ALGEBRAIC STRUCTURES RELATED TO THE COMBINATION OF BELIEF FUNCTIONS

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Abstract. Based on a new demand — the commutativity of belief functions combination with refinement/coarsening of the frame of discernment — the role of the disjunctive rule of combination has increased. To compare the nature of this rule with a more frequent but also more controversional one, i.e. with Dempster's rule, an algebraic analysis was used.

The basic necessary definitions both from the Dempster-Shafer theory and from algebra are recalled. An algebraic investigation of the Dempster's semigroup — the algebraic structure of binary belief functions with the Dempster's rule of combination is briefly recalled as well.

After this, a new algebraic structure of binary belief functions with the disjunctive rule of combination is defined. The structure is studied, and the results are discussed in a comparison with those ones of the classical Dempster's rule.

In the end, an impact of new algebraic results to the field of decision making and some ideas for future research are presented.

1 Introduction When combining two or more belief functions, there are generally accepted requirements of associativity and commutativity of an operation of their combination. A new requirement of commutativity of a combination with refinement/coarsening of the frame of discernment was introduced in [4]. There are three sources of this requirement: it arises from some applications of belief functions (namely in cases of subjective beliefs which are not constructed from probabilities), it furthermore arises from logical studies on belief functions, see [9], and it is motivated by the utilization of a method of representing a n-dimensional belief function by a set of two-dimensional ones, see [3].

The classical Dempster-Shafer theory uses the Dempster's (conjunctive) rule of combination $\oplus$, while the Transferable Belief Model [14, 15] uses its non-normalized version $\textcircled{\cap}$. To meet the new requirement it is necessary to use the disjunctive rule of combination $\textcircled{\cup}$, which is the only known associative and commutative combination of belief functions which commutes with coarsening of the frame of discernment (while $\oplus$ and $\textcircled{\cap}$ commute with refinement only).

An algebraic structure of binary belief functions with Demspter's rule $\oplus$, called the Dempster's semigroup, was in detail studied in a series of publications, e.g. [1, 2, 10, 11, 16]. The new importance of the disjunctive rule of combination $\textcircled{\cup}$ is the motivation for a study of algebraic structures of belief functions with $\textcircled{\cup}$ to obtain a better theoretical comparison of both approaches.

The next section briefly recalls the basic definitions. An algebraic analysis of the Dempster's semigroup which is used as a methodology for the presented investigation is overviewed in the third section.

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In Section 4, a new algebraic structure — the algebraic structure of belief functions with operation of combination $\circledcirc$ (disjunctive rule of combination) — is defined. The structure is analyzed there. The results are discussed and compared with those of the Dempster's semigroup in Section 5.

In Section 6, the disjunctive rule of combination $\circledcirc$ is considered from a decision making approach and its impact to this area is presented. In the end, some ideas for future research are outlined as well.

2 Preliminaries Let us recall some basic algebraic notions and some basic notions from the Dempster-Shafer theory before we begin a description of its algebra.

A *commutative semigroup* (called also an *Abelian semigroup*) is a structure $\mathbf{X} = (X, \oplus)$ formed by the set X and a binary operation $\oplus$ on X which is commutative and associative ($x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ holds for all $x, y, z \in X$). If it holds that $x \oplus x = x$ for $x \in X$, then x is called an *idempotent* of $(X, \oplus)$, if moreover holds $y \oplus x = x$ for all $y \in X$, then x is an *absorbing idempotent* of $(X, \oplus)$. A *commutative group* is a structure $\mathbf{X} = (X, \oplus, -, o)$ such that $(X, \oplus)$ is a commutative semigroup, o is a *neutral element* ($x \oplus o = x$) and $-$ is a unary operation of the inverse ($x \oplus -x = o$). An *ordered Abelian (semi)group* consists of a commutative (semi)group $\mathbf{X}$ as above and a linear ordering $\leq$ of its elements satisfying monotonicity ($x \leq y$ implies $x \oplus z \leq y \oplus z$ for all $x, y, z \in X$). A subset of X which is a (semi)group itself is called a *sub(semi)group.* A subsemigroup $(\{x|x \geq o, x \in X\}, \oplus, o)$ is called *a positive cone* of the ordered Abelian group (OAG) X, similarly *a negative cone* of OAG X for $x \leq o$.

For uncertainty processing, we extend OAG with *extremal elements* $\top$ and $\bot$ representing *True* and *False*, $\top \oplus x = \top,\ \bot \oplus x = \bot,\ \top \oplus \bot$ not defined. [1]

A *homomorphism* $p : (X, \oplus_1) \longrightarrow (Y, \oplus_2)$ is a mapping which preserves structure, i.e. $p(x \oplus_1 y) = p(x) \oplus_2 p(y)$ for each $x, y \in X$. The special cases are *automorphisms*, which are bijective morphisms from a structure onto itself. Morphisms which also preserve ordering of elements are called *ordered morphisms*, see [8].

Ordered structures and ordered morphisms are very important for a comparative approach to uncertainty management and decision making.

Let us consider a two-element frame of discernment $\Omega = \{0, 1\}$. Let us denote its power set as $\mathcal{P}(\Omega)$. A *basic belief assignment* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, such that $\sum_{A \subseteq \Omega} m(A) = 1$. A *belief function* is a mapping $bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. In our special case $bel(1) = m(1),\ bel(0) = m(0),\ bel(\{0, 1\}) = m(1) + m(0) + m(\{0, 1\}) = 1$. Each basic belief assignment determines a pair $(m(1), m(0))$ and conversely, each pair $(m(1), m(0))$ uniquely determines a basic belief assignment.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are singletons (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function.* A *probabilistic transformation* is a mapping $t : Bel_\Omega \longrightarrow Prob_\Omega$, such that $bel(X) \leq t(bel)(X) \leq 1 - bel(\overline{X})$. Thus the probabilistic transformation assigns a Bayesian belief function (i.e. probability function) to every general one. The fundamental example of probabilistic transformation is the pignistic transformation introduced by Smets.

The Dempster's conjunctive rule of combination is given as
$(bel_1 \oplus bel_2)(A) = \sum_{X \cap Y = A} \frac{1}{K} m_1(X) m_2(Y)$, where $K = \sum_{X \cap Y \neq \emptyset} m_1(X) m_2(Y)$, see [13], while *the disjunctive rule of combination* is given by the formula

[1] Some examples are OAG$^+$ $\mathbf{PP} = ([0, 1], \oplus_{PP}, 1 - x, \frac{1}{2}, \leq)$ and $\mathbf{MC} = ([-1, 1], \oplus_{MC}, -, 0, \leq)$ corresponding to the combining structures of the classical expert systems PROSPECTOR and EMYCIN, see [10], where $x \oplus_{PP} y = \frac{xy}{xy + (1-x)(1-y)}$ and $x \oplus_{MC} y = x + y - xy$ for $x, y \geq 0$, $x + y + xy$ for $x, y \leq 0$ and $\frac{x+y}{1 - min(|x|, |y|)}$ for $xy \leq 0$.

$(bel_1 \circledcirc bel_2)(A) = \sum_{X \cup Y = A} m_1(X) m_2(Y)$, see [7]. Specially for $(m_1(1), m_1(0)) = (a, b)$, $(m_2(1), m_2(0)) = (c, d)$ we have $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ and $(a, b) \circledcirc (c, d) = (ac, bd)$.

3 On the Dempster's semigroup Now we introduce some principal notions according to [10].

Definition 1 *A* Dempster's pair *(or d-*pair*) is a pair of reals such that $a, b \geq 0$ and $a + b \leq 1$. A d-pair (a, b) is* Bayesian *if $a + b = 1$, (a, b) is* simple *if $a = 0$ or $b = 0$, in particular,* extremal *d-pairs are pairs (1,0) and (0,1). (Definitions of Bayesian and simple d-pairs correspond evidently to the usual definitions of Bayesian and simple belief assignments [10], [13]). Let D_0 denote a set of all non-extremal d-pairs.*

Definition 2 *The* (standard/conjunctive) Dempster's semigroup[2] $\mathbf{D_0} = (D_0, \oplus)$ *is the set of all non-extremal Dempster's pairs, endowed with the operation $\oplus$ and two distinguished elements $0 = (0, 0)$ and $0' = (\frac{1}{2}, \frac{1}{2})$, where the operation $\oplus$ is defined by*

$$(a, b) \oplus (c, d) \quad = \quad \left(1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)}\right).$$

Definition 3 *For $(a, b) \in D_0$ we define*

$$\begin{array}{rcl} -(a,b) & = & (b,a), \\ h(a,b) & = & (a,b) \oplus 0' = (\frac{1-b}{2-a-b},\ \frac{1-a}{2-a-b}), \\ h_1(a,b) & = & \frac{1-b}{2-a-b}, \\ f(a,b) & = & (a,b) \oplus (b,a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2},\ \frac{a+b-a^2-b^2-ab}{1-a^2-b^2}). \end{array}$$

For $(a, b), (c, d) \in D_0$ we further define
$(a, b) \leq (c, d)$ iff [$h_1(a, b) < h_1(c, d)$ or $h_1(a, b) = h_1(c, d)$ and $a \leq c$].

Let G denote the set of all Bayesian non-extremal d-pairs. Let us denote the set of all simple d-pairs such that $b = 0$ ($a = 0$) as S_1 (S_2). Furthermore, put $S = \{(a, a) : 0 \leq a \leq 0.5\}$.
(Note: $h(a, b)$ is an abbreviation for $h((a, b))$, etc.)

Theorem 1 *(i) The Dempster's semigroup with the relation $\leq$ is an ordered commutative semigroup with the neutral element 0; $0'$ is the only nonzero idempotent of it.*

(ii) The set G with the ordering $\leq$ is an ordered Abelian group $(G, \oplus, -, 0', \leq)$ which is isomorphic to the PROSPECTOR group **PP** *(cf. [10]) and consequently isomorphic to the additive group of reals with usual ordering.*

(iii) The sets S, S_1 and S_2 with the operation $\oplus$ and the ordering $\leq$ form ordered commutative semigroups with neutral element 0, and all are isomorphic to the semigroup of non-negative elements (positive cone) of the MYCIN group **MC**.

(iv) The mapping h is an ordered homomorphism of the ordered Dempster's semigroup onto its subgroup G (i.e. onto **PP***).*

[2] A generalization of a notion of the Dempster's semigroup is described in [16], see also [10]. The resulting algebraic structure is called *a dempsteroid.* It has a similar relation to the Dempster's semigroup as OAG has to **PP** or **MC**. The special case — the standard dempsteroid $\mathbf{D_0} = (D_0, \oplus, -, 0, 0', \leq)$ is defined by the Dempster's semigroup.

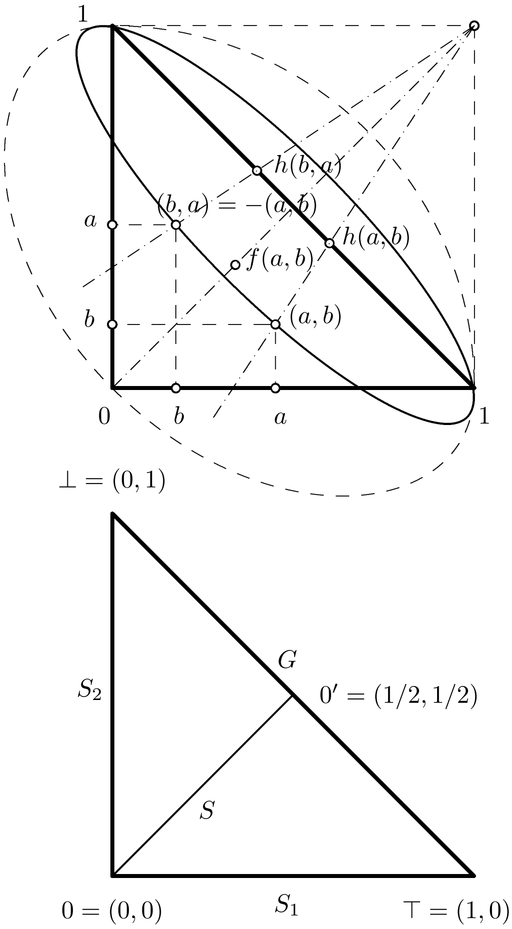

Figure 1: ***Dempster's semigroup.*** *Homomorphism h is in this representation a projection to group G along the straight lines running through the point $(1, 1)$. All the Dempster's pairs lying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.*

(v) The mapping f is a homomorphism of the Dempster's semigroup onto its subsemigroup S (but it is not an ordered homomorphism).

For proofs see [10], [11], [16]. Let us denote $h^{-1}(a) = \{x|h(x) = a\}$, and similarly $f^{-1}(a) = \{x|f(x) = a\}$. Using the theorem, see (iv) and (v), we can express

$$(a \oplus b) = h^{-1}(h(a) \oplus h(b)) \cap f^{-1}(f(a) \oplus f(b)).$$

4 The Disjunctive Dempster's semigroup Let us turn our attention to an algebra of a binary frame of discernment with the disjunctive rule of combination ☺. As ☺ is a commutative and associative operation, we can speak about an Abelian semigroup again.

Because of the different nature of the operation, $0' = (\frac{1}{2}, \frac{1}{2})$ does not play the analogical role as in the case of the (standard) Dempster's semigroup ($0'$ is not an idempotent). The other idempotent $0 = (0, 0)$ of the Dempster's semigroup is idempotent again, but it is not neutral element in this case. To obtain a neutral element we add to D_0 a technical pair $1 = (1,1)$ which is not a d-pair (it does not correspond to any basic belief assignment). Analogically, it is useful to consider all pairs (a,a) for $a \geq 1$ (or for all $a \geq 0$, where (a,a) for $\frac{1}{2} < a < 1$ do not play any important role in the presented theory).

Definition 4 *Let $D_{\circledcirc}$ denote $D_0 \cup \{(1,0),(0,1),(1,1)\}$. The* disjunctive Dempster's semigroup *$\mathbf{D}_{\circledcirc} = (D_{\circledcirc}, \circledcirc)$ is the set of all Dempster's pairs extended by $1 = (1,1)$, endowed with the operation $\circledcirc$ and two distinguished elements $0 = (0,0)$ and $1 = (1,1)$, where the operation $\circledcirc$ is defined by*

$$(a,b)\circledcirc(c,d) = (ac,\ bd)^3.$$

Definition 5 *For $(a,b) \in D_{\circledcirc}$ we define*

$$\begin{aligned} -(a,b) &= (b,a), \\ u(a,b) &= (a,b)\circledcirc(\tfrac{1}{a+b},\ \tfrac{1}{a+b}) = (\tfrac{a}{a+b},\ \tfrac{b}{a+b}), \quad \text{if } a+b>0, \\ u_1(a,b) &= \tfrac{a}{a+b}, \quad \text{if } a+b>0, \\ v(a,b) &= (a,b)\circledcirc(b,a) = (ab,ab). \end{aligned}$$

For $(a,b),(c,d) \in D_{\circledcirc}$ we further define
$(a,b) \leq_{\circledcirc} (c,d)$ iff [$u_1(a,b) < u_1(c,d)$ or $u_1(a,b) = u_1(c,d)$ and $a < c$ or $u_1(a,b) = u_1(c,d)$ and $a = c$ and $b \leq d$].
Note: $u(a,b)$, $u_1(a,b)$ are defined on $(D_{\circledcirc} - \{(0,0)\})$, i.e. it is not defined $u(0,0)$ which should be $(\frac{0}{0},\frac{0}{0})$, similarly for $u_1(0,0)$.

Lemma 1 *Let x,y (or $(a,b),(c,d)$) be elements of the disjunctive Dempster's semigroup. The following holds:*

(o) $1 = (1,1)$ *is neutral element in $\mathbf{D}_{\circledcirc}$, while $0 = (0,0)$ is an absorbing idempotent there, $\perp = (0,1)$ and $\top = (1,0)$ are idempotents which are neither neutral nor absorbing in $\mathbf{D}_{\circledcirc}$,*

(i) $-(x\circledcirc y) = -x\circledcirc - y$ *(i.e. $-((a,b)\circledcirc(c,d)) = (b,a)\circledcirc(d,c)$),*

(ii) $-(-x) = x$ *(i.e. $-(-(a,b)) = (a,b)$),*

(iii) $-x$ *is not an inverse to x, i.e. the equation $(a,b)\circledcirc(c,d) = (1,1)$ has no solution in $\mathbf{D}_{\circledcirc}$ for $(a,b) \neq (1,1)$,*

(iv) $u(x) = 0'$ *iff* $0 \neq x = -x$ *iff* $0 < x \leq 0'$ [4] *iff* $x \in S - \{0\}$,

(v) $x\circledcirc\top = (p_1(x),0)$, *i.e.* $(a,b)\circledcirc(1,0) = (p_1(a,b),0) = (a,0)$, *where* $p_1(x,y) = x$,
$x\circledcirc\perp = (0,p_2(x))$, *i.e.* $(a,b)\circledcirc(0,1) = (0,p_2(a,b)) = (0,b)$, *where* $p_2(x,y) = y$.

Proof:
(o): $(a,b)\circledcirc(1,1) = (1a,1b) = (a,b)$,
$(0,0)\circledcirc(0,0) = (0\cdot 0, 0\cdot 0) = (0,0)$, $(a,b)\circledcirc(0,0) = (0a,0b) = (0,0)$;
$(1,0)\circledcirc(1,0) = (1\cdot 1,\ 0\cdot 0) = (1,0)$, $(a,b)\circledcirc(1,0) = (1a,0b) = (a,0)$,
$(0,1)\circledcirc(0,1) = (0\cdot 0,\ 1\cdot 1) = (0,1)$, $(a,b)\circledcirc(0,1) = (0a,1b) = (0,a)$;
(i): $-((a,b)\circledcirc(c,d)) = -(ac,bd) = (bd,ac) = (b,a)\circledcirc(d,c)$;

[3] $\circledcirc$-sum of two d-pairs $(a,b)\circledcirc(c,d)$ is defined for all d-pairs from $D_{\circledcirc}$, $0 = (0,0)$ is not a neutral element and $0' = (\frac{1}{2},\frac{1}{2})$ is not idempotent of $\mathbf{D}_{\circledcirc}$.

[4] We cannot write $0 <_{\circledcirc} x \leq_{\circledcirc} 0'$ because $\leq_{\circledcirc}$ is defined on $D_{\circledcirc} - \{0\}$, but we can write $0 < x \leq_{\circledcirc} 0'$ due to $\leq_{\circledcirc}$ is equivalent to $\leq$ on $S - \{0\}$.

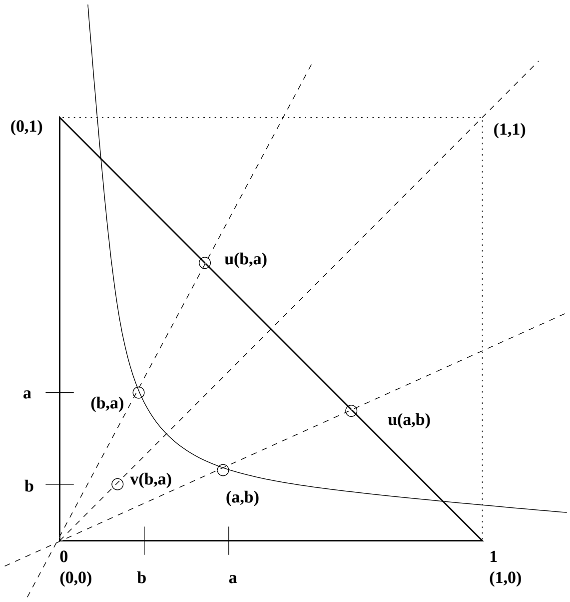

Figure 2: ***Disjunctive Dempster's semigroup*** *The homomorphism u is, in this representation, a projection to group G along the straight lines running through the point $(0,0)$. All Dempster's pairs lying on the same hyperbole are, by the homomorphism v, mapped to the same d-pair in semigroup S.*

(ii): $-(-(a,b)) = -(b,a) = (a,b)$, (the same proof as in the case of $\mathbf{D}_0$);
(iii): $(a,b)\circledcirc(c,d) = (ac,bd) = (1,1)$ *iff* $ac = 1$ & $bd = 1$ iff $a = c = b = d = 1$ because of $a,b,c,d \in [0,1]$.
(iv): $(a,b)\circledcirc(\frac{1}{a+b},\frac{1}{a+b}) = (\frac{a}{a+b},\frac{b}{a+b}) = (\frac{1}{2},\frac{1}{2})$ *iff* $\frac{a}{a+b} = \frac{1}{2},\ \frac{b}{a+b} = \frac{1}{2}$ *iff* $2a = a+b,\ 2b = a+b$ and $a+b \neq 0$ *iff* $a = b \neq 0$;
(v): $(a,b)\circledcirc(1,0) = (1a,0b) = (a,0) = (p_1(a,b),0)$,
$(a,b)\circledcirc(0,1) = (0a,1b) = (0,b) = (0,p_2(a,b))$.

Theorem 2

(i) *The disjunctive Dempster's semigroup $\mathbf{D}_{\circledcirc}$ is a commutative semigroup* [5] *with the neutral element 1; where 0, $\perp$, and $\top$ are all the other idempotents of it.*

(ii-a) *The set G of Bayesian d-pairs is not closed under operation $\circledcirc$.*

(ii-b) *The set G with the ordering $\leq_{\circledcirc}$ and with the operation $\circledcirc_G = \circledcirc \circ u$, where $(a,b)\circledcirc_G(c,d) = u(ac,bd) = (\frac{ac}{ac+bd},\frac{bd}{ac+bd})$, is an ordered Abelian group $\mathbf{G}_{\circledcirc_G} = (G,\circledcirc_G,-,0',\leq_{\circledcirc})$ which is isomorphic to the PROSPECTOR group* **PP** *(cf. [10]) and consequently, it is isomorphic to the additive group of reals with usual ordering.*

(iii) *The sets $S \cup \{1\}$, $S_1 \cup \{\top\}$ and $S_2 \cup \{\perp\}$ with the operation $\circledcirc$ and the ordering $\leq_{\circledcirc}$ form ordered commutative semigroups with neutral elements 1, $\top$, or $\perp$ respectively. We can define $0 \leq_{\circledcirc} x$ for any x on all these subalgebras. $S_1 \cup \{\top\}$ and $S_2 \cup \{\perp\}$ are*

[5] The same holds also for restrictions of $\mathbf{D}_{\circledcirc}$ to D_0 and to $D_0 \cup \{\perp,\top\}$.

isomorphic to the negative cone of the extended (with 0 *and* ∞*) multiplicative group of positive reals*[6]*, and* $S \cup \{1\}$ *is isomorphically embeddable there.*

(iv-a) *The mapping* u *is not a homomorphism of the disjunctive Dempster's semigroup onto its subalgebra* G*.*

(iv-b) *The mapping* u *is an ordered homomorphism of the disjunctive Dempster's semigroup without* 0 *(i.e. not defined for* $0 = (0,0)$*) onto group* $\mathbf{G}_{\circledcup_G}$ *(i.e. onto* $\mathbf{PP}$*), which is a subset of* $\mathbf{D}_{\circledcup}$*.*

(v) *The mapping* v *is a homomorphism of the disjunctive Dempster's semigroup onto its subsemigroup* S *(but it is not an ordered homomorphism).*

(vi) *An algebra* $(D_{\circledcup} - (S_1 \cup S_2), \circledcup, \leq_{\circledcup})$ *is a subalgebra of* $\mathbf{D}_{\circledcup}$*, it is an ordered Abelian semigroup*[7] *with the only idempotent* 1 *which is the neutral element of the algebra.*

Proof: (i): $a, b, c, d \in [0,1]$, thus $ac, bd \in [0,1]$, $ac \leq a, c$ and $bd \leq b, d$ for all $a, b, c, d \in [0,1]$, thus $ac + bd \leq a + b, c + d \leq 1$, hence $D_{\circledcup}$ is closed with respect to $\circledcup$. Associativity and commutativity follow properties of $\circledcup$. Neutral element 0 and idempotency of $0'$ follow (o) from the previous lemma. $(a,b)\circledcup(a,b) = (aa, bb) = (a,b)$ iff $aa = a,\ bb = b$ iff $a, b \in \{0,1\}$, thus there are just four idempotents $0,\ 1,\ \perp$, and $\top$, $(a,b)\circledcup(1,0) = (a,0)$, hence $\top$ is neither neutral nor absorbing, similarly for $\perp$.

(ii-a): $(a, 1{-}a)\circledcup(b, 1{-}b) = (ab, (1{-}a)(1{-}b)) = (X, Y),\ X{+}Y = ab{+}1{-}a{-}b{+}ab,\ X{+}Y = 1$ iff $1 = 1 + 2ab - a - b$ iff $2ab - a = b$ iff $a = \frac{b}{2b-1}$ iff $a = b = 0$ or $a = b = 1$.

(ii-b): $\circledcup_G = \circledcup \circ u$, where $(a,b)\circledcup_G(c,d) = u((a,b)\circledcup(c,d)) = u(ac, bd) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd})$;
closeness: $(a,b), (c,d) \in D_{\circledcup}$, hence $(a,b)\circledcup(c,d) \in D_{\circledcup}$, thus $(a,b)\circledcup_G(c,d) = u((a,b)\circledcup(c,d)) \in G$, i.e. $\circledcup_G$ maps $\mathbf{D}_{\circledcup}$ to G, hence $G \subset \mathbf{D}_{\circledcup}$ is closed with respect to $\circledcup_G$,
commutativity: $(c,d)\circledcup_G(a,b) = u(ca, db) = (\frac{ca}{ca+db}, \frac{db}{ca+db}) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd}) = (a,b)\circledcup_G(c,d)$,
associativity: $((a,b)\circledcup_G(c,d))\circledcup_G(e,f) = (\frac{\frac{ace}{ac+bd}}{\frac{ace}{ac+bd} + \frac{bdf}{ac+bd}}, \frac{\frac{bdf}{ac+bd}}{\frac{ace}{ac+bd} + \frac{bdf}{ac+bd}}) = (\frac{ace}{ace+bdf}, \frac{bdf}{ace+bdf}) = (\frac{a\frac{ce}{ce+df}}{\frac{ace}{ce+df} + \frac{bdf}{ce+df}}, \frac{b\frac{df}{ce+df}}{\frac{ace}{ce+df} + \frac{bdf}{ce+df}}) = (a,b)\circledcup_G((c,d)\circledcup_G(e,f))$,
neutral element: $(a, 1-a)\circledcup_G(\frac{1}{2}, \frac{1}{2}) = (\frac{\frac{a}{2}}{\frac{a}{2} + \frac{1-a}{2}}, \frac{\frac{1-a}{2}}{\frac{a}{2} + \frac{1-a}{2}}) = (\frac{a}{a+1-a}, \frac{1-a}{a+1-a}) = (a, 1-a)$,
inverse: $(a, 1-a)\circledcup_G(1-a, a) = (\frac{a(1-a)}{a(1-a)+(1-a)a}, \frac{(1-a)a}{(1-a)a+a(1-a)} = (\frac{1}{2}, \frac{1}{2})$;
monotonicity of $\leq_{\circledcup}$: for $(x, 1-x) \in G$ it holds that $u(x, 1-x) = (x, 1-x)$ and $(x, 1-x) \leq_{\circledcup} (y, 1-y)$ iff $x \leq y$; if $(a, 1-a) < (b, 1-b)$ i.e. $a < b$, then $az < bz$, hence $(a, 1-a)\circledcup(z, 1-z) < (b, 1-b)\circledcup(z, 1-z)$; similarly if $(a, 1-a) = (b, 1-b)$ i.e. $a = b$ then $az = bz$, hence $(a, 1-a)\circledcup(z, 1-z) = (b, 1-b)\circledcup(z, 1-z)$.
An isomorphism from G to $\mathbf{PP}$ is the projection $p_1(a, 1-a) = a$:
$p_1((a, 1-a)\circledcup_G(b, 1-b)) = p_1(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)}) = \frac{ab}{ab+(1-a)(1-b)} = a \oplus_{PP} b = p_1(a, 1-a) \oplus_{PP} p_1(b, 1-b)$; $p_1(-(a, 1-a)) = p_1(1-a, a) = 1-a = -_{PP}(a) = -_{PP}(p_1(a, 1-a))$; $p_1(0') = p_1(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} = 0_{PP}$; $(a, 1-a) \leq (b, 1-b)$ iff $u_1(a, 1-a) \leq u_1(b, 1-b)$ iff $p_1(a, 1-a) \leq p_1(b, 1-b)$, hence p_1 is an ordered isomorphism.

(iii): Commutativity and associativity follow properties of $\circledcup$ in all the three cases. Ordered isomorhpisms onto the negative cone of $(\mathbf{Re}_m^{>0})^+ = (Re^{>0} \cup \{0, \infty\}, \cdot, \frac{1}{x}, 1, \leq)$, (i.e. $([0,1], \cdot, 1, \leq)$), are the following projections $p_1(a, a) = a$ ($p_1(a, 0) = a$ and $p_2(0, a) = a$) for

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[6] Negative cone of $(\mathbf{Re}_\mathbf{m}^{\geq\mathbf{0}})^+ = ((0, \infty) \cup \{0, \infty\}, \cdot, \frac{1}{x}, 1, \leq)$, i.e. $([0,1], \cdot, 1, \leq)$.

[7] The same holds also for restriction of $\mathbf{D}_{\circledcup}$ to $D_0 - (S_1 \cup S_2)$, i.e. for $(D_0 - (S_1 \cup S_2), \circledcup, \leq_{\circledcup})$.
The set $D_{\circledcup} - \{0\}$ is not closed under $\circledcup$, thus it does not define a subalgebra of $\mathbf{D}_{\circledcup}$. Any extension of definition of u to 0 does not satisfy the monotonicity condition, hence whole $\mathbf{D}_{\circledcup}$ cannot be an OAG.

S (S_1 and S_2 respectively). Proofs are analogous to the case (ii-b).
$(a,a)\circledcirc(b,b) = (ab,ab)$, $a,b \in [0,\frac{1}{2}] \cup [1]$, thus $ab \in [0,\frac{1}{2}] \cup [1]$, $(a,0)\circledcirc(b,0) = (ab,0)$, and $(0,a)\circledcirc(0,b) = (0,ab)$, hence all three sets are closed under $\circledcirc$; for neutral elements see (i).

(iv-a): We know from (ii-a) that G is not closed with respect to the operation $\circledcirc$, hence it is not subalgebra of $\mathbf{D}_{\circledcirc}$,
moreover: $u((a,b)\circledcirc(c,d)) = u(ac,bd) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd})$; while $u(a,b)\circledcirc u(c,d) = (\frac{a}{a+b}, \frac{b}{a+b})\circledcirc (\frac{c}{c+d}, \frac{d}{c+d}) = (\frac{ac}{(a+b)(c+d)}, \frac{bd}{(a+b)(c+d)}) = (\frac{ac}{(ac+bd+ad+bc)}, \frac{bd}{(ac+bd+ad+bc)})$;

(iv-b): $u((a,b)\circledcirc(c,d)) = u(ac,bd) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd})$; $u(a,b)\circledcirc_G u(c,d) = (\frac{a}{a+b}, \frac{b}{a+b})\circledcirc_G (\frac{c}{c+d}, \frac{d}{c+d}) = (\frac{\frac{ac}{(a+b)(c+d)}}{\frac{ac}{(a+b)(c+d)} + \frac{bd}{(a+b)(c+d)}}, \frac{\frac{bd}{(a+b)(c+d)})}{\frac{ac}{(a+b)(c+d)}, \frac{bd}{(a+b)(c+d)}}) = (\frac{ac}{ac+bd}, \frac{bd}{ac+bd}) = u((a,b)\circledcirc(c,d))$;

(v): $v((a,b)\circledcirc(c,d)) = v(ac,bd) = (ac,bd)\circledcirc(bd,ac) = (acbd,acbd)$; $v(a,b)\circledcirc(c,d) = ((a,b)\circledcirc(b,a))\circledcirc((c,d),\circledcirc(d,c)) = ((ab,ab))\circledcirc((cd),(cd)) = (acbd,acbd) = v((a,b)\circledcirc(c,d))$.

(vi): $(a,c)\circledcirc(c,d) = (ac,bd) = (0,x) \in S_1$ iff $a=0$ or $c=0$, similarly for S_2, thus $D_{\circledcirc} - (S_1 \cup S_2)$ is closed under $\circledcirc$, hence our algebra is commutative subsemigroup of $\mathbf{D}_{\circledcirc}$, where 1 is neutral idempotent, the other idempotents of $\mathbf{D}_{\circledcirc}$ are not in the subalgebra; monotonicity of $\leq_{\circledcirc}$: let us suppose that $(a,b) \leq (c,d)$; if $\frac{a}{a+b} = u_1(a,b) < u_1(c,d) = \frac{c}{c+d}$, then $u_1((a,b)\circledcirc(x,y)) = u_1(a,b)\circledcirc u_1(x,y) = \frac{a}{a+b}\frac{x}{x+y} < \frac{c}{c+d}\frac{x}{x+y} = u_1(c,d)\circledcirc u_1(x,y) = u_1((c,d)\circledcirc(x,y))$; similarly if $u_1(a,b) = u_1(c,d)$, then $u_1((a,b)\circledcirc(x,y)) = u_1((c,d)\circledcirc(x,y))$, in this case it holds that $b \leq d$ thus also $by \leq dy$, hence $(a,b)\circledcirc(x,y) \leq_{\circledcirc} (c,d)\circledcirc(x,y)$, the monotonicity holds, and the subalgebra is OAG.

Corollary 1 *Let us denote $u^{-1}(a) = \{x|u(x) = a\}$, and similarly $v^{-1}(a) = \{x|v(x) = a\}$. Using the theorem, see (iv) and (v), we can express*

$$(a\circledcirc b) = u^{-1}(u(a)\circledcirc u(b)) \cap v^{-1}(v(a)\circledcirc v(b)).$$

5 A comparison of the disjunctive Dempster's semigroup with the standard (conjunctive) one

Both the algebraic structures have a lot of **similarities:**
Both of them are ordered Abelian semigroups with a neutral element.
There is the same operation $-$, which is not inverse in both cases.
Both the structures have subsemigroups S, S_1, S_2 respectively $S \cup 1, S_1 \cup \top, S_2 \cup \bot$ with neutral elements. The orderings $\leq$ and $\leq_{\circledcirc}$ are the same on S and S_1.
Both of them have an OAG defined on G. The orderings $\leq$ and $\leq_{\circledcirc}$ are the same on G.
Both of them have a surjective homomorphism $D_0 \longrightarrow G$, resp. $D_0 - \{0\} \longrightarrow G$.
Both of them have a surjective homomorphism $D_0 \longrightarrow S$.
Both the semigroup operations $\oplus$ and $\circledcirc$ are expressible using these homomorphisms, their pre-images and operations restricted to S and G.

Differences:
$\oplus$ is not defined for $\top \oplus \bot$, while $\circledcirc$ is defined on the whole extended $D_0^+ \cup \{(1,1)\}$, on the other hand the homomorphism u is not defined for $0 = (0,0)$, hence $u(x\circledcirc y)$ is not defined if $x \in S_1$ & $y \in S_2$ or $x \in S_2$ & $y \in S_1$ or $x = 0$ or $y = 0$.
0 is a neutral element in $\mathbf{D}_0$, while it is an absorbing element in $\mathbf{D}_{\circledcirc}$.
The neutral element $1 = (1,1)$ of $\mathbf{D}_{\circledcirc}$ is out of D_0.
$0' = (\frac{1}{2}, \frac{1}{2})$ is not an idempotent of $\mathbf{D}_{\circledcirc}$.
Extremal elements of $\mathbf{D}_0$ are not absorbing in $\mathbf{D}_{\circledcirc}$, $\bot$ is not the $\leq_{\circledcirc}$-least element of $\mathbf{D}_{\circledcirc}$.
The ordering $\leq_{\circledcirc}$ is inverse to $\leq$ on S_2, i.e. $(0,x) \leq_{\circledcirc} (0,y)$ iff $(0,x) \geq (0,y)$.
If we add $1 = (1,1)$ into $\mathbf{D}_0$ we obtain a new absorbing element, where $(a, 1-a) \oplus 1$ is not

defined for all $(a, 1-a) \in G$.
$\mathbf{G}_{\circledcirc_G}$ is not a subalgebra of $\mathbf{D}_{\circledcirc}$.

The principal is the following.

Both combinations $\oplus$ and $\circledcirc$ of two elements (d-pairs) $\geq 0'$ (or two ones ≤ 0) are on homomorphic straight lines further from S (than those, which contain the original elements (d-pairs)). We can reformulate this as that the certainty which is represented by belief functions is increased by both combining rules $\oplus$ and $\circledcirc$.

$\oplus$ combination of any two elements (d-pairs) is on an ellipse further from 0, i.e. vagueness is decreased by the Dempster's rule $\oplus$, while $\circledcirc$ combination of any two elements is on a hyperbole closer to 0, i.e. vagueness is increased by the disjunctive rule $\circledcirc$.

$\mathbf{S}_{\oplus} = (S, \oplus, 0, \leq)$ is o-isomorphic to a positive cone of OAGs, while $\mathbf{S}_{\circledcirc} = (S \cup \{1\}, \circledcirc, 1, \leq)$ is o-isomorphic to a negative cone of OAGs (ordered such that $0 \leq 0'$). In other words, inverse elements of S in a group defined by $\mathbf{S}_{\oplus}$ are in $\{(a,a) | a \leq 0\}$. While inverse elements of S in a group defined by $\mathbf{S}_{\circledcirc}$ are in $\{(a,a) | a \geq 1\}$.

6 Impact to decision making Summarizing the results of comparison of the disjunctive Dempster's semigroup with the standard (conjunctive) one, we obtain the following originally surprising theorem:

Theorem 3 *The groups* $\mathbf{G}_{\oplus} = (G, \oplus, -, 0', \leq)$ *and* $\mathbf{G}_{\circledcirc_G} = (G, \circledcirc_G, -, 0', \leq_{\circledcirc})$ *are identical; especially* $x \oplus y = x \circledcirc_G y$ *for all* $x, y \in G$. *(The same holds also for* $\mathbf{G}^+_{\oplus}$ *and* $\mathbf{G}^+_{\circledcirc_G}$*.)*

Proof: G is same in $\mathbf{G}_{\oplus}$ and $\mathbf{G}_{\circledcirc_G}$;
$(a, 1-a) \oplus (b, 1-b) = (1 - \frac{(1-a)(1-b)}{1-(a(1-b)+(b(1-a))}, 1 - \frac{ab}{1-(a(1-b)+(b(1-a))}) =$
$(\frac{1-(a-ab+b-ab)-(1-a-b+ab)}{1-(a-ab+b-ab)}, \frac{1-(a-ab+b-ab)-ab}{1-(a-ab+b-ab)}) = (\frac{ab}{1-a-b+2ab}, \frac{1-a-b+ab}{1-a-b+2ab}) =$
$(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)})$,
$\circledcirc_G = \circledcirc \circ u$:
$(a, 1-a) \circledcirc_G (b, 1-b) = u((a, 1-a) \circledcirc (b, 1-b)) = u((ab, (1-a)(1-b)) =$
$(\frac{ab}{ab+(1-a)(1-b)}, \frac{(1-a)(1-b)}{ab+(1-a)(1-b)})$;
hence $x \oplus y = x \circledcirc_G y$ for all $x, y \in G$, this holds also for extremal elements:
$(a, 1-a) \oplus (1,0) = (1,0)$, $(a, 1-a) \circledcirc_G (1,0) = u(a,0) = (\frac{a}{a}, \frac{0}{a}) = (1,0)$, analogically for $(0,1)$, $(1,0) \oplus (0,1)$ is not defined, $(1,0) \circledcirc_G (0,1) = u((1,0) \circledcirc (0,1)) = u(0,0)$, and it is also not defined;
the operation '$-$' and $0'$ are same for both $\mathbf{G}_{\oplus}$ and $\mathbf{G}_{\circledcirc_G}$;
$(a, 1-a) \leq (b, 1-b)$ *iff* $h_1((a, 1-a)) < h_1((b, 1-b))$ or if $h_1((a, 1-a)) = h_1((b, 1-b))$ and $a \leq b$ *iff* $\frac{1-(1-a)}{2-(a+1-a)} = \frac{a}{1} < \frac{b}{1} = \frac{1-(1-b)}{2-(b+1-b)}$ or $a = b$ and $a \leq b$ *iff* $a \leq b$;
$(a, 1-a) \leq_{\circledcirc} (b, 1-b)$ *iff* $u_1((a, 1-a)) < u_1((b, 1-b))$ or if $u_1((a, 1-a)) = u_1((b, 1-b))$ and $a \leq b$ *iff* $\frac{a}{a+1-a} = a < b = \frac{b}{b+1-b}$ or $a = b$ and $a \leq b$ *iff* $a \leq b$;
hence $(a, 1-a) \leq (b, 1-b)$ *iff* $a \leq b$ *iff* $(a, 1-a) \leq_{\circledcirc} (b, 1-b)$ for all $a, b \in [0,1]$, i.e. $x \leq y$ *iff* $p_1(x) \leq p_1(y)$ *iff* $x \leq_{\circledcirc} y$ for all $x, y \in G \cup \{(0,1), (1,0)\}$.
Thus we have $\mathbf{G}_{\oplus}$ is just the same as $\mathbf{G}_{\circledcirc_G}$, and $\mathbf{G}^+_{\oplus}$ is the same as $\mathbf{G}^+_{\circledcirc_G}$.

Corollary 2 *It holds that:*

$$\begin{array}{ccccc} h(a \oplus b) & = & h(a) \oplus h(b) & = & h(a) \circledcirc_G h(b) \\ u(a \circledcirc b) & = & u(a) \circledcirc_G u(b) & = & u(a) \oplus u(b) \end{array}$$

From the point of view of decision making, the difference between $\oplus$ *and* $\circledcirc$ *is given by their homomorphic projections* h *and* u *from* D_0 *onto* G. *(There is no difference on* G *because it holds* $h(x) = u(x) = x$ *for all* $x \in G$*).*

The theorem and its corollary express the importance of projection of D_0 onto G from the point of view of decision making. There are two homomorphic projections h and u (homomorphic with respect to operations $\oplus$ and $\circledcirc$). Another similar projection is the pignistic transformation defined in Transferable Belief Model (TBM), see e.g. [14]. Such projections are useful for decision making using belief functions. Hence, it would be an interesting and useful task to make a comparative study of these projections.

7 Conclusion A new algebraic structure — the disjunctive Dempster's semigroup — is defined on a binary frame of discernment and analyzed in this text. It is compared with the standard Dempster's semigroup. The high principal importance of homomorphic projections of general belief (d-pairs) onto Bayesian ones was shown. And consequently great importance, from the point of view of decision making, of probabilistic transformations and of all general Bayesian projections was mentioned.

8 Perspectives for future research There are the following fields for future research.

A comparative study of probabilistic transformations which could be motivated both by this algebraic analysis and by looking for a combination of belief functions which commutes with refinement coarsening, see [4]. The first results in this field has already been published in [6].

A study of automorphisms of the disjunctive Dempster's semigroup, analogous to the study of the standard case, see [2].

An algebraic study of subjective logic by Jøsang [12] and the comparison of the algebraic structure given on a binary frame of discernment by Jøsang's consensus operator (an algebraisation of his opinion space) with both the standard and the disjunctive Dempster's semigroup. This topic is just under development, for comparison of the Jøsang's semigroup with the standard Dempster's semigroup, see [7].

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References

[1] Milan Daniel. *Dempster's Semigroup and Uncertainty Processing in Rule Based Expert Systems.* Ph.D. thesis, Inst. of Comp. Sci., Academy of Sciences of the Czech Republic, Prague, 1993, (in Czech).

[2] Milan Daniel. Algebraic structures related to Dempster-Shafer theory. In B. Bouchon-Meunier, R. R. Yager, and L. A. Zadeh, editors, *Advances in Intelligent Computing - IPMU'94; Lecture Notes in Computer Science 945*, pages 51–61, Paris, France, 1995. Springer-Verlag.

[3] Milan Daniel. Composition and Decomposition of Belief Functions. In R. Jiroušek and J. Vejnarová, ed., *Proc. of 4th Czech-Japan Sem. on Data Analysis and Decision Making under Uncertainty* Jindřichův Hradec, 2001, 21–32.

[4] Milan Daniel. Combination of Belief Functions and Coarsening/Refinement. In Proceedings Ninth International conference IPMU, Université de Savoie, Annecy, 2002, Vol. I., pages 587 – 594.

[5] Milan Daniel. Algebraic structures related to the Consensus Operator for Combining of Beliefs. In Nielsen, T. D., Zhang, N. L., editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2003); LNAI 2711*, Springer-Verlag, 2003, pp. 332 – 344.

[6] Milan Daniel. Transformations of Belief Functions to Probabilities. In Vejnarová, J., editor, *Proceedings of WUPES 2003* (6th Worskop on Uncertainty Processing); VŠE - Oeconomica Publishers, 2003, pp. 77–90.

[7] Didier Dubois and Henri Prade. A Set-Theoretic View of Belief Functions. *Int. J. General Systems*, 12:193–226, 1986.

[8] Lazslo Fuchs. *Partially ordered algebraic systems.* Pergamon Press, 1963.

[9] Petr Hájek and Dagmar Harmancová. A note on Dempster's rule. Technical Report 891, Institute of computer Science, Academy of Sciences of the Czech Republic, Prague, 2001.

[10] Petr Hájek, Tomáš Havránek, and Radim Jiroušek. *Uncertain Information Processing in Expert Systems.* CRC Press, Boca Raton, Florida, 1992.

[11] Petr Hájek and Julio J. Valdés. Generalized algebraic foundations of uncertainty processing in rule-based expert systems (dempsteroids). *Computers and Artificial Intelligence*, 10:29–42, 1991.

[12] Audun Jøsang. A Logic for Uncertain Probabilities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9:279–311, 2001.

[13] Glen Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, New Jersey, 1976.

[14] Philippe Smets. The combination of evidence in the transferable belief model. *IEEE-Pattern analysis and Machine Intelligence*, 12:447–458, 1990.

[15] Philippe Smets and Robert Kennes. The transferable belief model. *Artificial Intelligence*, 66:191–234, 1994.

[16] Julio J. Valdés. *Algebraic and logical foundations of uncertainty processing in rule-based expert systems of Artificial Intelligence*. Ph.D. thesis, Czechoslovak Academy of sciences, Prague, 1987.

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