ON IDEALS OF AN IDEAL IN A BCI-ALGEBRA

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Abstract. The concept of ideals of an ideal in a BCI-algebra is introduced and some isomorphic theorems are obtained by using this concept.

§1. Introduction.

The concept of an ideal in a BCI-algebra was first introduced by K. Iséki in [1].

Definition 1[1]. Let \( X = (X; *, 0) \) be a BCI-algebra and \( \emptyset \neq I \subseteq X \), \( I \) is called an ideal of \( X \) if it satisfies the following conditions:

(i) 0 \( \in I \);

(ii) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \) (here \( x, y \in X \)).

We denote this fact by \( I \triangleleft X \). (\( I \triangleleft X \) means that \( I \triangleleft X \) and \( I \neq X \).)

If \( H \) is a subalgebra of a BCI-algebra \( X \), we denote it by \( H \leq X \), and \( H < X \) means that \( H \leq X \) and \( H \neq X \).

The concept of ideals has played an important role in the study of the theory of BCI-algebras. In a BCI-algebra \( X \), an ideal \( I \) need not be a subalgebra of \( X \). If the ideal \( I \) is also a subalgebra of \( X \), then it has better algebraic properties. Therefore C.S. Hoo and P.V. Ramana introduced the concept of closed ideals in [2].

Definition 2[2]. An ideal \( I \) of a BCI-algebra \( X \) is called a closed ideal if it is also a subalgebra of \( X \).

In this case it is denoted by \( I \trianglelefteq X \). (\( I \trianglelefteq X \) means that \( I \trianglelefteq X \) and \( I \neq X \).)

If \( I \) is a closed ideal of a BCI-algebra \( X \), then \( I \) is a BCI-algebra itself. So we may consider the ideals of \( I \). If \( I \) is an ideal of \( X \), but it is not closed, then \( I \) itself is not a BCI-algebra. Hence it has no ideals in the sense of Definition 1. However, we may also consider the "ideals" in the interior of \( I \). In this paper we introduced the concept of such "ideals" and give some isomorphic theorems by using this concept.

§2. Preliminaries. For the basic theory of BCK-and BCI-algebras the reader is referred to [8],[9] or [10],[11].

Let \( X = (X; *, 0) \) be a BCI-algebra and \( I \) be an ideal of \( X \). For \( x, y \in X \), define \( x \sim y \iff x * y, y * x \in I \), then \( \sim \) is a congruence relation on \( X \). The congruence class

\[ a + I = \{ a * x : x \in I \} \]

is a congruent class of \( a \) and \( a + I \) is a congruent class of \( a \). The congruence class \( a + I \) is called an ideal of \( I \) if it satisfies the following conditions:

(i) 0 \( \in I \);

(ii) \( x * y \in I \) and \( y \in I \) imply \( x \in I \) (here \( x, y \in X \)).
containing $x$ is denoted by $I_x$. In the quotient algebra $X/I$ the multiplication is defined as follows: $I_x * I_y = I_{xy}$. It is well defined since $\sim$ induced by $I$ is a congruence relation on $X$.

In this paper hereafter, $X$ always denotes a BCI-algebra.

Now we list some well known facts in the theory of BCI-algebras which we use in this paper.

**Proposition 1** [11]. Let $X = (X; *, 0)$ and $X' = (X'; *', 0')$ be BCI-algebras and $f : X \rightarrow X'$ be a homomorphism, then $f(0) = 0'$.

**Proposition 2** [11]. $I \triangleleft X \Rightarrow I_0 \subseteq I$ and $I_0 \triangleleft' X$.

$I_0$ is called the closed kernel of $I$.

**Proposition 3** [5]. If $I \triangleleft X$ and $A \triangleleft X$ such that $I_0 \subseteq A \subseteq I$, then $I_0$ induces the same congruence relation on $X$ just as $A$ does, hence we have $(X/A; *, A_0) = (X/I_0; *, I_0)$. Especially, $(X/I; *, I_0) = (X/I_0; *, I_0)$.

**Proposition 4** [6]. Let $f : X \rightarrow X'$ be a BCI-epimorphism. If $I \triangleleft X$, then $f(x) \triangleleft X'$.

**Proposition 5** [7]. Let $X$ and $X'$ be BCI-algebras and $f : X \rightarrow X'$ be a homomorphism. Set $\text{Ker}_f = \{x \in X | f(x) = 0'\}$, then $\text{Ker}_f \triangleleft X$. Here $0'$ is the zero element of $X'$. $\text{Ker}_f$ is called the homomorphic kernel of $f$.

**Proposition 6** [11]. If $I \triangleleft X$ and set

$$\varphi : X \rightarrow X/I$$

$$x \mapsto I_x$$

then $\varphi$ is an BCI-epimorphism and $\text{Ker}_f = I_0$. $\varphi$ is called the natural homomorphism from $X$ on $X/I$.

**Proposition 7** [11]. Let $X$ and $X'$ be BCI-algebras and $\eta : X \rightarrow X'$ be a homomorphism. Suppose that $\text{Ker}_f \leq A \triangleleft X$, then $\eta^{-1}(\eta(A)) = A$.

**Proposition 8** [11]. Let $f : X \rightarrow X'$ be a BCI-epimorphism, then $X/\text{Ker}_f \cong X'$.

Here by "$f : X \rightarrow X'$ be a BCI-epimorphism" we mean that both $X$ and $X'$ are BCI-algebras and $f$ is an epimorphism.

§3. Ideals of an ideal in a BCI-algebra.

**Definition 3.** Suppose that $I \triangleleft X$ and $\emptyset \neq A \subseteq I$. $A$ is called an ideal of $I$ if it satisfies the following conditions:

(i) $0 \in A$; (ii) for $x, y \in I$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

This fact is also denoted by $A \triangleleft I$. ($A \triangleleft I$ means that $A \triangleleft X$ and $A \neq X$.)

We can also define the concept of a subalgebra of an ideal $I$, although $I$ itself need not be a subalgebra of $X$. 
Definition 4. Suppose that $I \triangleleft X$ and $\emptyset \neq H \subseteq I$. $H$ is called a subalgebra of $I$ if $x, y \in H \Rightarrow x \ast y \in H$.

We denote this case by $H \leq I$. ($H < I$ means that $H \leq I$ and $H \neq I$.)

Definition 5. Suppose that $I \triangleleft X$ and $\emptyset \neq A \subseteq I$. If $A$ is an ideal of $I$ (in the sense of Definition 3) and $A$ is a subalgebra of $I$ as well, then $A$ is called a closed ideal of $I$. We denote this fact by $A \triangleleft^c I$. ($A \triangleleft^c I$ means that $A \triangleleft^c I$ and $A \neq I$).

Now we prove the transitive property of ideals.

Theorem 1. $A \triangleleft I \triangleleft X \Rightarrow A \triangleleft X$.

Proof. Clearly

(1) $\emptyset \neq A \subseteq X$

From $A \triangleleft I$ and Definition 3 we have

(2) $0 \in A$

For $x, y \in X$, suppose that

(3) $x \ast y \in A, y \in A$

Clearly,

(4) $A \subseteq I$

since $A \triangleleft I$. By (3) and (4) we have

(5) $x \ast y \in I, y \in I$

From (5) and $I \triangleleft X$ it follows that

(6) $x \in I$

Therefore, now we can consider the problem in $I$ since $x \in I$ and $y \in I$. Owing to the fact that $x \ast y \in A, y \in A$ and $A \triangleleft I$, by Definition 3 we obtain $x \in A$. The above argument shows that for $x, y \in X$,

(7) $x \ast y \in A, y \in A$ imply $x \in A$

By (1), (2), (7) and Definition 1 we get $A \triangleleft X$. □

Corollary 2. Suppose that $I \triangleleft X$ and $\emptyset \neq A \subseteq I$, then $A \triangleleft I \iff A \triangleleft X$.

Let $I$ be an ideal of a BCI-algebra $X$ and $A$ be an ideal of $I$ in the sense of Definition 3. For $x, y \in I$ define $x \sim_A y \iff x \ast y, y \ast x \in A$. (If no confusion may arise by context, one may use $\sim$ in stead of $\sim_A$.) One can easily verify that $\sim_A$ is a congruence relation on $I$ and so it gives a partition of $I$. For $x \in I$, by $A^{(I)}_x$ we denote the congruence class containing $x$. If no confusion may arise by context, one may use the brief symbol $A_x$ in stead of $A^{(I)}_x$. By Theorem 1 $A$ is also an ideal of $X$. Hence $A$ also induces a congruence relation on $X$. We have
Theorem 3. Suppose that $A \unlhd I \unlhd X$ and $u \in I$. Set 
\[ A_u^X = \{ x \in X | x * u, u * x \in A \} \]
\[ A_u^I = \{ x \in I | x * u, u * x \in A \} \]
then $A_u^I = A_u^X$.

Proof. By $A \unlhd I \unlhd X$ and Theorem 1 we obtain $A \unlhd X$, so $A$ also induces a congruence relation on $X$. From the definition it is clear that
\[ (1) \quad A_u^I \subseteq A_u^X \]
On the other hand, $\forall x \in A_u^X \implies x \in X$ and $x * u \in A$. It follows that $x * u \in I$ since $A \unlhd I$. Now $x * u \in I$, $u \in I$ and $I \unlhd X$ imply $x \in I$, therefore $x \in A_u^I$. This shows that 
\[ (2) \quad A_u^X \subseteq A_u^I \]
By (1) and (2) we obtain $A_u^I = A_u^X$. □

Theorem 3 shows that in this situation we can write $A_u^I = A_u^X$ briefly as $A_u$.

Corollary 4. $A \unlhd I \unlhd X \implies \emptyset \neq I/A \subseteq X/A$.

Theorem 5. Suppose that $A \unlhd X$, $A \subseteq H \subseteq X$, then

(i) $A \unlhd H$;
(ii) for $h \in H$, $A_h^H = H \cap A_h^X \subseteq A_h^X$;
(iii) if $H$ is not an ideal of $X$, then $A_h^H$ may be a proper subset of $A_h^X$.

Proof. (i) and (ii) hold obviously. We only need to give examples to show that (iii) holds.

The first example is $X = B_{4-2-1} = \{0, 1, 2, 3\}$ (see [3]). $X$ is in fact a BCK-algebra and so certainly a BCI-algebra as well. Its multiplication table is showed by Table 1.

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Table 1 Table 2

Set $A = \{0, 1\}$, $H = \{0, 1, 2\}$, and $h = 2 \in H$. Then $A \unlhd X$, $A \subset H \subset X$. $H$ is not an ideal of $X$. We have $A_h^X = \{2\}$, $A_h^X = \{2, 3\}$. So $A_h^X$ is a proper subset of $A_h^X$.

The second example is $X = I_{4-2-1} = \{0, 1, 2, 3\}$ (see [4]). It is a proper BCI-algebra. Its multiplication table is showed by Table 2. Set $A = \{0, 1\}$, $H = \{0, 1, 2\}$, $h = 2 \in H$. Then $A \unlhd X$, $A \subset H \subset X$. $H$ is not an ideal of $X$. It is easy to see that $A_h^X = \{2\}$, $A_h^X = \{2, 3\}$, hence $A_h^X$ is a proper subset of $A_h^X$. □
Definition 6. Suppose that $A \triangleleft I \triangleleft X$, then we define

$$I/A = \{A_u | u \in I\}.$$  

(Notice: By Theorem 3 we have $A_u^{(I)} = A_u^{(X)} = A_u$.)

Lemma 6. Suppose that $A \triangleleft I \triangleleft X$, $x \in X$, then $A_x \in I/A \iff x \in I$.

Proof. If $x \in I$, then by Definition 6 we have $A_x \in I/A$.

Now, suppose that $A_x \in I/A$, then by Definition 6 there exists $u \in I$ such that $A_x = A_u$, therefore $x * u \in A \subseteq I \implies x * u \in I$. From $u \in I$ and $I \triangleleft X$ it follows that $x \in I$. □

Corollary 7. $A \triangleleft I \triangleleft X$ and $A_x \in I/A \implies A_x \subseteq I$.

Theorem 8. $A \triangleleft I \triangleleft X = \implies I/A \triangleleft X/A$.

Proof. It is clear by Corollary 4 that

(1) $\emptyset \neq I/A \subseteq X/A$

$0 \in I$ since $I \triangleleft X$. Hence by Definition 6 we have

(2) $A_0 \in I/A$

Suppose that $A_x, A_y \in X/A$ such that $A_x * A_y \in I/A$ and $A_y \in I/A$. It follows that $A_{x+y} \in I/A$. By Lemma 6 we have $x * y \in I$ and $y \in I$. Thus we get $x \in I$ since $I \triangleleft X$. Therefore $A_x \in I/A$.

The above argument shows that

(3) for $A_x, A_y \in X/A$, $A_x * A_y \in I/A$, $A_y \in I/A \implies A_x \in I/A$

From (1),(2),(3) and Definition 1 we obtain $I/A \triangleleft X/A$. □

Corollary 9. $A \triangleleft I \triangleleft X \implies I/A \triangleleft X/A$.

Proof. $A \triangleleft I \triangleleft X \implies A \triangleleft I \triangleleft X$. Thus by Theorem 8 we obtain

(1) $I/A \triangleleft X/A$

$I \triangleleft X$ means that $I$ is also a subalgebra of $X$. Suppose that $A_x, A_y \in I/A$, then by Lemma 6 we get $x, y \in I$, it follows that $x * y \in I$ since $I \triangleleft X$. Therefore $A_x * A_y = A_{x+y} \in I/A$. This shows that

(2) $I/A \leq X/A$

By (1) and (2) we have $I/A \triangleleft X/A$. □

Theorem 10. Suppose that $A \triangleleft I \triangleleft X$ and $x \in X$, then we have

(i) $A_x \subseteq I_x$;
(ii) $y \in I_x \implies A_y \subseteq I_x$;
(iii) $I_x = \bigcup_{y \in I_x} A_y$. 

Proof. (i) By definition we have
\[ A_x = \{ y \in X | y \ast x, x \ast y \in A \} \]
\[ I_x = \{ y \in X | y \ast x, x \ast y \in I \} \]
By \( A \preceq I \) we have
\[ A \subseteq I \]
From (1), (2) and (3) it is clear that \( A_x \subseteq I_x \).

(ii) If \( y \in I_x \), then
\[ y \sim^I x \]
\[ u \sim^I y \]
\[ \forall u \in A_y \implies u \ast y, y \ast u \in A \implies u \ast y, u \ast y \in I \text{ since } A \subseteq I \] It follows that
\[ u \sim^I y \]
By (4) and (5) we obtain \( u \sim^I x \), i.e., \( u \in I_x \). Hence \( A_y \subseteq I_x \).

(iii) By (ii) it is obvious that
\[ \bigcup_{y \in I_x} A_y \subseteq I_x \]
On the other hand, \( \forall y \in I_x \implies y \in A_y \subseteq \bigcup_{y \in I_x} A_y \), therefore
\[ I_x \subseteq \bigcup_{y \in I_x} A_y \]
Combining (6) and (7) it follows that \( I_x = \bigcup_{y \in I_x} A_y \). □

Theorem 11. Let \( f : X \longrightarrow X' \) be BCI-epimorphism and \( \text{Ker} f \leq A \preceq X \), then

(i) \( f(A) \preceq X' \);

(ii) \( X/A \cong X'/f(A) \).

Proof. (i) By the given conditions and Proposition 4 we have \( f(A) \preceq X' \).

(ii) Set \( A' = f(A) \), then \( A' \preceq X' \).
Define
\[ \psi : X \longrightarrow X'/A' \]
\[ x \mapsto A'_{f(x)} \]
here \( A'_{f(x)} \) denotes the congruence class containing \( f(x) \) in the quotient algebra \( X'/A' \).
Then it follows that \( \psi(x \ast y) = A'_{f(x \ast y)} = A'_{f(x)} \ast A'_{f(y)} = A'_{f(x)} \ast A'_{f(y)} = \psi(x) \ast \psi(y) \), hence \( \psi \) is a homomorphism.

For any \( A_y' \in X'/A' \), where \( y \in X' \), there exists \( x \in X \) such that \( y = f(x) \), since \( f \) is surjective. So we have \( A_y' = A'_{f(x)} = \psi(x) \), this shows that \( \psi \) is surjective.

By the given conditions and Proposition 7 we get
\[ f^{-1}(A') = f^{-1}(f(A)) = A \]
Then we have
\[ x \in \ker \psi \iff A'_{f(x)} = A'_0, \quad (0' \text{ is the zero element of } X'). \]
\[ \iff f(x) * 0' \in A' \text{ and } 0' * f(x) \in A' \]
\[ \iff f(x) * f(0) \in A' \text{ and } f(0) * f(x) \in A' \text{ (by Proposition 1)} \]
\[ \iff f(x * 0) \in A' \text{ and } f(0 * x) \in A' \]
\[ (1) \iff x * 0 \in A \text{ and } 0 * x \in A \]
\[ \iff x \in A_0 \]

This shows that \( \ker \psi = A_0 \). Then by Proposition 8 it follows that
\[ (2) \quad X/A_0 \cong X'/A' \]

By Proposition 3 we get
\[ (3) \quad X/A = X/A_0 \]

From (2) and (3) we have \( X/A \cong X'/A' \). □

**Theorem 12.** Suppose that \( A \triangleleft I \triangleleft X \), then \( X/I \cong (X/A)/(I/A) \).

**Proof.** Let
\[ \varphi : X \longrightarrow X/A \]
\[ x \longmapsto A_x \]

be the natural homomorphism. By Proposition 6 we have \( \ker \varphi = A_0 \). By Proposition 2 it follows that \( A_0 \subseteq A \). Therefore \( \ker \varphi = A_0 \subseteq A \subseteq I \).

From Definition 6 we have \( \varphi(I) = \{ A_x \mid x \in I \} = I/A \). Using above conditions and Theorem 11 it follows that \( X/I \cong (X/A)/(\varphi(I)) = (X/A)/(I/A) \). □

**References**


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