CCR C^*-ALGEBRAS AS INDUCTIVE LIMITS

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Abstract. As the main result we prove that CCR C^*-algebras are approximately subhomogeneous (ASH).

0. Introduction

Inductive limits of CCR (or liminary) C^*-algebras have been of great interest recently. They include inductive limits of finite dimensional C^*-algebras, which are called AF-algebras, and inductive limits of subhomogeneous (or homogeneous) C^*-algebras, which are called ASH-algebras (or AH-algebras) (cf.[EE], [Ln], [RS]). On the other hand, in the papers [Sd1] and [Sd2] we began to study the inductive limit structure of group C^*-algebras of Lie groups. In detail, we proved there that the unitizations of the C^*-algebras of all simply connected solvable Lie groups of non-type R are not ASH (approximately subhomogeneous), and also obtained that the C^*-algebras of certain simply connected nilpotent Lie groups (of type R in the sense of [AM, Chapter V] (cf. [OV, 6.5])) such as the real Heisenberg Lie group are ASH and the C^*-algebras of certain simply connected solvable Lie groups (of type R and non-type I) such as the Mautner group are AL (approximately liminary).

As the main result of this paper, it is shown that any CCR (or liminary) C^*-algebra is ASH. Consequently, we obtain that the group C^*-algebras of all CCR locally compact groups including all connected nilpotent Lie groups and connected semi-simple Lie groups are ASH. Note that type I C^*-algebras are not always ASH. For example, the Toeplitz C^*-algebra is of type I but not ASH since it contains a Fredholm operator with index nonzero (cf. [Mp], [Sd1]). Also the unitizations of the C^*-algebras of all simply connected solvable Lie groups of non-type R such as the real $ax + b$ group are non-CCR and not ASH ([Sd1]). Furthermore, it is deduced from our main result that AL-algebras are inductive limits of ASH-algebras.

Notation. Let $\hat{\mathfrak{A}} = \mathfrak{A}^\wedge$ denote the spectrum of a C^*-algebra $\mathfrak{A}$ of its all irreducible representations up to unitary equivalence. Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$ C^*-algebras ([Dx, Chapter 10]), and $\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a bounded continuous field on X. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on an infinite dimensional separable Hilbert space. Let $C^*(G)$ be the (full) group C^*-algebra of a Lie (or locally compact) group G (cf. [Dx]).

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1. The main results

Recall that a C^*-algebra is subhomogeneous if its all irreducible representations are finite dimensional and their dimensions are bounded (cf. [RS, 3.4]). Recall that a C^*-algebra is CCR (or liminary) if the image under its any irreducible representation is either a matrix algebra or the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space.

Theorem 1. *Let $\mathfrak{A}$ be a CCR C^*-algebra. Then $\mathfrak{A}$ is ASH.*

Proof. By [Pd, Theorem 6.2.11], there exists a composition series $\{\mathfrak{I}_\lambda\}_{\lambda\in\Lambda}$ of ideals of $\mathfrak{A}$ such that $\mathfrak{I}_\lambda/\mathfrak{I}_{\lambda-1}$ are of continuous trace and $\mathfrak{I}_{\lambda-1}$ is an essential closed ideal of $\mathfrak{I}_\lambda$ for each λ. We may also assume that $\Lambda = \mathbb{N}$ countable if necessary by using transfinite induction. Since $\mathfrak{I}_1$, $\mathfrak{I}_2/\mathfrak{I}_1$ are of continuous trace, we have

$$\mathfrak{I}_1 \cong \Gamma_0(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge}),$$
$$\mathfrak{I}_2/\mathfrak{I}_1 \cong \Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{\mathfrak{B}_\pi^2\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge}),$$

where the fibers $\mathfrak{B}_\pi^1, \mathfrak{B}_\pi^2$ are isomorphic to either $M_n(\mathbb{C})$ or $\mathbb{K}$ (where n may vary). Consider the following commutative diagram (cf. [Wo, 3.2]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_2 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} & & \\
0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_1) & \xrightarrow{q} & M(\mathfrak{I}_1)/\mathfrak{I}_1 & \longrightarrow & 0
\end{array}$$

where $M(\mathfrak{I}_1)$ means the multiplier algebra of $\mathfrak{I}_1$, τ is the canonical embedding, q is the canonical quotient and σ is the Busby invariant associated with the extension $\mathfrak{I}_2$ of $\mathfrak{I}_1$ by $\mathfrak{I}_2/\mathfrak{I}_1$. Note that the map τ is injective since $\mathfrak{I}_1$ is essential in $\mathfrak{I}_2$. Then it is well known that the extension $\mathfrak{I}_2$ is isomorphic to the pull-back algebra defined by

$$M(\mathfrak{I}_1) \oplus_{(q,\sigma)} \mathfrak{I}_2/\mathfrak{I}_1 = \{(f,g) \in M(\mathfrak{I}_1) \oplus \mathfrak{I}_2/\mathfrak{I}_1 \,|\, q(f) = \sigma(g)\},$$

which is a C^*-subalgebra of the direct sum $M(\mathfrak{I}_1) \oplus \mathfrak{I}_2/\mathfrak{I}_1$. Moreover, by [APT, Theorem 3.3], we have

$$M(\mathfrak{I}_1) = M(\Gamma_0(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge})) \cong \Gamma^b(\mathfrak{I}_1^\wedge, \{M(\mathfrak{B}_\pi^1)\}_{\pi\in\mathfrak{I}_1^\wedge}).$$

However, since $\mathfrak{I}_2$ is CCR, it is embedded in the C^*-subalgebra $\Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge})$ of $\Gamma^b(\mathfrak{I}_1^\wedge, \{M(\mathfrak{B}_\pi^1)\}_{\pi\in\mathfrak{I}_1^\wedge})$. Therefore, the above pull-back algebra is in fact isomorphic to

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge}) \oplus_{(q,\sigma)} \Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{\mathfrak{B}_\pi^2\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge}).$$

Then the direct sum $\Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge}) \oplus \Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{\mathfrak{B}_\pi^2\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge})$ is an inductive limit of subhomogeneous algebras of the form:

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{M_{n_\pi}(\mathbb{C})\}_{\pi\in\mathfrak{I}_1^\wedge}) \oplus \Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{M_{m_\pi}(\mathbb{C})\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge}),$$

where if $\mathfrak{B}_\pi^1, \mathfrak{B}_\pi^2$ are unital, then $\mathfrak{B}_\pi^1 = M_{n_\pi}(\mathbb{C})$ and $\mathfrak{B}_\pi^2 = M_{m_\pi}(\mathbb{C})$, and if $\mathfrak{B}_\pi^1, \mathfrak{B}_\pi^2$ are nonunital, then $\mathfrak{B}_\pi^1 = \mathbb{K} \supset M_{n_\pi}(\mathbb{C})$ and $\mathfrak{B}_\pi^2 = \mathbb{K} \supset M_{m_\pi}(\mathbb{C})$. Note that the numbers n_π, m_π can be bounded since

$$\Gamma_0(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge}) \subset \Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge}),$$

and $\Gamma_0(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_\pi^1\}_{\pi\in\mathfrak{I}_1^\wedge})$, $\Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{\mathfrak{B}_\pi^2\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge})$ are of continuous trace so that there exist their continuous operator fields of finite ranks (locally) and we can cut down their general continuous operator fields by the multiplication by these operator fields of finite ranks. Also note that

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{M_{n_\pi}(\mathbb{C})\}_{\pi\in\mathfrak{I}_1^\wedge}) \cong M(\Gamma_0(\mathfrak{I}_1^\wedge, \{M_{n_\pi}(\mathbb{C})\}_{\pi\in\mathfrak{I}_1^\wedge})),$$

which implies that $\Gamma^b(\mathfrak{I}_1^\wedge, \{M_{n_\pi}(\mathbb{C})\}_{\pi\in\mathfrak{I}_1^\wedge})$ is subhomogeneous since subhomogeneous algebras are closed under taking their multiplier algebras (cf. [RS, Proposition 3.4.3]). Moreover, it follows that the pull-back algebra $\mathfrak{I}_2$ is an inductive limit of subhomogeneous algebras of the form:

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{M_{n_\pi}(\mathbb{C})\}_{\pi\in\mathfrak{I}_1^\wedge}) \oplus_{(q,\sigma)} \Gamma_0((\mathfrak{I}_2/\mathfrak{I}_1)^\wedge, \{M_{m_\pi}(\mathbb{C})\}_{\pi\in(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge})$$

(note that subhomogeneous algebras are closed under taking subalgebras (cf. [RS, Proposition 3.4.3]), and if all the above subhomogeneous algebras of pull-back type are zero, the pull-back algebra $\mathfrak{I}_2$ must be zero).

For convenience, we consider the case $n = 3$. Then we have the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\tau_1} & & \downarrow{\scriptstyle\sigma_1} & & \\ 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_1) & \xrightarrow{q_1} & M(\mathfrak{I}_1)/\mathfrak{I}_1 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_2 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\tau_2} & & \downarrow{\scriptstyle\sigma_2} & & \\ 0 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_2/\mathfrak{I}_1) & \xrightarrow{q_2} & M(\mathfrak{I}_2/\mathfrak{I}_1)/(\mathfrak{I}_2/\mathfrak{I}_1) & \longrightarrow & 0 \end{array}$$

Note that τ_2 is injective since $\mathfrak{I}_2/\mathfrak{I}_1$ is essential in $\mathfrak{I}_3/\mathfrak{I}_1$. Let $\mathfrak{I}_j/\mathfrak{I}_{j-1} = \mathfrak{D}_j$. Then the quotient $\mathfrak{I}_3/\mathfrak{I}_1$ is isomorphic to the pull-back algebra:

$$\Gamma^b(\mathfrak{D}_2^\wedge, \{\mathfrak{B}_{\pi_2}\}_{\pi_2\in\mathfrak{D}_2^\wedge}) \oplus_{(q_2,\sigma_2)} \Gamma_0(\mathfrak{D}_3^\wedge, \{\mathfrak{B}_{\pi_3}\}_{\pi_3\in\mathfrak{D}_3^\wedge}),$$

which is ASH by using the same reasoning as above. Then the ideal $\mathfrak{I}_3$ is isomorphic to the following successive pull-back algebra:

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_{\pi_1}\}_{\pi_1\in\mathfrak{I}_1^\wedge}) \oplus_{(q_1,\sigma_1)} [\Gamma^b(\mathfrak{D}_2^\wedge, \{\mathfrak{B}_{\pi_2}\}_{\pi_2\in\mathfrak{D}_2^\wedge}) \oplus_{(q_2,\sigma_2)} \Gamma_0(\mathfrak{D}_3^\wedge, \{\mathfrak{B}_{\pi_3}\}_{\pi_3\in\mathfrak{D}_3^\wedge})],$$

which is also ASH by using the same reasoning as above.

In general, we use the induction and consider the following diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{D}_j & \longrightarrow & \mathfrak{I}_n/\mathfrak{I}_{j-1} & \longrightarrow & \mathfrak{I}_n/\mathfrak{I}_j & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\tau_j} & & \downarrow{\scriptstyle\sigma_j} & & \\ 0 & \longrightarrow & \mathfrak{D}_j & \longrightarrow & M(\mathfrak{D}_j) & \xrightarrow{q_j} & M(\mathfrak{D}_j)/\mathfrak{D}_j & \longrightarrow & 0 \end{array}$$

for $1 \le j \le n-1$, where $\mathfrak{D}_j = \mathfrak{I}_j/\mathfrak{I}_{j-1}$ and $\mathfrak{I}_0 = \{0\}$. Then the ideal $\mathfrak{I}_n$ is isomorphic to the following successive pull-back algebra:

$$\Gamma^b(\mathfrak{I}_1^\wedge, \{\mathfrak{B}_{\pi_1}\}_{\pi_1\in\mathfrak{I}_1^\wedge}) \oplus_{(q_1,\sigma_1)} [\Gamma^b(\mathfrak{D}_2^\wedge, \{\mathfrak{B}_{\pi_2}\}_{\pi_2\in\mathfrak{D}_2^\wedge}) \oplus_{(q_2,\sigma_2)} [\cdots$$
$$[\Gamma^b(\mathfrak{D}_{n-1}^\wedge, \{\mathfrak{B}_{\pi_{n-1}}\}_{\pi_{n-1}\in\mathfrak{D}_{n-1}^\wedge}) \oplus_{(q_{n-1},\sigma_{n-1})} \Gamma_0(\mathfrak{D}_n^\wedge, \{\mathfrak{B}_{\pi_n}\}_{\pi_n\in\mathfrak{D}_n^\wedge})]\cdots]].$$

Finally, as a composition series of $\mathfrak{A}$ we choose certain subhomogeneous subalgebras of $\mathfrak{I}_n$ ($n \geq 1$) constructed above inductively such that their union is dense in $\mathfrak{A}$. Note that by the construction the subhomogeneous subalgebras of $\mathfrak{I}_n$ can be embedded into the subhomogeneous subalgebras of $\mathfrak{I}_{n+1}$ (see the case $n = 2$). □

Remark. If $\mathfrak{A}$ is a unital CCR C^*-algebra, then $\pi(\mathfrak{A})$ for $\pi \in \hat{\mathfrak{A}}$ is isomorphic to $M_n(\mathbb{C})$ for some n since $\pi(\mathfrak{A})$ is unital. Note that CCR C^*-algebras are not closed under taking extensions by themselves and even taking unitizations. Indeed, the unitization of $\mathbb{K}$ by adding the identity operator is not CCR (or liminary) (cf. [Mp, Section 5.6]). Also, ASH algebras are not closed under taking extensions. For example, the Toeplitz algebra is an extension of the C^*-algebra of all continuous functions on the torus by $\mathbb{K}$ but not ASH since it is generated by the unilateral shift operator with Fredholm index -1 (cf. [Mp, Section 3.5] and [Sd1]).

Corollary 2. *Let G be a CCR locally compact group. Then $C^*(G)$ is ASH.*

More specifically,

Corollary 3. *Let G be either a connected nilpotent Lie group, a simply connected solvable Lie group of type R and type I, or a connected semi-simple Lie group. Then $C^*(G)$ is ASH.*

Proof. Note that the C^*-algebras of connected nilpotent Lie groups, simply connected solvable Lie groups of type R and type I and connected semi-simple Lie groups are CCR (cf. [Dx, 13.11.12], [AM, Chapter V]). □

Corollary 4. *Let $\mathfrak{A}$ be an inductive limit of CCR C^*-algebras. Then $\mathfrak{A}$ is an inductive limit of ASH-algebras.*

Proof. Use Theorem 1. □

Remark. It is known that inductive limits of CCR C^*-algebras are nuclear and quasidiagonal, but quasidiagonal nuclear C^*-algebras are not always ASH (cf. [Sl]). Note also that AH-algebras are not closed under taking their inductive limits (cf. [DE], [RS, Proposition 3.1.9]). It might be true that ASH-algebras are closed under inductive limits. If so, AL-algebras are in fact ASH.

Question. Is it true that the C^*-algebra of a simply connected solvable Lie group of type R is ASH ?

Remark. In [Sd1] we obtained that the unitizations of the C^*-algebras of all simply connected solvable Lie groups of non-type R are not ASH. Since ASH algebras are not closed under taking extensions, it might be true that the C^*-algebras of all simply connected solvable Lie groups are ASH.

However, it is obtained that

Proposition 5. *Let $\mathfrak{A}$ be a C^*-algebra of type I and non CCR. Then $\mathfrak{A}$ is not ASH.*

Proof. Since $\mathfrak{A}$ is of type I and non CCR, there exists $\pi \in \hat{\mathfrak{A}}$ such that $\pi(\mathfrak{A})$ strictly contains the C^*-algebra $\mathbb{K}_\pi$ of all compact operators on the representation space of π. Suppose that $\mathfrak{A}$ is ASH. Then $\mathfrak{A}$ is an inductive limit of subhomogeneous algebras $\mathfrak{B}_\lambda$. Then $\pi(\mathfrak{B}_\lambda)$ are also subhomogeneous. Since $\mathbb{K}_\pi$ is a closed ideal of $\pi(\mathfrak{A})$ and the union of $\pi(\mathfrak{B}_\lambda)$ is dense in $\pi(\mathfrak{A})$, it follows that the intersection $\mathbb{K}_\pi \cap \pi(\mathfrak{B}_\lambda)$ for some λ must be non-empty and $\pi(\mathfrak{B}_\lambda)$ strictly contains $\mathbb{K}_\pi \cap \pi(\mathfrak{B}_\lambda)$ so that $\pi(\mathfrak{B}_\lambda)$ contains an operator of infinite rank. Since the representation of $\pi(\mathfrak{B}_\lambda)$ corresponding to the closed ideal $\mathbb{K}_\pi \cap \pi(\mathfrak{B}_\lambda)$ (or its irreducible representation) is irreducible, this contradicts to $\pi(\mathfrak{B}_\lambda)$ being subhomogeneous. □

Corollary 6. *Let G be a simply connected solvable Lie group of non-type R and type I. Then $C^*(G)$ is not ASH.*

Proof. By assumption, $C^*(G)$ is non CCR and of type I (cf. [AM, Chapter V]). □

Remark. In [Sd2] we obtained that the C^*-algebras of the generalized Mautner groups of type R and non-type I (cf. [AM, Chapter 3]) are inductive limits of CCR C^*-algebras. By using the method for the proof of Theorem 1 and the inductive limit structure given in [Sd2], we conclude that those algebras are in fact ASH. This supports our conjecture.

Finally, as one more remark,

Remark. By [Kt, Theorem 1], the C^*-algebra $C^*(G)$ of a connected locally compact group G has the real rank zero (cf. [BP]) if and only if G is compact. Hence, if G is non-compact, $C^*(G)$ is not AF. Note that AF-algebras have the real rank zero (that is, an approximation property by projections with finite spectrums).

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