Scientiae Mathematicae Japonicae , No. 10, (2004), 493–500

ON IDEALS OF AN IDEAL IN A BCI-ALGEBRA

Jiang Hao and Chen Xue Li

Abstract. The concept of ideals of an ideal in a BCI-algebra is introduced and some isomorphic theorems are obtained by using this concept.

§1. Introduction.

The concept of an ideal in a BCI-algebra was first introduced by K.Iséki in [1].

Definition 1[1]**.** Let $X = (X; *, 0)$ be a BCI-algebra and $\emptyset \neq I \subseteq X$, I is called an ideal of X if it satisfies the following conditions:

(i) $0 \in I$;

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$(here $x, y \in X$).

We denote this fact by $I \trianglelefteq X$.($I \triangleleft X$ means that $I \trianglelefteq X$ and $I \neq X$.)

If H is a subalgebra of a BCI-algebra X, we denote it by $H \leq X$, and $H < X$ means that $H \leq X$ and $H \neq X$.

The concept of ideals has played an important role in the study of the theory of BCI-algebras. In a BCI-algebra X, an ideal I need not be a subalgebra of X. If the ideal I is also a subalgebra of X, then it has better algebraic properties. Therefore C.S.Hoo and P.V.Ramana introduced the concept of closed ideals in [2].

Definition 2[2]**.** An ideal I of a BCI-algebra X is called a closed ideal if it is also a subalgebra of X.

In this case it is denoted by $I \overset{c}{\trianglelefteq} X$.($I \overset{c}{\triangleleft} X$ means that $I \overset{c}{\trianglelefteq} X$ and $I \neq X$.)

If I is a closed ideal of a BCI-algebra X, then I is a BCI-algebra itself. So we may consider the ideals of I. If I is an ideal of X, but it is not closed, then I itself is not a BCI-algebra. Hence it has no ideals in the sense of Definition 1. However, we may also consider the "ideals" in the interior of I. In this paper we introducd the concept of such "ideals" and give some isomorphic theorems by using this concept.

§2. Preliminaries. For the basic theory of BCK-and BCI-algebras the reader is referred to [8],[9] or [10],[11].

Let $X = (X; *, 0)$ be a BCI-algebra and I be an ideal of X. For $x, y \in X$, define $x \sim y \Longleftrightarrow x * y, y * x \in I$, then $\sim$ is a congruence relation on X. The congruence class

2000 *Mathematics Subject Classification.* 06F35.
*Key Words and Phrases.*BCI-algebra, BCK-algebra, ideal, isomorphism.
Project 102028 supported by Natural Science Foundation of Zhejiang Province, P.R. China

containing x is denoted by I_x. In the quotient algebra X/I the multiplication is defined as follows: $I_x * I_y = I_{x*y}$. It is well defined since $\sim$ induced by I is a congruence relation on X.

In this paper hereafter, X always denotes a BCI-algebra.

Now we list some well known facts in the theory of BCI-algebras which we use in this paper.

Proposition 1[11]**.** *Let* $X = (X;\ *,\ 0)$ *and* $X' = (X';\ *',\ 0')$ *be BCI-algebras and* $f : X \longrightarrow X'$ *be a homomorphism, then* $f(0) = 0'$.

Proposition 2[11]**.** $I \trianglelefteq X \Longrightarrow I_0 \subseteq I$ *and* $I_0 \overset{c}{\trianglelefteq} X$.

I_0 is called the closed kernel of I.

Proposition 3[5]**.** *If* $I \trianglelefteq X$ *and* $A \trianglelefteq X$ *such that* $I_0 \subseteq A \subseteq I$, *then* I_0 *induces the same congruence relation on* X *just as* A *does, hence we have* $(X/A;\ *,\ A_0) = (X/I_0;\ *,\ I_0)$. *Especially,*$(X/I;\ *,\ I_0) = (X/I_0;\ *,\ I_0)$.

Proposition 4[6]**.** *Let* $f : X \longrightarrow X'$ *be an BCI-epimorphism. If* $I \trianglelefteq X$, *then* $f(x) \trianglelefteq X'$.

Proposition 5[7]**.** *Let* X *and* X' *be BCI-algebras and* $f : X \longrightarrow X'$ *be a homomorphism. Set* $Kerf = \{x \in X | f(x) = 0'\}$, *then* $Kerf \overset{c}{\trianglelefteq} X$. *Here* $0'$ *is the zero element of* X'. $Kerf$ *is called the homomorphic kernel of* f.

Proposition 6[11]**.** *If* $I \trianglelefteq X$ *and set*

$$\begin{aligned} \varphi :& X \longrightarrow X/I \\ & x \longmapsto I_x \end{aligned}$$

then φ *is an BCI-epimorphism and* $Ker\ f = I_0$. φ *is called the natural homomorphism from* X *on* X/I.

Proposition 7[11]**.** *Let* X *and* X' *be BCI-algebras and* $\eta : X \longrightarrow X'$ *be a homomorphism. Suppose that* $Kerf \le A \trianglelefteq X$, *then* $\eta^{-1}(\eta(A)) = A$.

Proposition 8[11]**.** *Let* $f : X \longrightarrow X'$ *be a BCI-epimorphism, then* $X/Kerf \cong X'$.

Here by "$f : X \longrightarrow X'$ be a BCI-epimorphism" we mean that both X and X' are BCI-algebras and f is a epimorphism.

§3. Ideals of an ideal in a BCI-algebra.

Definition 3. Suppose that $I \trianglelefteq X$ and $\emptyset \neq A \subseteq I$. A is called an ideal of I if it satisfies the following conditions:

(i) $0 \in A$; (ii) for $x, y \in I$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

This fact is also denoted by $A \trianglelefteq I$.($A \triangleleft I$ means that $A \trianglelefteq X$ and $A \neq X$.)

We can also define the concept of a subalgebra of an ideal I, although I itself need not be a subalgebra of X.

Definition 4. Suppose that $I \trianglelefteq X$ and $\emptyset \neq H \subseteq I$. H is called a subalgebra of I if $x, y \in H \Rightarrow x * y \in H$.

We denote this case by $H \leq I$.($H < I$ means that $H \leq I$ and $H \neq I$.)

Definition 5. Suppose that $I \trianglelefteq X$ and $\emptyset \neq A \subseteq I$. If A is an ideal of I (in the sense of Definition 3) and A is a subalgebra of I as well, then A is called a closed ideal of I. We denote this fact by $A \overset{c}{\trianglelefteq} I$.($A \overset{c}{\triangleleft} I$ means that $A \overset{c}{\trianglelefteq} I$ and $A \neq I$).

Now we prove the transitive property of ideals.

Theorem 1. $A \trianglelefteq I \trianglelefteq X \Longrightarrow A \trianglelefteq X$.

Proof. Clearly

(1) $$\emptyset \neq A \subseteq X$$

From $A \trianglelefteq I$ and Definition 3 we have

(2) $$0 \in A$$

For $x, y \in X$, suppose that

(3) $$x * y \in A,\ y \in A$$

Clearly,

(4) $$A \subseteq I$$

since $A \trianglelefteq I$. By (3) and (4) we have

(5) $$x * y \in I,\ y \in I$$

From (5) and $I \trianglelefteq X$ it follows that

(6) $$x \in I$$

Therefore, now we can consider the problem in I since $x \in I$ and $y \in I$. Owing to the fact that $x * y \in A,\ y \in A$ and $A \trianglelefteq I$, by Definition 3 we obtain $x \in A$. The above argument shows that for $x, y \in X$,

(7) $$x * y \in A,\ y \in A \text{ imply } x \in A$$

By (1),(2),(7) and Definition 1 we get $A \trianglelefteq X$. □

Corollary 2. *Suppose that $I \trianglelefteq X$ and $\emptyset \neq A \subseteq I$, then $A \trianglelefteq I \Longleftrightarrow A \trianglelefteq X$.*

Let I be an ideal of a BCI-algebra X and A be an ideal of I in the sense of Definition 3. For $x, y \in I$ define $x \overset{A}{\sim} y \Longleftrightarrow x * y, y * x \in A$.(If no confusion may arise by context, one may use $\sim$ in stead of $\overset{A}{\sim}$.) One can easily verify that $\overset{A}{\sim}$ is a congruence relation on I and so it gives a partion of I. For $x \in I$, by $A_x^{(I)}$ we denote the congruence class containing x. If no confusion may arise by context, one may use the brief symbol A_x in stead of $A_x^{(I)}$. By Theorem 1 A is also an ideal of X. Hence A also induces a congruence relation on X. We have

Theorem 3. *Suppose that $A \trianglelefteq I \trianglelefteq X$ and $u \in I$. Set*

$$A_u^{(X)} = \{x \in X | x * u, u * x \in A\}$$
$$A_u^{(I)} = \{x \in I | x * u, u * x \in A\}$$

then $A_u^{(I)} = A_u^{(X)}$.

Proof. By $A \trianglelefteq I \trianglelefteq X$ and Theorem 1 we obtain $A \trianglelefteq X$, so A also induces a congruence relation on X. From the definition it is clear that

$$A_u^{(I)} \subseteq A_u^{(X)} \tag{1}$$

On the other hand, $\forall x \in A_u^{(X)} \Longrightarrow x \in X$ and $x * u \in A$. It follows that $x * u \in I$ since $A \subseteq I$. Now $x * u \in I$, $u \in I$ and $I \trianglelefteq X$ imply $x \in I$, therefore $x \in A_u^{(I)}$. This shows that

$$A_u^{(X)} \subseteq A_u^{(I)} \tag{2}$$

By (1) and (2) we obtain $A_u^{(I)} = A_u^{(X)}$. □

Theorem 3 shows that in this situation we can write $A_u^{(I)} = A_u^{(X)}$ briefly as A_u.

Corollary 4. $A \trianglelefteq I \trianglelefteq X \Longrightarrow \emptyset \neq I/A \subseteq X/A$.

Theorem 5. *Suppose that $A \trianglelefteq X$, $A \subseteq H \leq X$, then*

(i) $A \trianglelefteq H$;
(ii) *for* $h \in H$, $A_h^{(H)} = H \cap A_h^{(X)} \subseteq A_h^{(X)}$;
(iii) *if H is not an ideal of X, then $A_h^{(H)}$ may be a proper subset of $A_h^{(X)}$.*

Proof. (i) and (ii) hold obviously. We only need to give examples to show that (iii) holds. The first example is $X = B_{4-2-1} = \{0, 1, 2, 3\}$ (see [3]). X is in fact a BCK-algebra and so certainly a BCI-algebra as well. Its multiplication table is showed by Table 1.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

T*able* 1

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0

T*able* 2

Set $A = \{0, 1\}$, $H = \{0, 1, 2\}$, and $h = 2 \in H$. Then $A \triangleleft X$, $A \subset H < X$. H is not an ideal of X. We have $A_2^{(H)} = \{2\}$, $A_2^{(X)} = \{2, 3\}$. So $A_2^{(H)}$ is a proper subset of $A_2^{(X)}$.

The second example is $X = I_{4-2-1} = \{0, 1, 2, 3\}$ (see [4]). It is a proper BCI-algebra. Its multiplication table is showed by Table 2. Set $A = \{0, 1\}$, $H = \{0, 1, 2\}$, $h = 2 \in H$. Then $A \triangleleft X$, $A \subset H < X$. H is not an ideal of X. It is easy to see that $A_2^{(H)} = \{2\}$, $A_2^{(X)} = \{2, 3\}$, hence $A_2^{(H)}$ is a proper subset of $A_2^{(X)}$. □

Definition 6. Suppose that $A \trianglelefteq I \trianglelefteq X$, then we define

$$I/A = \{A_u | u \in I\}.$$

(Notice:By Theorem 3 we have $A_u^{(I)} = A_u^{(X)} = A_u$.)

Lemma 6. *Suppose that* $A \trianglelefteq I \trianglelefteq X,\ x \in X,$ *then* $A_x \in I/A \Longleftrightarrow x \in I$.

Proof. If $x \in I$, then by Definition 6 we have $A_x \in I/A$.

Now, suppose that $A_x \in I/A$, then by Definition 6 there exists $u \in I$ such that $A_x = A_u$, therefore $x * u \in A \subseteq I \Longrightarrow x * u \in I$. From $u \in I$ and $I \trianglelefteq X$ it follows that $x \in I$. □

Corollary 7. $A \trianglelefteq I \trianglelefteq X$ *and* $A_x \in I/A \Longrightarrow A_x \subseteq I$.

Theorem 8. $A \trianglelefteq I \trianglelefteq X \Longrightarrow I/A \trianglelefteq X/A$.

Proof. It is clear by Corollary 4 that

$$\emptyset \neq I/A \subseteq X/A \tag{1}$$

$0 \in I$ since $I \trianglelefteq X$. Hence by Definition 6 we have

$$A_0 \in I/A \tag{2}$$

Suppose that $A_x, A_y \in X/A$ such that $A_x * A_y \in I/A$ and $A_y \in I/A$. It follows that $A_{x*y} \in I/A$. By Lemma 6 we have $x * y \in I$ and $y \in I$. Thus we get $x \in I$ since $I \trianglelefteq X$. Therefore $A_x \in I/A$.

The above argument shows that

$$\text{for } A_x, A_y \in X/A,\ A_x * A_y \in I/A,\ A_y \in I/A \Longrightarrow A_x \in I/A \tag{3}$$

From (1),(2),(3) and Definition 1 we obtain $I/A \trianglelefteq X/A$. □

Corollary 9. $A \trianglelefteq I \overset{c}{\trianglelefteq} X \Longrightarrow I/A \overset{c}{\trianglelefteq} X/A$.

Proof. $A \trianglelefteq I \overset{c}{\trianglelefteq} X \Longrightarrow A \trianglelefteq I \trianglelefteq X$. Thus by Theorem 8 we obtain

$$I/A \trianglelefteq X/A \tag{1}$$

$I \overset{c}{\trianglelefteq} X$ means that I is also a subalgebra of X. Suppose that $A_x, A_y \in I/A$, then by Lemma 6 we get $x, y \in I$, it follows that $x * y \in I$ since $I \leq X$. Therefore $A_x * A_y = A_{x*y} \in I/A$. This shows that

$$I/A \leq X/A \tag{2}$$

By (1) and (2) we have $I/A \overset{c}{\trianglelefteq} X/A$. □

Theorem 10. *Suppose that* $A \trianglelefteq I \trianglelefteq X$ *and* $x \in X$, *then we have*

(i) $A_x \subseteq I_x$;
(ii) $y \in I_x \Longrightarrow A_y \subseteq I_x$;
(iii) $I_x = \bigcup_{y \in I_x} A_y$.

Proof. (i) By definition we have

$$A_x = \{y \in X | y * x, x * y \in A\} \tag{1}$$

$$I_x = \{y \in X | y * x, x * y \in I\} \tag{2}$$

By $A \trianglelefteq I$ we have

$$A \subseteq I \tag{3}$$

From (1),(2) and (3) it is clear that $A_x \subseteq I_x$.

(ii) If $y \in I_x$, then

$$y \overset{I}{\sim} x \tag{4}$$

$\forall u \in A_y \Longrightarrow u * y, y * u \in A \Longrightarrow y * u, u * y \in I$ since $A \subseteq I$. It follows that

$$u \overset{I}{\sim} y \tag{5}$$

By (4) and (5) we obtain $u \overset{I}{\sim} x$, i.e., $u \in I_x$. Hence $A_y \subseteq I_x$.

(iii) By (ii) it is obvious that

$$\bigcup_{y \in I_x} A_y \subseteq I_x \tag{6}$$

On the other hand, $\forall y \in I_x \Longrightarrow y \in A_y \subseteq \bigcup_{y \in I_x} A_y$, therefore

$$I_x \subseteq \bigcup_{y \in I_x} A_y \tag{7}$$

Combining (6) and (7) it follows that $I_x = \bigcup_{y \in I_x} A_y$. □

Theorem 11. *Let $f : X \longrightarrow X'$ be BCI-epimorphism and $Kerf \leq A \trianglelefteq X$, then*

(i) $f(A) \trianglelefteq X'$;

(ii) $X/A \cong X'/f(A)$.

Proof. (i) By the given conditions and Proposition 4 we have $f(A) \trianglelefteq X'$.

(ii) Set $A' = f(A)$, then $A' \trianglelefteq X'$.

Define

$$\psi : X \longrightarrow X'/A'$$
$$x \longmapsto A'_{f(x)}$$

here $A'_{f(x)}$ denotes the congruence class containing $f(x)$ in the quotient algebra X'/A'. Then it follows that $\psi(x * y) = A'_{f(x*y)} = A'_{f(x)*f(y)} = A'_{f(x)} * A'_{f(y)} = \psi(x) * \psi(y)$, hence ψ is a homomorphism.

For any $A'_y \in X'/A'$, where $y \in X'$, there exists $x \in X$ such that $y = f(x)$, since f is surjective. So we have $A'_y = A'_{f(x)} = \psi(x)$, this shows that ψ is surjective.

By the given conditions and Proposition 7 we get

$$f^{-1}(A') = f^{-1}(f(A)) = A \tag{1}$$

Then we have

$$\begin{aligned}
& x \in \mathrm{Ker}\psi \\
\Longleftrightarrow\ & A'_{f(x)} = A'_{0'}, \text{ (0' is the zero element of } X'.) \\
\Longleftrightarrow\ & f(x) * 0' \in A' \text{ and } 0' * f(x) \in A' \\
\Longleftrightarrow\ & f(x) * f(0) \in A' \text{ and } f(0) * f(x) \in A' \text{ (by Proposition 1)} \\
\Longleftrightarrow\ & f(x * 0) \in A' \text{ and } f(0 * x) \in A' \\
\overset{(1)}{\Longleftrightarrow}\ & x * 0 \in A \text{ and } 0 * x \in A \\
\Longleftrightarrow\ & x \in A_0
\end{aligned}$$

This shows that $\mathrm{Ker}\psi = A_0$. Then by Proposition 8 it follows that

$$X/A_0 \cong X'/A' \tag{2}$$

By Proposition 3 we get

$$X/A = X/A_0 \tag{3}$$

From (2) and (3) we have $X/A \cong X'/A'$. □

Theorem 12. *Suppose that $A \trianglelefteq I \trianglelefteq X$, then $X/I \cong (X/A)/(I/A)$.*

Proof. Let

$$\begin{aligned}
\varphi : & X \longrightarrow X/A \\
& x \longmapsto A_x
\end{aligned}$$

be the natural homomorphism. By Proposition 6 we have Ker $\varphi = A_0$. By Proposition 2 it follows that $A_0 \subseteq A$. Therefore Ker $\varphi = A_0 \subseteq A \subseteq I$.

From Definition 6 we have $\varphi(I) = \{A_x | x \in I\} = I/A$. Using above conditions and Theorem 11 it follows that $X/I \cong (X/A)/(\varphi(I)) = (X/A)/(I/A)$. □

References

[1] K.Iséki, *On BCI-algebras*, Math. Sem. Notes, 8(1980),125-130.

[2] C.S.Hoo and P.V.Ramana, *Quasi-commutative p-semisimple BCI-algebras*, Math. Japon., 32(1987),889-894.

[3] Jiang Hao, *Computational methods in the study of finite BCK-algebras with low orders*, Kobe J. Math., 7(1990), 33-46.

[4] Jiang Hao, *Atlas of proper BCI-algebras of order $n \leq 5$*, Math. Japon., 38(1993), 589-591.

[5] H.C.Chen and Y.Q.Zhu, *Remarks on quotient algebras of a BCI-algebra*, J. Huang Gang Normal College, Nat. Sci. Edi., 1989, No.4, 2-4.

[6] X.J.Zhou, *BCI-epimorphism reserves ideals*, Selected Papers of Graduates of Zhejiang Teachers University, April, 1985, 107-109.

[7] Z.M.Chen and H,X.Wang, *Closed ideals and congruences on BCI-algebras*, Kobe J. Math., 8(1991),1-9.

[8] K.Iséki and S.Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon., 23(1978),1-26.

[9] C. S. Hoo, *A survey of BCK and BCI-algebras*, Southeast Asian Bull. Math., 12(1988), 1-9.

[10] J. Meng and Y.B.Jun, *BCK-algebras*, Kyung Moon Sa Co., Soul, Korea, 1994.

[11] J. Meng and Y.L.Liu, *An introduction to BCI-algebras* (Chinese), Shănxi Scientific and Technological Press, Xi'an, China, 2001.

Department of Mathematics, Xixi campus of d Zhejiang University, Hangzhou 310028 P.R. China

E-mail address: jmhty@mail.hz.zj.cn