T$ is paranormal if and only if

$$T^2 - 2\lambda T + \lambda^2 I \geq 0$$

for all $\lambda > 0$. A generalization of this result for absolute-$k$-paranormal operators was shown in [10, Theorem 6]. Here we show a further generalization for absolute-$(p, r)$-paranormal operators as follows.

Proposition 3. The following assertions hold for each $p > 0$ and $r > 0$:

(i) $T$ is absolute-$(p, r)$-paranormal if and only if

$$(2.9) \quad r |T^*|^r |T|^{2p} |T^*|^r - (p + r) \lambda^p |T^*|^{2r} + p \lambda^{p+r} I \geq 0 \quad \text{for all } \lambda > 0.$$ 

(ii) $T$ is $p$-paranormal if and only if

$$(2.10) \quad |T^*|^p |T|^{2p} |T^*|^p - 2\lambda |T^*|^{2p} + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0.$$ 

We use the following well-known fact in the proof of Proposition 3.

Lemma A. For positive real numbers $a > 0$ and $b > 0$,

$$\lambda a + \mu b \geq a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$. 

The following assertions hold for each Proposition 6.

Proof of Proposition 4.

Proof of (i). (2.2) is equivalent to the following (2.11):  
\[(2.11) \quad \frac{r}{p + r} \cdot \lambda^{-p}((T^\ast|T|^{2p}|T^\ast|^{r}x, x) \frac{p}{p + r} \lambda^{r}(x, x) \geq ((|T^\ast|^{2r}x, x) \quad \text{for all } x \in H.\]

By Lemma A,  
\[(|T^\ast|^{r}T^\ast|T^\ast|^{2p}|T^\ast|^{r}x, x) \frac{p}{p + r} \lambda^{r}(x, x) = \left\{ \lambda^{-p}((T^\ast|T|^{2p}|T^\ast|^{r}x, x) \frac{p}{p + r} \lambda^{r}(x, x) \right\} \frac{p}{p + r} \leq \frac{r}{p + r} \cdot \lambda^{-p}((T^\ast|T|^{2p}|T^\ast|^{r}x, x) + \frac{p}{p + r} \cdot \lambda^{r}(x, x) \]  
holds for all \(x \in H\) and \(\lambda > 0\), so that (2.11) implies the following (2.12):  
\[(2.12) \quad \frac{r}{p + r} \cdot \lambda^{-p}((T^\ast|T|^{2p}|T^\ast|^{r}x, x) + \frac{p}{p + r} \cdot \lambda^{r}(x, x) \geq ((|T^\ast|^{2r}x, x) \quad \text{for all } x \in H \text{ and } \lambda > 0.\]

Conversely, (2.11) follows from (2.12) by putting \(\lambda = \left\{ \frac{(|T^\ast|^{r}T^\ast|T^\ast|^{2p}|T^\ast|^{r}x, x)}{(x, x)} \right\} \frac{p}{p + r} > 0\) in case  
\((|T^\ast|^{r}T^\ast|T^\ast|^{2p}|T^\ast|^{r}x, x) \neq 0\), and letting \(\lambda \to +0\) in case \((|T^\ast|^{r}T^\ast|T^\ast|^{2p}|T^\ast|^{r}x, x) = 0\). Hence (2.11) is equivalent to (2.12). Consequently, the proof of Proposition 3 is complete since (2.12) is equivalent to (2.9).

Proof of (ii). Put \(r = p > 0\) and replace \(\lambda^{p}\) with \(\lambda\) in (i), then we have (ii) by (ii) of Corollary 2.

It was shown in [8] and [11] that if \(T\) is invertible and paranormal, then \(T^{-1}\) is also paranormal. Here we show the following generalization of this well-known result.

Proposition 4. The following assertions hold for each \(p > 0\) and \(r > 0\):

(i) If \(T\) is invertible and \((p, r)\)-paranormal, then \(T^{-1}\) is \((r, p)\)-paranormal.

(ii) If \(T\) is invertible and \(p\)-paranormal, then \(T^{-1}\) is also \(p\)-paranormal.

We prepare the following lemma to give a proof of Proposition 4.

Lemma 5. Let \(T\) be an invertible operator. For each \(p > 0\) and \(r > 0\), \(T\) is \((p, r)\)-paranormal if and only if  
\[(2.13) \quad |||T||^{p}x||^{p}|||T^{-1}||^{r}x||^{r} \geq 1 \quad \text{for every unit vector } x.\]

Proof. (2.2) is equivalent to the following (2.14) by putting \(y = |T^\ast|^{r}x\) since \(R(|T^\ast|^{r}) = H\):  
\[(2.14) \quad |||T||^{p}y||^{p}|||T^{-1}||^{r}y||^{r} \geq ||y||^{p+r} \quad \text{for all } y \in H.\]

(2.14) is equivalent to the following (2.15):  
\[(2.15) \quad |||T||^{p}y||^{p}|||T^{-1}||^{r}y||^{r} \geq 1 \quad \text{for every unit vector } y.\]

(2.15) is equivalent to (2.13) since \(|T^\ast|^{r} = |T^{-1}|^{p}||^{r}x, x\), so that the proof is complete.

Proof of Proposition 4.

(i) Obvious by Lemma 5.

(ii) Put \(r = p > 0\) in (i), then we have (ii) by (ii) of Corollary 2.

At the end of this section, we show the following parallel result to Proposition 4 for class \(AI(p, r)\) operators.

Proposition 6. The following assertions hold for each \(p > 0\) and \(r > 0\):

(i) If \(T\) belongs to class \(AI(p, r)\), then \(T^{-1}\) belongs to class \(AI(r, p)\).

(ii) If \(T\) belongs to class \(AI(p, p)\), then \(T^{-1}\) also belongs to class \(AI(p, p)\).

(iii) If \(T\) is invertible and belongs to class \(A\), then \(T^{-1}\) also belongs to class \(A\).
We use the following lemma in the proof of Proposition 6.

**Lemma F** ([9]). Let $A > 0$ and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^\frac{\lambda}{2}(A^\frac{\lambda}{2}B^\frac{\lambda}{2}BA^\frac{\lambda}{2})^{\lambda - 1}A^\frac{\lambda}{2}B^*$$

holds for any real number $\lambda$.

**Proof of Proposition 6.**

**Proof of (i).** Assume that $T$ belongs to class $AI(p, r)$ for $p > 0$ and $r > 0$, i.e.,

$$(1.4) \quad (|T^*|^r |T|^p |T^*|^r)^{\frac{\lambda}{p+r}} \geq |T^*|^{2r}.$$  

By Lemma F, we have

$$(|T^*|^r |T|^p |T^*|^r)^{\frac{\lambda}{p+r}} = |T^*|^r |T|^p (|T|^p |T^*|^{2r} |T|^p)^{\frac{\lambda}{p+r}} |T|^p |T^*|^r,$$

so that (1.4) implies the following (2.16):

$$\quad (2.16) \quad |T|^{2p} \geq (|T|^p |T^*|^{2r} |T|^p)^{\frac{\lambda}{p+r}}.$$  

We remark that $|T| = |T^{-1}|^{-1}$ and $|T^*| = |T^{-1}|^{-1}$. Applying these facts to (2.16), we have

$$|T^{-1}|^{-2p} \geq (|T^{-1}|^{-p} |T^{-1}|^{-2r} |T^{-1}|^{-p})^{\frac{\lambda}{p+r}} = (|T^{-1}|^p |T^{-1}|^{2r} |T^{-1}|^p)^{\frac{\lambda}{p+r}},$$

so that

$$(|T^{-1}|^p |T^{-1}|^{2r} |T^{-1}|^p)^{\frac{\lambda}{p+r}} \geq |T^{-1}|^{2p}.$$  

Hence $T^{-1}$ belongs to class $AI(r, p)$.

**Proof of (ii).** Put $r = p > 0$ in (i), then we have (ii).

**Proof of (iii).** Put $p = 1$ in (ii), then we have (iii) since class $A(1, 1)$ equals class $A$.  

**Remark.** Aluth and Wang [2] introduced $w$-hyponormal operators such that

$$|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|,$$

where the polar decomposition of $T$ is $T = U|T|$ and $\tilde{T} = |T|^\frac{1}{2} U|T|^\frac{1}{2}$. $w$-Hyponormal operators are studied in [1], [3] and [12]. It was shown in [12] [15] that the class of invertible $w$-hyponormal operators equals class $AI(\frac{1}{2}, \frac{1}{2})$, so that it turns out by Proposition 6 that if $T$ is invertible and $w$-hyponormal, then $T^{-1}$ is also $w$-hyponormal.

### 3. Inclusion relations among the related classes

We cite the following result which plays an important role to give proofs of the results in this section.

**Theorem H-M** (Hölder-McCarthy inequality [14]). Let $A$ be a positive operator. Then the following inequalities hold for all $x \in H$:

(i) $$(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$$ for $0 < r \leq 1$.

(ii) $$(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$$ for $r \geq 1$.

We remark that (i) and (ii) of Theorem H-M can be rewritten as follows:

(i)' $$(A^r x) \leq (Ax)^r \|x\|^{1-r}$$ for $0 < r \leq 1$.

(ii)' $$(A^r x) \geq (Ax)^r \|x\|^{1-r}$$ for $r \geq 1$.

Firstly, we show the monotonicity of the classes of absolute-($p, r$)-paranormal operators for $p > 0$ and $r > 0$ as generalizations of [7, Theorem 4.1] and [10, Theorem 4].
Theorem 7. Let $T$ be absolute-$(p_0,r_0)$-paranormal for $p_0 > 0$ and $r_0 > 0$. Then $T$ is absolute-$(p,r)$-paranormal for any $p \geq p_0$ and $r \geq r_0$. Moreover, for each $r \geq r_0$ and unit vector $x$,

$$f_r(p) = \|\|T|^p|T^*|^r x\|_{\frac{r}{r_0 + r}}$$

is increasing for $p \geq p_0$.

Theorem 7 can be considered as a parallel result to the following Theorem B which states the monotonicity of class $AI(p,r)$ for $p > 0$ and $r > 0$.

Theorem B ([7]). If $T$ belongs to class $AI(p_0,r_0)$ for $p_0 > 0$ and $r_0 > 0$, then $T$ belongs to class $AI(p,r)$ for any $p \geq p_0$ and $r \geq r_0$.

Proof of Theorem 7. Assume that $T$ is absolute-$(p_0,r_0)$-paranormal for $p_0 > 0$ and $r_0 > 0$, i.e.,

$$\|\|T|^p_0|T^*|^r_0 y\|_{r_0} \geq \|\|T|^p_0|T^*|^r_0 x\|_{r_0}$$

Then for each $r \geq r_0$ and unit vector $x$,

$$\|\|T|^{p_0}|T^*|^r_0 x\|_{r_0}$$

$$= \|\|T|^{p_0}|T^*|^r_0 x\|_{r_0}$$

$$\geq \|\|T|^p_0|T^*|^r_0 x\|_{r_0}$$

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so that we have

$$\|\|T|^p_0|T^*|^r_0 x\|_{r_0} \geq \|\|T|^r_0 x\|.$$ 

Hence for each $p \geq p_0$, $r \geq r_0$ and unit vector $x$,

$$\|\|T|^p_0|T^*|^r_0 x\|$$

$$\geq \|\|T|^p_0|T^*|^r_0 x\|$$

$$= \|\|T|^p_0|T^*|^r_0 x\|$$

$$\geq \|\|T|^p_0|T^*|^r_0 x\|$$

$$= \|\|T|^p_0|T^*|^r_0 x\|$$

so that we have

$$\|\|T|^p_0|T^*|^r_0 x\|_{\frac{r}{r_0 + r}} \geq \|\|T|^r_0 x\|.$$ 

(3.4) assures that $T$ is absolute-$(p,r)$-paranormal for any $p \geq p_0$ and $r \geq r_0$, and for each $r \geq r_0$ and unit vector $x$, $f_r(p) = \|\|T|^p|T^*|^r x\|_{\frac{r}{r_0 + r}}$ is increasing for $p \geq p_0$. Consequently, the proof of Theorem 7 is complete.

Secondly, we show inclusion relations among the class of absolute-$(p,r)$-paranormal operators and the related classes.

Theorem 8. The following assertions hold for each $p > 0$ and $r > 0$:

(i) Every class $A(p,r)$ operator is absolute-$(p,r)$-paranormal.

(ii) Every absolute-$(p,r)$-paranormal operator is normaloid.
(i) of Theorem 8 is a generalization of [7, Theorem 3.5] and [10, Theorem 4], and (ii) is a generalization of [10, Theorem 5] and the following result.

**Theorem C ([7]).** Every $p$-paranormal operator is normaloid for $p > 0$.

**Proof of Theorem 7.**

**Proof of (i).** Assume that $T$ belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, i.e.,

$$
\left( |T^*| |T|^{2p} |T^*|^r \right)^{\frac{1}{p+r}} \geq |T^*|^{2r}.
$$

Then for every unit vector $x$,

\[
\left\| |T^*|^r x \right\|^2 = \left( |T^*|^{2r} x, x \right) \\
\leq \left( \left( |T^*| |T|^{2p} |T^*|^r \right)^{\frac{1}{p+r}} x, x \right) \quad \text{by (1.4)} \\
\leq \left( |T^*| |T|^{2p} |T^*|^r x, x \right)^{\frac{1}{p+r}} \quad \text{by (i) of Theorem H-M for } \frac{1}{p+r} \in (0, 1) \\
= \left\| |T|^p |T^*|^r x \right\|^{\frac{2}{p+r}},
\]

so that we have

$$
\left\| |T|^p |T^*|^r x \right\| \geq \left\| |T^*|^r x \right\|^{\frac{p}{p+r}} \quad \text{for every unit vector } x,
$$

i.e., $T$ is absolute-$(p, r)$-paranormal.

**Proof of (ii).** Assume that $T$ is absolute-$(p, r)$-paranormal. Put $q = \max\{p, r\} > 0$, then $T$ is absolute-$(q, q)$-paranormal by Theorem 7, i.e., $T$ is $q$-paranormal by (ii) of Corollary 2. Hence $T$ is normaloid by Theorem C.

Lastly, we introduce a characterization of log-hyponormal operators via absolute-$(p, r)$-paranormality as an extension of [16, Theorem 1].

**Theorem 9.** The following assertions are mutually equivalent:

(i) $T$ is log-hyponormal.

(ii) $T$ is invertible and $p$-paranormal for all $p > 0$.

(iii) $T$ is invertible and absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$.

In [16], we gave a proof in terms of norm inequalities. Here we give a proof in terms of operator inequalities by using Proposition 3.

**Proof of Theorem 9.** (i) $\iff$ (ii) is [16, Theorem 1] itself. It is pointed out in [7] that every log-hyponormal operator belongs to class $AI(p, r)$ for all $p > 0$ and $r > 0$, so that (i) $\implies$ (iii) holds by (i) of Theorem 8. Hence we have only to prove (iii) $\implies$ (i).

Assume that $T$ is absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$. By (i) of Proposition 3, (2.9) holds particularly for $\lambda = 1$, that is,

$$
|T^*|^r |T|^{2p} |T^*|^{r} - (p + r) |T^*|^{2r} + pI \geq 0 \quad \text{for all } p > 0 \text{ and } r > 0.
$$

Since $T$ is invertible, (3.5) can be rewritten as the following (3.6):

$$
\frac{|T|^{2p} - I}{p} \geq \frac{|T^*|^{-2r} - I}{-r} \quad \text{for all } p > 0 \text{ and } r > 0.
$$

By letting $p \to +0$ and $r \to +0$ in (3.6), we have

$$
\log |T|^2 \geq \log |T^*|^2,
$$

i.e., $T$ is log-hyponormal. □
The following diagram represents the inclusion relations among the classes discussed in this section.

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References


DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE, SCIENCE UNIVERSITY OF TOYO, 1-3 KAGURAZAKA, SHINJUKU, TOKYO 162-8601, JAPAN

E-mail address, Takeaki Yamazaki: tyamaz@am.kagu.sut.ac.jp
E-mail address, Masahiro Yanagida: yanagida@am.kagu.sut.ac.jp