ABSTRACT GENERALIZED QUASILINEARIZATION METHOD FOR COINCIDENCES WITH APPLICATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. An abstract unified theory of both monotone iterative and generalized quasilinearization methods is presented for operator equations of coincidence type in ordered Banach spaces. Applications are given for strong solutions of semilinear elliptic problems.

1. Introduction

In [9], a joint paper with Radu Precup, we develop an abstract theory of the generalized quasilinearization method for semilinear operator equations of coincidence type, i.e.

$$Lu = N(u), \ u \in D, \tag{1}$$

in ordered Banach spaces. Our theory contains as a particular case, the monotone iterative method (see [**?**, 5, 13, 23]) and, in the same time, is a monotone and nonsmooth version of Newton method for approximating roots of nonlinear equations. This version unifies some old ideas, as are presented in abstract setting in [26] (see also the references therein) or [3, 12, 21, 25], and recent ideas, as they are used in various applications in [14, 15, 18] or [1, 6, 10, **?**, 19, 20, 24]. It is worth to mention that when applied to differential equations, the Newton method is also known as quasilinearization method. A remarkable contribution in this direction has been the monograph of Bellman and Kalaba [4]. Interesting new extensions are due to Lakshmikantham [14]. In [8, 9] we give details regarding the history of the subject and the place of our theory in relation with similar results.

In this paper we apply the abstract theory on the quasilinearization method for strong solutions of semilinear elliptic problems. We obtain two monotone, L^p-convergent sequences of approximate solutions, which satisfy some corresponding linear problems. We estimate that the order of convergence is two. Our results complement in some sense, and intersect, but do not include, the ones existing in the literature, mainly given by Lakshmikantham-Vatsala in [17] and by Lakshmikantham-Leela in [16]. In a forthcoming paper [7] we intend to consider also the case of fully nonlinear elliptic boundary value problems. By our knowledge, the quasilinearization method has not been initialized until now to this kind of problems.

2. Abstract theory

In this section we present our abstract theory on quasilinearization, as developed in [9]. The first result represents a generalization of the monotone iterative technique for coincidences.

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Theorem 2.1. *Let X be an ordered Banach space, Z be an ordered topological linear space, D a linear subspace of X and $\alpha_0, \beta_0 \in D$. Let $L : D \to Z$ be a linear operator and $N : X \to Z$ be a mapping. Assume that the following conditions are satisfied:*

(i): $\alpha_0 \le \beta_0$, $L\alpha_0 \le N(\alpha_0)$ *and* $L\beta_0 \ge N(\beta_0)$;

(ii): *for every $u, v \in X$ with $\alpha_0 \le u \le v \le \beta_0$, there is a linear operator $P(u,v) : X \to Z$ such that $L - P(u,v) : D \to Z$ is bijective with positive inverse,*

$$N(u) \le N(v) - P(u,v)(v-u) \tag{2}$$

and

$$-P(u,v)\,z \le -P(\alpha,\beta)\,z \tag{3}$$

for all $\alpha, \beta, u, v, z \in X$ with $\alpha_0 \le \alpha \le u \le v \le \beta \le \beta_0$ and $z \ge 0$;

(iii): *either*

(a) the positive cone of X is regular and the operators

$$\begin{cases} (L - P(\alpha_0,\beta_0))^{-1} N,\ (L - P(\alpha_0,\beta_0))^{-1} P(\alpha_0,\beta_0), \\ (L - P(\alpha_0,\beta_0))^{-1} P(u,u),\ u \in X,\ \alpha_0 \le u \le \beta_0 \end{cases} \tag{4}$$

are continuous on $[\alpha_0, \beta_0]$,

or

(b) the positive cone of X is normal and the operators (4) are completely continuous on $[\alpha_0, \beta_0]$.

Then the sequences $(\alpha_n), (\beta_n)$ given by the iterative schemes

$$L\alpha_{n+1} = N(\alpha_n) + P(\alpha_n, \beta_n)(\alpha_{n+1} - \alpha_n), \tag{5}$$

$$L\beta_{n+1} = N(\beta_n) + P(\alpha_n, \beta_n)(\beta_{n+1} - \beta_n) \tag{6}$$

($n \in \mathbf{N}$) are well and uniquely defined in D. In addition, they are monotonically convergent in X to the minimal and, respectively, to the maximal solution in $[\alpha_0, \beta_0]$ of (1).

Remark 2.1. *If N and $P(u,v)$ are continuous then the assumption on operators (4) in (iii) is satisfied if $(L - P(\alpha_0,\beta_0))^{-1}$ is continuous, in case (a), and if N is bounded and $(L - P(\alpha_0,\beta_0))^{-1}$ is completely continuous, in case (b).*

Remark 2.2. *In particular, if $P(u,v) = 0$ for every u, v, Theorem 2.1 reduces to the monotone iterative method for the operator equation $Lu = N(u)$ with an increasing mapping N. The reader can see that in this case, (iii) (b) requires that $L^{-1}N$ is completely continuous.*

The next result gives conditions so that $(\alpha_n), (\beta_n)$ converge quadratically to the unique solution in $[\alpha_0, \beta_0]$ of (1).

Theorem 2.2. *Assume all the assumptions of Theorem 2.1 hold. If*

(iv): *for every $u, v \in D$ with $\alpha_0 \le u \le v \le \beta_0$, there exists a mapping $R(v,u) : D \to Z$ such that*

$$N(u) \ge N(v) - R(v,u)(v-u); \tag{7}$$

(v): *$L - R(v,u)$ is inverse positive, i.e. $(L - R(v,u))\,z \ge 0$ implies $z \ge 0$,*

then (1) has a unique solution u^* in $[\alpha_0, \beta_0]$.

In addition assume that the following conditions are satisfied:

(vi): $(L - P(u,u))^{-1} : Z \to X$ *is continuous for every $u \in D$, $\alpha_0 \le u \le \beta_0$;*

(vii): *there exist two constants $c_1, c_2 > 0$ such that*

$$|(R(w,\alpha) - P(\alpha,\beta))\,z|_Z \le c_1 |w - \alpha|_X |z|_X + c_2 |\alpha - \beta|_X |z|_X \tag{8}$$

for all $\alpha, \beta, w, z \in D$, $\alpha_0 \le \alpha \le w \le \beta \le \beta_0$, $z \ge 0$.

Then the convergence of $(\alpha_n), (\beta_n)$ *to* u^* *is quadratic.*

Remark 2.3. *All the above results are valid if the operator* N *is defined only on* $[\alpha_0, \beta_0] \cap D$, *instead on the whole space* X.

As a consequence of Theorem 2.2, we obtain the following abstract version of Lakshmikantham's generalized quasilinearization method for the semilinear operator equation (1).

Theorem 2.3. *Let* X *be an ordered Banach space,* Z *be another ordered Banach space,* D *a linear subspace of* X *and* $\alpha_0, \beta_0 \in D$. *Let* $L : D \to Z$ *be a linear operator and* $N : X \to Z$ *be a mapping. Assume that the following conditions are satisfied:*

(a): $\alpha_0 \leq \beta_0$, $L\alpha_0 \leq N(\alpha_0)$ *and* $L\beta_0 \geq N(\beta_0)$;

(b): $N = N_1 - N_2$, *where* $N_1, N_2 : X \to Z$ *are* C^1*-Gâteaux differentiable mappings which are convex on* $[\alpha_0, \beta_0]$, *and for every* $u, v, z \in X$ *with* $\alpha_0 \leq u \leq v \leq \beta_0$ *and* $z \geq 0$,
$$N_i'(u)z \leq N_i'(v)z, \ i = 1, 2;$$

(c): $L - N_1'(u) + N_2'(v) : D \to Z$ *are bijective with positive inverse for every* $u, v \in [\alpha_0, \beta_0]$ *with* $u \leq v$ *or* $v \leq u$;

(d): *either*

(1) the positive cone of X *is regular and the operator*
$$(L - N_1'(\alpha_0) + N_2'(\beta_0))^{-1} \tag{9}$$
is continuous on $[\alpha_0, \beta_0]$,

or

(2) the positive cone of X *is normal, the mapping* N *is bounded and the operator (9) is completely continuous on* $[\alpha_0, \beta_0]$.

Then (1) has a unique solution u^* *in* $[\alpha_0, \beta_0]$ *and the sequences* (α_n), (β_n) *given by the iterative schemes*
$$L\alpha_{n+1} = N(\alpha_n) + (N_1'(\alpha_n) - N_2'(\beta_n))(\alpha_{n+1} - \alpha_n), \tag{10}$$
$$L\beta_{n+1} = N(\beta_n) + (N_1'(\alpha_n) - N_2'(\beta_n))(\beta_{n+1} - \beta_n) \tag{11}$$
$(n \in \mathbf{N})$ *are well and uniquely defined in* D *and they are monotonically convergent in* X *to* u^*.

If in addition $(L - N'(u))^{-1} : Z \to X$ *is continuous for every* $u \in D$, $\alpha_0 \leq u \leq \beta_0$, *and* N_1', N_2' *are Lipschitz on* $[\alpha_0, \beta_0]$, *then the convergence of* $(\alpha_n), (\beta_n)$ *in* X *is quadratic.*

Remark 2.4. *The hypothesis (b) can be replaced by the assumption that* N_1 *and* N_2 *are twice uniformly differentiable on every segment of* X, *the positive cone of* Z *is normal, and* $N_i''(u) \geq 0$ *for every* $u \in X$ *and* $i = 1, 2$ *(see [26]).*

3. Applications to elliptic problems

Let $1 < p < \infty$, Ω be a C^2 bounded domain of $\mathbb{R}^n$. We denote by $\mathcal{M}_n$ the space of $n \times n$ real matrices; $|\cdot|_m$ is the euclidean norm in $\mathbb{R}^m$. The Sobolev spaces $W^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are as defined in [2]. We denote by B the following linear elliptic operator in nondivergence form
$$Bu = \sum_{i,j=1}^{n} l_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} l_i(x) \frac{\partial u}{\partial x_i} \tag{12}$$

where $L = (l_{ij}) \in C(\overline{\Omega}, \mathcal{M}_n)$, $l = (l_i) \in L^\infty(\Omega, \mathbb{R}^n)$, and

$$\sum_{i,j=1}^{n} l_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \quad , \forall\, x \in \Omega, \ \xi \in \mathbb{R}^n.$$

We consider the semilinear elliptic problem

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \ -Bu = f(x,u), \quad \text{for a.e.} \quad x \in \Omega \tag{13}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory.
We will work in the presence of lower and upper solutions.

Definition 3.1. α_0 *is a lower solution of (13) if*

$$\alpha_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \ -B\alpha_0 \leq f(x,\alpha_0), \quad \textit{for a.e.} \quad x \in \Omega.$$

Whenever the oposite inequality holds for β_0*, we say that* β_0 *is an upper solution.*

We study the approximate solutions for (13) given by the following iterative schemes.

$$-B\alpha_{n+1} = f(x,\alpha_n) + P(x,\alpha_n,\beta_n)\,(\alpha_{n+1} - \alpha_n) \tag{14}$$

$$-B\beta_{n+1} = f(x,\beta_n) + P(x,\alpha_n,\beta_n)\,(\beta_{n+1} - \beta_n)\,. \tag{15}$$

Our main result states that, under some additional assumptions on f, these schemes give monotone and quadratically convergent sequences of approximate solutions. As consequences we obtain two results. One contains similar ideas to those used by Lakshmikantham et al. (see [18, 10, 17]) as regards the conditions for the nonlinear part and the form of the function P in the iterative schemes. The basic condition for f is some convexity, and P is given in terms of derivatives of f. The second consequence of our main result use for P an expression in terms of divided differences and it can be used when f is not differentiable.

In what follows, for two functions $\alpha_0, \beta_0 \in L^p(\Omega)$ with $\alpha_0 \leq \beta_0$, we consider the order interval $[\alpha_0, \beta_0]$ given by

$$[\alpha_0, \beta_0] = \{u \in L^p(\Omega) \ : \ \alpha_0(x) \leq u(x) \leq \beta_0(x) \ \text{ for a.e. } x \in \Omega\}\,.$$

The next theorem is the main result of this section.

Theorem 3.2. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function. Assume that*

(i): *there exist* α_0 *and* β_0 *a lower and, respectively, an upper solution of (13) with* $\alpha_0 \leq \beta_0$ *a.e. in* Ω *and* $f(\cdot,\alpha_0(\cdot))$, $f(\cdot,\beta_0(\cdot)) \in L^p(\Omega)$*;*

(ii): *there exists a Carathéodory function* $P : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$f\,(x,u) \leq f\,(x,v) - P\,(x,u,v)\,(v-u) \tag{16}$$

for $\alpha_0\,(x) \leq u \leq v \leq \beta_0\,(x)\,,$ *a.e. in* Ω*. Also, there exists a real number* $M_1 \geq 0$ *such that*

$$0 \leq -P\,(x,u,v) \leq -P\,(x,\alpha,\beta) \leq M_1 \tag{17}$$

for $\alpha_0\,(x) \leq \alpha \leq u \leq v \leq \beta \leq \beta_0\,(x)$*, a.e. in* Ω*;*

Then the sequences (α_n) *and* (β_n) *given by the iterative schemes (14) and (15) are well and uniquely defined in* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$*, are monotone and converge in the* L^p *-norm to the minimal and respectively maximal solution of (13) in the order interval* $[\alpha_0, \beta_0]$*.*

If in addition the following conditions are satisfied

(iii): *there exists a Carathéodory function* $b : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$f(x,u) \geq f(x,v) - b(x,v,u)(v-u) \tag{18}$$

for $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$, *a.e. in* Ω*;*

(iv): $0 \leq -b(t,v,u) \leq M_2$ *for all* $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$ *a.e. in* Ω*, and for some* $M_2 \geq 0$.

then (13) has a unique solution in the order interval $[\alpha_0, \beta_0]$.

Moreover, the next condition

(v): *there exist two constants* $c_1, c_2 \geq 0$ *such that*

$$b(x,u,\alpha) - P(x,\alpha,\beta) \leq c_1(u-\alpha) + c_2(\beta-\alpha) \tag{19}$$

for $\alpha_0(x) \leq \alpha \leq u \leq \beta \leq \beta_0(x)$, *a.e. in* Ω,

assures that the convergence of (α_n) *and* (β_n) *in* $L^p(\Omega)$ *is quadratic.*

Proof. We divide the proof into several steps.

1) All hypotheses of Theorem 2.1 are fulfilled.

Using notations of Theorem 2.1, let us consider

$$X = Z = L^p(\Omega),\ D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$Lu = -Bu,\ N(u) = f(\cdot, u(\cdot)),\ Q(u,v)z = P(\cdot, u(\cdot), v(\cdot))z,$$

for $u, v \in D \cap [\alpha_0, \beta_0]$ with $u \leq v$, and $z \in D$.

The linear operator $Q(u,v)$ is well defined and continuous between D and $L^p(\Omega)$, since the function P is Carathéodory and satisfies some boundedness condition (17), which assure that $P(\cdot, u(\cdot), v(\cdot)) \in L^\infty(\Omega)$. The fact that the nonlinear operator N is well defined and continuous between the set $\{u \in D : \alpha_0 \leq u \leq \beta_0\}$ and $L^p(\Omega)$ follows by the inequality (16) and the Lebesgue dominated convergence theorem.

It is easy to see that hypothesis (i), relations (2) and (3) of Theorem 2.1 are valid. Also, for every $u, v \in D$ with $\alpha_0 \leq u \leq v \leq \beta_0$, the mapping $L - Q(u,v)$ from D to $L^p(\Omega)$, in fact

$$w \longmapsto -Bw - l(\cdot)w, \quad \text{where } l(x) = P(x, u(x), v(x)) \leq 0, \quad \text{a.e. in } \Omega$$

is bijective, with positive and continuous inverse ([11], Theorem 9.15 and Lemma 9.17). Let us remember that the positive cone of $L^p(\Omega)$ is regular. Let us notice now that, with these notations, relations (14)-(15) coincide with (5)-(6).

We apply now Theorem 2.1 and deduce that the sequences (α_n) and (β_n) given by the iterative schemes (14) and (15) are well and uniquely defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, are monotone and converge in the L^p -norm to the minimal and respectively maximal solution of (13) in the order interval $[\alpha_0, \beta_0]$.

2) The solution is unique in $[\alpha_0, \beta_0]$.

Let us denote by u_* the minimal solution, by u^* the maximal solution and also put $l_*(x) = b(x, u^*(x), u_*(x))$. Using (18) and the above notations, we obtain that $-Bu_* \geq -Bu^* - l_*(x)(u^* - u_*)$ a.e. in Ω. Then

$$-Bu_* - l_*(x)u_* \geq -Bu^* - l_*(x)u^*,\ \text{a.e. in } \Omega,$$

where $l_* \in L^\infty(\Omega)$ and $l_*(x) \leq 0$. The weak maximum principle implies that $u_* \geq u^*$. But $u_* \leq u^*$. Hence, $u_* = u^*$ and the solution is unique in $[\alpha_0, \beta_0]$, indeed.

3) The convergence is quadratic.

Let us denote

$$p_n = u^* - \alpha_n \text{ and } q_n = \beta_n - u^*.$$

Using (16), (17), (18), (14) and (19) we obtain the following inequalities.

$$\begin{array}{rcl} -Bp_{n+1}-P\left(x,u^{*},u^{*}\right)p_{n+1} & \leq & -Bp_{n+1}-P\left(x,\alpha_n,\beta_n\right)p_{n+1} \\ & = & -P\left(\alpha_n,\beta_n\right)p_n-f\left(x,\alpha_n\right)+f\left(x,u^{*}\right) \\ & \leq & \left(b\left(x,u^{*},\alpha_n\right)-P\left(x,\alpha_n,\beta_n\right)\right)p_n \\ & \leq & c_1p_n^2+c_2(\beta_n-\alpha_n)p_n \\ & = & c_1p_n^2+c_2(q_n+p_n)p_n \\ & \leq & c_3p_n^2+c_4q_n^2. \end{array}$$

Whenever $p_n^2, q_n^2 \in L^p(\Omega)$, using that the linear operator $-B-P(\cdot,u^*,u^*)I$ has a bounded inverse, we obtain that

$$||p_{n+1}||_{L^p} \leq C_1||p_n^2||_{L^p}+C_2||q_n^2||_{L^p}.$$

□

The next theorem is a consequence of the previous one and contains similar ideas to those used by Lakshmikantham et al. (see [18, 10, 17]) as regards the conditions for the nonlinear part and the form of the function P in the iterative schemes. The basic condition for f is some convexity, and P is given in terms of derivatives of f.

Theorem 3.3. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and α_0,β_0 be a lower and, respectively, an upper solution of (13), such that $\alpha_0\leq\beta_0$ a.e. in Ω and $f(\cdot,\alpha_0(\cdot)), f(\cdot,\beta_0(\cdot))\in L^p(\Omega)$.*
Assume that $f=f_1-f_2$ where $f_1,f_2:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory, $f_1(x,\cdot)$ and $f_2(x,\cdot)$ are C^1 on $\mathbb{R}$ and convex on $[\alpha_0(x),\beta_0(x)]$ for a.a. $x\in\Omega$. In addition, assume that $\frac{\partial f_1}{\partial u}(x,\cdot)$ and $\frac{\partial f_2}{\partial u}(x,\cdot)$ are Lipschitz on $[\alpha_0(x),\beta_0(x)]$ with Lipschitz constants not depending on x, and

$$-M \leq \frac{\partial f_1}{\partial u}(t,u)-\frac{\partial f_2}{\partial u}(t,v)\leq 0$$

for all $u,v\in[\alpha_0(x),\beta_0(x)]$ and for a.a. $x\in\Omega$.
Then the sequences (α_n) and (β_n) given by the iterative schemes

$$-B\alpha_{n+1}=f\left(x,\alpha_n\right)+\left(\frac{\partial f_1}{\partial u}(x,\alpha_n)-\frac{\partial f_2}{\partial u}(x,\beta_n)\right)\left(\alpha_{n+1}-\alpha_n\right),$$

$$-B\beta_{n+1}=f\left(x,\beta_n\right)+\left(\frac{\partial f_1}{\partial u}(x,\alpha_n)-\frac{\partial f_2}{\partial u}(x,\beta_n)\right)\left(\beta_{n+1}-\beta_n\right)$$

are well and uniquely defined in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, and converge monotonically and quadratically in $L^p(\Omega)$ to the unique solution of (13) in $[\alpha_0,\beta_0]$.

Proof. Apply Theorem 3.2 for $P(t,u,v)=b(t,u,v)=\frac{\partial f_1}{\partial x}(t,u)-\frac{\partial f_2}{\partial x}(t,v)$. The differentiability of $f_1(x,\cdot)$ and $f_2(x,\cdot)$ and their convexity on $[\alpha_0(x),\beta_0(x)]$ imply that the following relations hold

$$f_1(x,v)-\frac{\partial f_1}{\partial u}(x,v)(v-u)\leq f_1(t,u)\leq f_1(x,v)-\frac{\partial f_1}{\partial u}(x,u)(v-u),$$

$$-f_2(x,v)+\frac{\partial f_2}{\partial u}(x,u)(v-u)\leq f_2(x,u)\leq -f_2(x,v)+\frac{\partial f_2}{\partial u}(x,v)(v-u),$$

for all $\alpha_0(x)\leq u\leq v\leq\beta_0(x)$. By summing up these inequalities, we obtain relation (16) and (18). Relation (17) is also valid, since the derivative of a convex function is monotone increasing. Now it is clear that the hypotheses (i)-(iv) of Theorem 3.2 are fullfiled. It remains to prove (v). This is valid, indeed as follows by the next inequalities. We use that f_1 and f_2 are convex and have Lipschitz derivatives on $[\alpha_0(x),\beta_0(x)]$.

$$\begin{aligned} b(x,u,\alpha) - P(x,\alpha,\beta) &= \frac{\partial f_1}{\partial u}(x,u) - \frac{\partial f_1}{\partial u}(x,\alpha) - \frac{\partial f_2}{\partial u}(x,\alpha) + \frac{\partial f_2}{\partial u}(x,\beta) \\ &\leq c_1(u-\alpha) + c_2(\beta-\alpha), \end{aligned}$$

for all $\alpha_0(x) \leq \alpha \leq u \leq \beta \leq \beta_0(t)$. This completes the proof. □

The second consequence of our main result use for P an expression in terms of divided differences and it can be used when f is not differentiable.

For a function $g : [c,d] \to \mathbb{R}$ and two given points $u, v \in [c,d]$, $u \neq v$, we let the divided difference of g on points u, v be defined by

$$[g; u, v] = \frac{g(u) - g(v)}{u - v}.$$

Recall if the function g is convex, then (by Jensen's inequality),

$$[g; u, v] \leq [g; u, w] \leq [g; v, w] \tag{20}$$

whenever $c \leq u \leq v \leq w \leq d$.

Theorem 3.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $\alpha_0, \beta_0 \in C(\Omega)$ be a lower and, respectively, an upper solution of (13), such that $\alpha_0 \leq \beta_0$ a.e. in Ω and $f(\cdot, \alpha_0(\cdot)), f(\cdot, \beta_0(\cdot)) \in L^p(\Omega)$. Let $\alpha_{-1}, \beta_{-1} \in C(\Omega)$ such that $\alpha_{-1}(x) < \alpha_0(x)$ and $\beta_0(x) < \beta_{-1}(x)$ for each $x \in \Omega$.*
Assume that $f = f_1 - f_2$ where $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory, $f_1(t, \cdot)$ and $f_2(t, \cdot)$ are convex on $[\alpha_{-1}(x), \beta_0(x)]$ and respectively on $[\alpha_0(x), \beta_{-1}(x)]$ for a.a. $x \in \Omega$. In addition, assume that

$$\begin{aligned} -M &\leq [f_1(x,\cdot); \alpha_{-1}(x), u] - [f_2(x,\cdot); v, \beta_{-1}(x)] \leq 0, \\ -M &\leq [f_1(x,\cdot); v, \beta_{-1}(x)] - [f_2(x,\cdot); \alpha_{-1}(x), u] \leq 0, \end{aligned}$$

for all $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$ and for a.a. $x \in \Omega$.
Then the sequences (α_n) and (β_n) given by the iterative schemes

$$\begin{aligned} -B\alpha_{n+1} &= f(x, \alpha_n) + ([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n])(\alpha_{n+1} - \alpha_n), \\ -B\beta_{n+1} &= f(x, \beta_n) + ([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n])(\beta_{n+1} - \beta_n) \end{aligned}$$

are well and uniquely defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and converge monotonically in $L^p(\Omega)$ to the unique solution of (13) in $[\alpha_0, \beta_0]$.

Proof. Apply Theorem 3.2 for

$$\begin{aligned} P(x,u,v) &= [f_1(x,\cdot); \alpha_{-1}(x), u] - [f_2(x,\cdot); v, \beta_{-1}(x)], \\ b(x,v,u) &= [f_1(x,\cdot); v, \beta_{-1}(x)] - [f_2(x,\cdot); \alpha_{-1}(x), u]. \end{aligned}$$

Using inequlities (20) we have

$$\begin{gathered} [f_1(x,\cdot); \alpha_{-1}(x), u] \leq [f_1(x,\cdot); u, v], \ [f_2(x,\cdot); v, \beta_{-1}(x)] \geq [f_2(x,\cdot); u, v], \\ [f_1(x,\cdot); \alpha_{-1}(x), u] \geq [f_1(x,\cdot); \alpha_{-1}(x), \alpha], \\ [f_2(x,\cdot); v, \beta_{-1}(x)] \leq [f_2(x,\cdot); \beta, \beta_{-1}(x)], \end{gathered}$$

whenever $\alpha_{-1}(x) < \alpha_0(x) \leq \alpha \leq u \leq v \leq \beta \leq \beta_0(x) < \beta_{-1}(x)$. Whence, by summing up the first two inequalities and the last two ones, we obtain (16) and (17), respectively.
Using again (20) we obtain

$$[f_1(x,\cdot); u, v] \leq [f_1(x,\cdot); v, \beta_{-1}], \ [f_2(x,\cdot); \alpha_{-1}(x), u] \leq [f_2(x,\cdot); u, v]$$

whenever $\alpha_0(t) \leq u \leq v \leq \beta_0(t)$. Whence, by summing up we get (18). □

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