AMENABILITY OF THE ALGEBRAS $R(S), F(S)$ OF A TOPOLOGICAL SEMIGROUP S

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Abstract. For a locally compact Hausdorff semigroup S, the L^∞-representation algebra $R(S)$ was extensively studied by Dunkl and Ramirez. The Fourier- Stieltjes algebra $F(S)$ of a topological semigroup was introduced and studied by Lau. The aim of this paper is to investigate the amenability of these algebras.

1. Introduction

Dunkl and Ramirez defined the subalgebra $R(S)$ of the algebra of weakly almost periodic functions on S, WAP(S) [2]. This algebra is called the L^∞-representation algebra of S. In fact $R(S)$ is the set of all functions $f(x) = \int_X (Tx)gd\mu$ $(x \in S)$ where (T, X, μ) is an L^∞-representation of S and $g \in L^1(X, \mu)$.

In [3] Lau studied the subalgebra $F(S)$ of WAP(S) of a topological semigroup S with involution. If S is commutative, then $F(S) \subseteq R(S)$ and in particular, if G is an abelian topological group, then $F(G) = R(G) = \widehat{M(\hat{G})}$ where $\hat{G}$ is the dual group of G. If S is a topological $*$- semigroup with an identity, then $F(S)$ is the linear span of positive definite functions on S. By [3, Theorem 3.2], $F(S)$ is a subalgebra of WAP(S). In this paper, we investigate the structure of $R(S), F(S)$ and $\overline{R(S)}, \overline{F(S)}$, the sup-norm closures of $R(S)$ and $F(S)$ and show that these are left introverted subalgebras of WAP(S) and study left (resp. right) amenability and amenability of these algebras. Also we show that SAP$(S) \subseteq \overline{F(S)}$, where SAP$(S)$ is the Banach algebra of strongly almost periodic functions on S, and then show that $\overline{F(S)}$ is amenable if and only if $\overline{F(S)} = \text{SAP}(S) \oplus C$, where C is a closed ideal of $\overline{F(S)}$.

2. Preliminaries

Let S be a topological semigroup. Let λ be a probability measure on a measurable space X. We know that $L^\infty(X, \lambda)$ is a commutative W^*-algebra. An L^∞-representation (T, X, μ) of S is a weak*-continuous homomorphism T of S into the unit ball of $L^\infty(X, \mu)$.

Definition 2.1. *The representation algebra $R(S)$ is the set of all functions f such that $f(x) = \int_X T_x(g)d\mu$, where (T, X, μ) is an L^∞-representation of S and $g \in L^1(X, \mu)$. We put $\|f\| = \inf \|g\|_1$, where infimum is taken over all elements $g \in L^1(X, \mu)$ in above presentation of f.*

Proposition 2.1. *$R(S)$ is a normed subalgebra of* WAP(S) *with pointwise multiplication. It is conjugate closed, translation invariant and contains constant functions. It is complete in its norm.*

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Proof See Theorem 2.1.6 and Proposition 2.1.4 of [2]. □

Definition 2.2. *Let $B(S)$ be the space of all bounded functions on S and $f \in B(S)$, $s \in S$. The left (resp. right) translation of f is defined for $t \in S$ by $L_s f(t) = f(st)$ (resp. $R_s f(t) = f(ts)$). A subset $\mathcal{F} \subseteq B(S)$ is said to be left (resp. right) translation invariant if $L_s\mathcal{F} \subseteq \mathcal{F}(R_s\mathcal{F} \subseteq \mathcal{F})$. $\mathcal{F}$ is said to be translation invariant if it is both left and right translation invariant. Let $\mathcal{F}$ be a left (resp. right) translation invariant, conjugate closed, linear subspace of $B(S)$ containing the constant functions. $\mu \in \mathcal{F}^*$ is called left (resp. right) invariant if $\mu(L_s f) = \mu(f)$ (resp. $\mu(R_s f) = \mu(f)$) for all $f \in \mathcal{F}, s \in S$. The functional $\mu \in \mathcal{F}^*$ is called a mean on $\mathcal{F}$ if $\mu(1) = 1$, and μ is positive, i.e. $\mu(f) \geq 0$ whenever $f \geq 0$. A mean μ on $\mathcal{F}$ is called left (resp. right) invariant if $\mu(L_s f)$ (resp. $\mu(R_s f)) = \mu(f)$ $(s \in S, f \in \mathcal{F})$. The set of all left (resp. right) means on $\mathcal{F}$ is denoted by $LIM(\mathcal{F})$ (resp. $RIM(\mathcal{F})$). $\mathcal{F}$ is called left (resp.right) amenable if $LIM(\mathcal{F})$ (resp. $RIM(\mathcal{F})$) is not empty. If $\mathcal{F}$ is translation invariant and $IM(\mathcal{F}) = LIM(\mathcal{F}) \cap RIM(\mathcal{F})$ is nonempty then $\mathcal{F}$ is called amenable. The semigroup S is called left amenable, right amenable, amenable if the appropriate property holds for $B(S)$.*

Definition 2.3. *Let $\mathcal{F}$ be a translation invariant subalgebra of $B(S)$. $\mathcal{F}$ is said left m-introverted if $T_\mu\mathcal{F} \subseteq \mathcal{F}$ $(\mu \in MM(\mathcal{F}))$ where $T_\mu f(x) = \mu(L_x f)$, $L_x f(y) = f(xy)$ and $MM(\mathcal{F})$ the set of all multiplicative means on $\mathcal{F}$. An m-admissible subalgebra of $B(S)$ is a normed closed, conjugate closed, translation invariant, left m-introverted subalgebra of $B(S)$ containing constant functions.*

Following [3] we have the following definition.

Definition 2.4. *Let S be a topological semigroup with continuous involution $*$. Let M be a W^*-algebra and $M_1 = \{x \in M : \|x\| \leq 1\}$. By a $*$-representation of S we mean (ω, M), where ω is a $*$-homomorphism of S into M_1. Let $\sigma = \sigma(M, M_*)$ and $\Omega(S)$ be the set of all σ-continuous $*$-representations of S such that $\overline{\langle\omega(S)\rangle}^{\sigma} = M$. Let $F(S)$ be the set of all functions f on S such that $f = \hat{\psi}$ for some $\psi \in M_*$ (predual of M), where $\hat{\psi} = \psi \circ \omega$. Let $f \in F(S)$, as in [3], we define $\|f\|_\Omega = \inf\{\|\psi\| \mid \psi \in M_*, \hat{\psi} = f$ for some $(\omega, M) \in \Omega(S)\}$.*

Proposition 2.2. *$F(S)$ is a commutative subalgebra of $\mathrm{WAP}(S)$ which is conjugate closed, translation invariant, containing the constant functions. Also, if $f \in F(S)$, then $f^* \in F(S)$, where $f^*(s) = \overline{f(s^*)}$. Furthermore, $\|\cdot\|_\Omega$ is a norm on $F(S)$ and $(F(S), \|\cdot\|_\Omega)$ is a commutative normed algebra with unit.*

Proof This is Theorem 3.2 of [3] where its proof was referred to [2], [4]. □

3. AMENABILITY OF $F(S), R(S), \overline{F(S)}, \overline{R(S)}$

In this section we assume that S is a topological semigroup with a continuous involution. Let $Y = \sigma(F(S))$ (the spectrum of $F(S)$) and $K(Y)$ be the minimal ideal of Y.

Proposition 3.1. *Y is a semigroup.*

Proof Proposition 5.4 in [3]. □

Theorem 3.1. *$F(S)$ is amenable if and only if $K(Y)$ is a topological group.*

Proof Assume that $F(S)$ is amenable and $\overline{F(S)}$ is the norm closure of $F(S)$ in $\mathrm{WAP}(S)$. Then clearly $\overline{F(S)} \subseteq \mathrm{WAP}(S)$. Let $m \in IM(F(S))$, then for each $f \in \overline{F(S)}$, there is a sequence $\{g_n\}$ in $F(S)$ such that $g_n \xrightarrow{\|\cdot\|_\infty} f$. Clearly $\{m(g_n)\}$ is a Cauchy sequence of scalars, so it is convergent. We define $\bar{m} \in (\overline{F(S)})^*$ by $\bar{m}(f) = \lim_{n\to\infty} m(g_n)$. This is clearly well defined. Since $\bar{m}$ is positive, $\bar{m}(1) = 1$, and for each $x \in S$, $\bar{m}(L_x f) =$

$\lim_{n\to\infty} m(_x g_n) = \lim_{n\to\infty} m(g_n) = \bar{m}(f)$, it follows that $\bar{m}$ is a left invariant mean on $\overline{F(S)}$. Similarly $\bar{m}$ is right translation invariant on $\overline{F(S)}$. Hence $\overline{F(S)}$ is amenable. Also by [3, Theorem 3.2] $F(S)$ is a translation invariant, conjugate closed subalgebra of WAP(S) and contains constant functions, hence $\overline{F(S)}$ is norm closed, translation invariant, conjugate closed subalgebra of WAP(S) containing constant functions. Therefore by [1, Corollary 4.2.7] $\overline{F(S)}$ is introverted. So, it is an m-admissible subalgebra of WAP(S). Hence, by [1, Theorem 4.2.14], $K(X)$ is a topological group, where X is $S^{\overline{F(S)}}$, the spectrum of $\overline{F(S)}$ and $K(X)$ is the minimal ideal of X. Now let $\pi = i^* : (\overline{F(S)})^* \longrightarrow F(S)^*$ be the adjoint mapping of i. Clearly π is a continuous mapping from X to Y. Now, $(\epsilon_1, Y), (\epsilon_2, X)$ are two compactifications of S, where ϵ_i $(i = 1, 2)$ is an evaluation mapping and $\pi \circ \epsilon_2 = \epsilon_1$, so by [1, Proposition 3.1.6] $\pi(X) = Y$. Thus $\pi(K(X)) = K(Y)$ is a topological group. Conversely, if $K(Y)$ is a topological group, then $K(X)$ is a topological group, so by [1, Proposition 4.2.14] $\overline{F(S)}$ is amenable, and so is $F(S)$. □

We have another consequence of [1, Theorem 4.2.14]. In fact, $\overline{F(S)}$ is left (right) amenable if and only if $K(X)$ is a minimal right (left) ideal of X.

Now, we extend the above theorem for left (resp. right) amenability of $F(S)$.

Theorem 3.2. *$F(S)$ is left (right) amenable if and only if $K(Y)$ is a minimal right (left) ideal of Y.*

Proof Let $F(S)$ be left amenable. By the proof of the above theorem, $\overline{F(S)}$ is also left amenable. By [1, Theorem 4.2.14] $K(X)$ is a minimal right ideal of X. Now, we have $\pi(K(X)) = K(Y)$, therefore by [1, Corollary 1.3.16] $K(Y)$ is a minimal right ideal of Y. Conversely, if $K(Y)$ is a minimal right ideal of Y, then by the same argument $K(X)$ is a minimal right ideal of X and therefore, again by [1, Theorem 4.2.14] $\overline{F(S)}$ is left amenable. Thus $F(S)$ is left amenable. The right version is proved in a similar way. □

Next we show that $F(S)$ is an F-algebra in the sense of [4].

Theorem 3.3. *If S is a unital topological semigroup with continuous involution, then $F(S)$ is an F-algebra.*

Proof First note that $F(S)$ is the predual of the von Neumann algebra $W^*(S)$ (see [3] for notation and proof). Each element of $F(S)$ is of the form $\hat{\psi} = \psi \circ \omega_\Omega$, for some $\psi \in (M_\Omega)_*$, where $(\omega_\Omega, M_\Omega)$, is the universal representation of S [3]. Take $\psi_1, \psi_2 \in (M_\Omega)_*$, then by the Gelfand-Naimark-Segal construction, for $i = 1, 2$, there are vectors $\xi_i, \eta_i \in H_\Omega$ such that $\psi_i(x) = \langle x\xi_i, \eta_i \rangle$ $(x \in M_\Omega)$. Therefore, for each $s \in S$

$$\begin{aligned} \hat{\psi}_1\hat{\psi}_2(s) &= \langle \omega_\Omega(s)\xi_1, \eta_1 \rangle \langle \omega_\Omega(s)\xi_2, \eta_2 \rangle \\ &= \langle \omega_\Omega(s) \otimes \omega_\Omega(s)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle. \end{aligned}$$

Hence, if $1 \in W^*(S)$ is the identity element , then

$$\langle 1, \hat{\psi}_1\hat{\psi}_2 \rangle = \langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle = \langle 1, \hat{\psi}_1 \rangle \langle 1, \hat{\psi}_2 \rangle,$$

and we are done. □

Corollary 3.1. *If S is a unital topological semigroup with continuous involution, then $W^*(S)$ has a topological left invariant mean.*

Proof This follows from above proposition and [4, Theorem 4.1]. Note that $F(S)$ is always left amenable in the sense of [4]. □

Remark 3.1. *(a) If S is an idempotent commutative topological semigroup with involution $s = s^*$, then by [3, 3.3(c)] $F(S) = R(S)$. Therefore in this special case the above results hold for $R(S)$ too.*

(b) By Proposition 2.1, $R(S)$ is right translation invariant, conjugate closed, containing constant functions. Therefore when S is commutative, $\overline{R(S)} \subseteq \mathrm{WAP}(S)$ is norm closed, translation invariant, conjugate closed subalgebra of $\mathrm{WAP}(S)$ *containing constant functions. Therefore $\overline{R(S)}$ is an m-admissible subalgebra of* $\mathrm{WAP}(S)$ *[1, Corollary 4.2.7]. Hence, the results of theorems 3.1 and 3.2 hold for $\overline{R(S)}$.*

(c) If G is a non-compact locally compact abelian group, then $\mathrm{WAP}(G) \neq \overline{R(G)} = \widehat{\overline{M(\hat{G})}}$ *[2, 5.2.10]. More generally by [2, 5.2.12], if S is a non-compact locally compact subsemigroup of a locally compact abelian group, then* $\mathrm{WAP}(S) \neq \overline{R(S)}$.

(d) Example 4.2 of [3] shows that $\overline{F(S)} \neq \overline{R(S)}$ may happen.

(e) When S is an abelian semigroup with involution, we have another interesting result. By [1, 4.3.8] $\mathrm{SAP}(S)$, *the Banach space of all strongly almost periodic functions on S [1, 4.3.2], is the closed linear span of characters. Therefore, by [3, 3.3.(f)]* $\mathrm{SAP}(S) \subseteq F(S) \subseteq \mathrm{WAP}(S)$.

Theorem 3.4. *Let S be a topological semigroup with a continuous involution. Consider the following statements:*

a) *$R(S)$ is amenable.*
b) *$F(S)$ is amenable.*
c) *$\overline{R(S)}$ is amenable.*
d) *$\overline{F(S)}$ is amenable.*
e) *$\overline{F(S)} = \mathrm{SAP}(S) \oplus C$, where C is a translation invariant, closed linear subspace of $\overline{F(S)}$.*

Then (a) is equivalent to (c). Also $(b), (d)$, and (e) are equivalent.

Proof First we show that

$$SAP(S) \subseteq \overline{F(S)} \subseteq WAP(S).$$

Let $u \in SAP(S)$. Then there is a finite dimensional unitary representation $(\pi, \mathcal{H}_\pi)$ of S such that $u(s) = \langle \pi(s)\xi, \eta\rangle (s \in S)$ for some $\xi, \eta \in \mathcal{H}_\pi$.

Put $M = \langle \pi(S)\rangle^{-\sigma} \subseteq B(\mathcal{H}_\pi)$, and $\alpha = (\pi, M) \in \Omega(S)$ (c.f. Definition 2.5). Consider $\psi(x) = \langle x\xi, \eta\rangle (x \in M)$. Then obviously $\psi \in M_*$. In fact we can show that ψ is ω^*-continuous on $B(\mathcal{H}_\pi)$. If $\{T_\alpha\} \subseteq B(\mathcal{H}_\pi)$ and $T_\alpha \xrightarrow{\omega^*} T \in B(\mathcal{H}_\pi)$, then by considering the rank one operator $\xi \otimes \eta$ in $B(H)_*$, we have

$$\langle T_\alpha \xi, \eta\rangle = \langle T_\alpha, \xi \otimes \eta\rangle \longrightarrow \langle T, \xi \otimes \eta\rangle = \langle T\xi, \eta\rangle$$

i.e. $\psi(T_\alpha) \longrightarrow \psi(T)$. Now, we have $\hat{\psi} = \psi \circ \pi = u$, hence $u \in F(S)$. Therefore $SAP(S) \subseteq \overline{F(S)}$. The proof of the other inclusion in contained in [3].

Now the rest of the proof is simple. In fact we have shown in the proof of Theorem 3.2, that the amenability of $F(S)$ and $\overline{F(S)}$ are equivalent. Also by the same argument the amenability of $R(S)$ and $\overline{R(S)}$ are equivalent. Now, since $\overline{F(S)}$ is an m-admissible subalgebra of $C(S)$ such that $SAP(S) \subseteq \overline{F(S)} \subseteq WAP(S)$, by [1, Theorem 4.3.13] $\overline{F(S)}$ is amenable if and only if $\overline{F(S)} = SAP(S) \oplus C$ where C is a closed ideal in $\overline{F(S)}$. □

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