EXTENDED COMPLEMENTARY DOMAIN OF THE FURUTA INEQUALITY

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Received March 3, 2004; revised April 5, 2004

Abstract. We discuss monotone properties of Furuta type inequalities: Let $A \geq B \geq 0$ and $0 \leq t \leq p$. Then for each $t \in [0, 1]$, $A^t \frac{1}{p-t} B^p$ is increasing for $p \geq 1$, and if $A^p \geq B^p$ for some fixed $p > 0$, then $A^t \frac{1}{p-t} B^p$ is increasing for $t \in [0, 1]$. Moreover, if $A^p \geq B^p$ for some fixed $p > 0$, then the following inequalities hold:

$$A^t \frac{1}{p-t} B^p \leq B^\delta \leq A^t \frac{1}{p-t} B^p,$$

for $0 \leq \delta \leq p < p_0 < 1$.

Consequently, it improves a recent result of Yang and Zuo.

1. Introduction. Recently, C.Yang and H.Zuo [25] have shown monotone properties of Furuta type inequalities on complementary domain. They point out a fresh aspect of operator means, however their results are depending on [18] too much and the proofs are confused a little. The $\alpha$-power mean of $A$ and $B$ introduced by Kubo-Ando [24] is given by

$$A^\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

for $0 \leq \alpha \leq 1$.

We can regard (1) as a path connecting $A (= A^0 B)$ and $B (= A^1 B)$, (cf.[17],[21]). We use the notation $\tilde{z}_r$ to distinguish from $\alpha$-power mean $\tilde{z}_\alpha$ ($\alpha \in [0, 1]$) as follows:

$$A \tilde{z}_r B = A^{\hat{r}} (A^{-\hat{r}} B A^{-\hat{r}})^r A^{\hat{r}}$$

for $r \notin [0, 1]$

Throughout this note, $A$ and $B$ are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. positive invertible) operator.

The main theorem of [25] is the following:

**Theorem A.** Let $A \geq B \geq 0$ and fix $0 < p_0 < 1$. Then $A^t \frac{1}{p-t} B^p$ is an increasing function for $t \in [0, p_0]$ and $p \in [p_0, 1]$.

This result shows the monotonicity of the paths $A^t \frac{1}{p-t} B^p$ from $A^t$ to $B^p$ at the fixed point $p_0$.

The Furuta inequality [13] (cf.[14]) can be written by the form of $\alpha$-power mean as follows ([2],[16]).
**Theorem 1.** Let \( \alpha \neq 0 \). In this context, Theorem A gives some answers to this. The following is valid:

\[
A^{-t} \frac{\alpha t + 1}{\alpha t + p} B^p \leq A \quad \text{and} \quad B \leq B^{-t} \frac{\alpha t + 1}{\alpha t + p} A^p
\]

holds for \( t \geq 0 \) and \( 1 \leq p \).

From this formulation, \( A^{-t} \frac{\alpha t}{\alpha t + p} B^p \) is a path from \( A^{-t} \) to \( B^p \) and (F) is understood as the comparison between \( A \) and the operator value \( A^{-t} \frac{\alpha t + 1}{\alpha t + p} B^p \) at the internally dividing point 1 of \([-t, p] \) with the ratio \( \frac{\alpha t + 1}{\alpha t + p} \).

The Furuta inequality (F) is an extension of the Löwner-Heinz inequality:

**(LH)**

\[
\text{If } A \geq B \geq 0, \text{ then } A^\alpha \geq B^\alpha \text{ for } 0 \leq \alpha \leq 1.
\]

In [16], we had rearranged (F) more precisely in one line by the form of \( \alpha \)-power mean (cf. [3]).

**Satellite theorem of Furuta inequality:** If \( A \geq B \geq 0, \) then

\[
A^{-t} \frac{\alpha t + 1}{\alpha t + p} B^p \leq B \leq A \leq B^{-t} \frac{\alpha t + 1}{\alpha t + p} A^p
\]

holds for all \( t \geq 0 \) and \( 1 \leq p \).

Whether similar relations to (F) hold for the case \( t \leq 0 \) is a subject easily drawn ([4],[5],[6],[18]). In this context, Theorem A gives some answers to this. The following is a further generalization of these.

**Theorem 1.** Let \( A \geq B \geq 0 \) and \( 0 \leq t \leq 1 \leq p \). Then

1. \[
A^t \frac{\alpha t + 1}{\alpha t + p} B^p \text{ is increasing for } p \geq 1.
\]

2. \[
\text{If } A^p \geq B^p, \text{ then } A^t \frac{\alpha t + 1}{\alpha t + p} B^p \text{ is increasing for } t \in [0, 1].
\]

**Proof.** (1) For \( 1 \leq p_1 \leq p_2, \)

\[
A^t \frac{\alpha t + 1}{\alpha t + p_1} B^{p_1} = A^t \frac{\alpha t + 1}{\alpha t + p_2} \left( A^t \frac{\alpha t + 1}{\alpha t + p_2} B^{p_2} \right) \geq A^t \frac{\alpha t + 1}{\alpha t + p_2} \left( B^t \frac{\alpha t + 1}{\alpha t + p_2} B^{p_2} \right) = A^t \frac{\alpha t + 1}{\alpha t + p_2} B^{p_2}.
\]

(2) For \( 0 \leq t_1 \leq t_2 \leq 1 \leq p \) with \( t_2 - t_1 \leq p - t_2, \)

\[
A^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} B^p = B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^{t_2} = B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} \left( B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^{t_2} \right)
\]

\[
= B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} \left( A^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} B^p \right) = B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^{t_2} \left( A^{-t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} B^{-p} \right) A^{t_2}
\]

\[
\geq B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^{t_2} \left( A^{-t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^p \right) A^{t_2} = B^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} A^{t_2} = A^{t_2} \frac{\alpha t_2 + 1}{\alpha t_2 + p} B^p.
\]

As an application, we have the following result which is an extension of Theorem A. As a matter of fact, the range of \( p \) is extended:

**Corollary** Let \( A \geq B \geq 0 \) and fix \( 0 < p_0 < 1 \). Then \( A^t \frac{\alpha t + 1}{\alpha t + p_0} B^p \) is an increasing function for \( t \in [0, p_0] \) and \( p \geq p_0 \).
**Proof.** Since \( A \geq B \geq 0, A_1 = A^{p_0} \geq B_1 = B^{p_0} \) for fixed \( p_0 \in (0,1) \) by (LH). By Theorem 1, \( A_1^t \#_{\frac{1}{p-1}} B_1^p \) is increasing for \( p \geq 1 \) and \( t \in [0,1] \). Putting \( t_1 = p_0 t, \; p_1 = p_0 p \), we have \( A_1^t \#_{\frac{1}{p-1}} B_1^p = A^{t_1} \#_{\frac{p_p - 1}{p_1 - 1}} B^{p_1} \) is increasing for \( p_1 \geq p_0 \) and \( t_1 \in [0,p_0] \).

### 2. Extended complementary domain

As a generalized form of (F), Furuta had shown the following grand Furuta inequality \([15]\), which is a parameteric one interpolating (F) and Ando-Hiai inequality equivalent to their majorization theorem \([1]\). We cite it here by the satellite form with the \( \alpha \)-power mean \((8),(20)\).

If \( A \geq B \geq 0 \), then for \( r \geq 0 \) and \( 0 \leq t \leq 1 \leq p \leq \beta \),

\[
\begin{align*}
\text{GF} & A^{-r} \#_{\frac{1}{p-1}} (A^t \#_{\frac{1}{p-1}} B^p) \leq (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \leq A \leq (B^t \#_{\frac{1}{p-1}} A^p)^{\frac{1}{p}} \leq B^{-r} \#_{\frac{1}{p-1}} (B^t \#_{\frac{1}{p-1}} A^p).
\end{align*}
\]

The following has shown in \([10]\) which is the key point for our proof of the grand Furuta inequality, and (2) has shown in \([8]\).

**Theorem B.** Let \( A \geq B \geq 0 \) and \( 0 \leq t \leq 1 \leq p \leq \beta \). Then the following (1) and (2) holds.

\[
\begin{align*}
(1) & \quad (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \leq B \leq A \\
(2) & \quad (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \leq B^p
\end{align*}
\]

Additionally, both \( (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \) and \( (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \) are decreasing for \( \beta(\geq p) \).

As complements of the Furuta inequality, we had investigated the part \( A^t \#_{\frac{1}{p-1}} B^p \) of (GF) and had the following results in \([18]\)(cf.\[4],[5],[6],[19]\)).

**Theorem C.** If \( A \geq B \geq 0 \), then

\[
\begin{align*}
(1) & \quad A^t \#_{\frac{1}{p-1}} B^p \leq B^{2p} \leq A^{2p} \text{ for } 0 \leq t < p \leq \frac{1}{2}, \\
(2) & \quad A^t \#_{\frac{1}{p-1}} B^p \leq B \leq A \text{ for } 0 \leq t < p \leq 1 \text{ and } p \geq \frac{1}{2}.
\end{align*}
\]

We here remark that \( A^t \#_{\alpha} B^p \) is a path connecting \( A^t \) with \( B^p \) and \( A^t \#_{\beta} B^p \) for \( \beta \geq p \) is regarded as the value of the path at the external point \( \beta \) of \([t,p]\) with ratio \( \frac{\beta - t}{p-1} \). From the viewpoint of this, Theorem C says that \( A^t \#_{\beta} B^p \) is comparable with \( A^\beta \) and \( B^\beta \), at the points \( \beta = 1 \) and \( \beta = 2p \), respectively.

C.Yang and H.Zuo \([25]\) have tried to see (GF) for the case \( r \leq 0 \). So we also follow it from the same viewpoint as theirs. The next is an application of Theorem B.

**Theorem 2.** Let \( A \geq B \geq 0 \) and \( 0 \leq t \leq 1 \leq p \). Then for each \( 1 \geq r \geq 0 \),

\[
A^r \#_{\frac{1}{p-1}} (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \text{ is decreasing for } \beta \geq p \text{ and}
\]

\[
A^r \#_{\frac{1}{p-1}} (A^t \#_{\frac{1}{p-1}} B^p)^{\frac{1}{p}} \leq A^r \#_{\frac{1}{p-1}} (A^t \#_{\frac{1}{p-1}} B^p)
\]
holds for $\beta \geq p$. If $A^p \geq B^p$, then $A^t \geq_{\frac{t}{t-r}} (A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r}$ is increasing for $r \in [0, 1]$.

**Proof.** Theorem B assures $B = (A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r} \leq A$ and $(A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r}$ is decreasing for $\beta \geq p$. Hence so is $A^t \geq_{\frac{t}{t-r}} (A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r}$. The desired inequality holds from

$$A^t \geq_{\frac{t}{t-r}} B^\beta_1 = A^t \geq_{\frac{t}{t-r}} (A^t \geq_{\frac{t}{t-r}} B^\beta_1) \geq A^t \geq_{\frac{t}{t-r}} (B^\beta_1 \geq_{\frac{t}{t-r}} B^\beta_1) = A^t \geq_{\frac{t}{t-r}} B^\beta_1.$$

Next, if $A^p \geq B^p$, then $A^p \geq B^p \geq B^\beta_1$ by Theorem B (2), so that $A^t \geq_{\frac{t}{t-r}} B_1^p = A^t \geq_{\frac{t}{t-r}} (A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r}$ is increasing for $r \in [0, 1]$ by Theorem 1.

Now, our discussions of the above are done under the assumptions of $A^p \geq B^p$ or $A \geq B$ for $p \geq 1 > t > 0$, so we next assume $A^1 \geq B^t$ as in below. For this, we need the following key fact, which was shown in the proof of Theorem B (9],[22],[23]).

**Lemma.** Let $A$, $B > 0$ and $0 \leq t < p \leq \beta \leq 2p - t$. If $A^t \geq B^t$, then

$$A^t \geq_{\frac{t}{t-r}} B^p \leq B^\beta_1.$$

**Proof.** We give a proof for convenience. Since $A \geq_{r-t} B = A(A^{-1} \geq_{r-t} B^{-1})A$, we have

$$A^t \geq_{\frac{t}{t-r}} B^p = B^p \geq_{\frac{t}{t-r}} A^t = B^p(B^{-p} \geq_{\frac{t}{t-r}} A^{-t})B^p \leq B^p(B^{-p} \geq_{\frac{t}{t-r}} B^{-t})B^p = B^\beta_1.$$

In [19], we had compared the order among the paths represented by $\alpha$-mean, which is our view point of the Furu type inequalities. The following is inspired by Theorem A.

**Theorem 3.** Let $A$, $B > 0$ and $0 \leq t < p \leq \beta$. If $A^t \geq B^t$, then the following inequalities hold.

(1) For $t \leq \delta \leq p$, $(A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r} \leq B^\delta \leq A^t \geq_{\frac{t}{t-r}} B^p$.

(2) For $p \leq \delta \leq \beta$, $(A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r} \leq A^t \geq_{\frac{t}{t-r}} B^p$.

**Proof.** (1) Since $A^t \geq_{\frac{t}{t-r}} B^p \geq B^\delta$, the right side of (1) holds. To show the first inequality, we have only to use Lemma inductively. For $p \leq \beta_1 \leq 2p - t$, Lemma says $A^t \geq_{\frac{t}{t-r}} B^p \leq B^{\beta_1}$ and we have $(A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r} \leq B^{\beta_1}$ by (LH). That is, (1) is proved for the case $p \leq \delta \leq 2p - t$.

If $\beta > 2p - t$, then we take $\beta_1 < \beta_2 < \ldots < \beta_{k-1} < \beta_k = \beta$ such that $\beta_{i+1} < 2\beta_i - t$ for $i = 1, \ldots, k - 1$ and put $B_i = (A^t \geq_{\frac{t}{t-r}} B^p)^\frac{t}{t-r}$ for $i = 1, \ldots, k$. Thus it suffices to show that $B_k^\delta \leq B^\delta$. For this, we prove $B_k^\delta \leq B_{k-1}^\delta \leq \ldots \leq B_1^\delta \leq B^\delta$ by induction. In the preceding discussion, it is shown that $B_1^\delta \leq B^\delta$. So we assume that $B_i^\delta \leq B^\delta$ for some $i$. Since $B_i^\delta \leq B_1^\delta \leq A^t$ and $\beta_i \leq \beta_{i+1} \leq 2\beta_i - t$, it follows from Lemma that $A^t \geq_{\frac{t}{t-r}} B_i^\beta \leq B_i^\beta_{i+1}$. Therefore we have

$$B_i^{\beta_{i+1}} = A^t \geq_{\frac{t}{t-r}} B_i^\beta \leq A^t \geq_{\frac{t}{t-r}} (A^t \geq_{\frac{t}{t-r}} B^p) = A^t \geq_{\frac{t}{t-r}} B_i^\beta \leq B_i^{\beta_{i+1}}$$
and so $B^i_{t+1} \leq B^i_t$ by (LH). By the induction, we have $B^i_0 \leq B^i_{k-1} \leq \ldots \leq B^i_i \leq B^i$. Proof of (2) is similar to (1), but we need two steps to catch $\beta$. First of all, we show that if we put $D = (A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\delta}{k}}$, then $D^i \leq A^i$. If $p \leq \delta \leq 2p - t$, then

$$D^i = A^t \otimes_{_{_{t=0}^{t=k}}} B^p \leq B^i$$

by Lemma and so $D^i \leq B^i \leq A^i$ by (LH) since $0 \leq t < p \leq \delta$. In the case $2p - t < \delta \leq \beta$, we devide $[p, \delta]$ into $p = \delta_0 < \delta_1 < \ldots < \delta_l = \delta$ such that $\delta_{i+1} \leq 2\delta_i - t (i = 0, \ldots, l - 1)$ and put $D_i = (A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\delta}{k}} (i = 0, \ldots, l)$. Then, as in the proof of (1), we have $D^{i+1}_{i+1} \leq D^{i+1}_i$ and so $D^i = D^i_i \leq D^i_{i-1} \leq \ldots \leq D^i_0 = B^i \leq A^i$.

Since we could prove $D^i \leq A^i$, we next apply it to Lemma. First, if $\delta \leq \beta \leq 2\delta - t$, then

$$A^t \otimes_{_{_{t=0}^{t=k}}} B^p = A^t \otimes_{_{_{t=0}^{t=k}}} (A^t \otimes_{_{_{t=0}^{t=k}}} B^p) = A^t \otimes_{_{_{t=0}^{t=k}}} D^i = D^t \leq D^\beta = (A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\beta}{k}}.$$

Hence we have the conclusion $(A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\delta}{k}} \leq A^t \otimes_{_{_{t=0}^{t=k}}} B^p (\leq B^t)$ by (LH). On the other hand, if $2\delta - t < \beta$, we use the same method as in (1) again: We take $\delta = \beta_0 < \beta_1 < \ldots < \beta_{k-1} < \beta_k = \beta$ such that $\beta_{i+1} \leq 2\beta_i - t$ for $i = 1, \ldots, k - 1$ and put $B_i = (A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\beta}{k}} (i = 0, \ldots, k)$. Thus we obtain $B^{i+1}_{i+1} \leq B^{i+1}_i$ and so $B^i_{i+1} \leq B^i_i, (i = 1, \ldots, k - 1)$. That is, we have the conclusion;

$$(A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\beta}{k}} \leq \ldots \leq (A^t \otimes_{_{_{t=0}^{t=k}}} B^p)^{\frac{\beta}{k}} \leq A^t \otimes_{_{_{t=0}^{t=k}}} B^p.$$

3. Chaotic order. In Theorem 3, we consider operator inequalities of Furuta type under the assumption $A^t \geq B^t$ for some $t$ with $0 < t \leq p$. We note that the chaotic order $\log A \geq \log B$, simply denoted by $A \gg B$, is regarded as the case $t = 0$ in the sense that $\log X = \lim_{t \to +0} X^t$ (cf.[7]). In this section, we discuss them under the chaotic order $A \gg B$. Some useful characterizations of the chaotic order have given in [23], (cf., [22]) as follows:

**Theorem D.** If $A \gg B$, then the following (1) and (2) hold.

1. $A^{-t} \otimes_{_{_{t=0}^{t=k}}} B^p \leq B^\delta$ and $A^t \geq B^{-t} \otimes_{_{_{t=0}^{t=k}}} A^p$ for $t \geq 0$ and $0 \leq \delta \leq p$.

2. $A^{-t} \otimes_{_{_{t=0}^{t=k}}} B^p \leq A^\alpha$ and $B^\alpha \leq B^{-t} \otimes_{_{_{t=0}^{t=k}}} A^p$ for $-t \leq \alpha \leq 0$ and $0 \leq p$.

Concerning to Theorem D, Fujii and Kim [12] point out the next characterization of chaotic order(cf.[23]).

**Theorem E.** Let $A, B > 0$. Then the following are equivalent.

1. $A \gg B$

2. $A^{-t} \otimes_{_{_{t=0}^{t=k}}} B^p \geq B^\beta$ for $0 \leq p \leq \beta \leq 2p, t \geq 0$

3. $A^\beta \geq B^{-t} \otimes_{_{_{t=0}^{t=k}}} A^p$ for $0 \leq p \leq \beta \leq 2p, t \geq 0$
We give additions to Theorem E in parallel with Theorem D. Under the assumptions $-2t \leq \alpha \leq -t$, $p \geq 0$, the following (4) and (5) are also equivalent to $A \gg B$.

\begin{align*}
(4) & \quad A^{-t} \frac{B^p}{\beta_{-\gamma} t} \geq A^\alpha \\
(5) & \quad B^\alpha \geq B^{-t} \frac{A^p}{\beta_{-\gamma} t}
\end{align*}

For convenience, we prove them. Since $-t \leq \alpha + t \leq 0$,

\begin{align*}
A^{-t} \frac{B^p}{\beta_{-\gamma} t} &= A^{-t}(A^t \frac{B^p}{\beta_{-\gamma} t} B^{-\gamma} A^{-t} = A^{-t}(A^t \frac{B^p}{\beta_{-\gamma} t} (A^t \frac{B^{-\gamma} A^{-t}}{p}) A^{-t} \\
&= A^{-t}(A^t \frac{B^p}{\beta_{-\gamma} t} A^{-t} (\frac{B^p}{\beta_{-\gamma} t} B^{-\gamma} A^{-t}) A^{-t} = A^\alpha.
\end{align*}

In Theorem 3, we can move $\beta(\geq p)$ freely, but now we must put some restriction on $\beta$ under the assumption of chaotic order $A^0 \geq B^0$.

**Theorem 4.** Let $A$, $B > 0$ and $0 \leq t \leq p \leq \beta \leq 2p$. If $A \gg B$, then the following (1) and (2) hold.

(1) For $0 \leq \delta \leq p$,

\begin{align*}
A^{-t} \frac{B^p}{\beta_{-\gamma} t} &\leq (A^{-t} \frac{B^p}{\beta_{-\gamma} t} B^{-\gamma} A^{-t} \leq A^\delta \leq B^{-t} \frac{A^p}{\beta_{-\gamma} t} A^{-t} \\
(2) &\quad \text{For } -t \leq \gamma \leq 0
\end{align*}

\begin{align*}
A^{-t} \frac{B^p}{\beta_{-\gamma} t} &\leq (A^{-t} \frac{B^p}{\beta_{-\gamma} t} B^{-\gamma} A^{-t} \leq B^\gamma \leq B^{-t} \frac{A^p}{\beta_{-\gamma} t} A^{-t}
\end{align*}

**Proof.** (1) $A^{-t} \frac{B^p}{\beta_{-\gamma} t} \leq B^\delta$ and $A^\delta \leq B^{-t} \frac{A^p}{\beta_{-\gamma} t} A^{-t}$ are obtained by Theorem D and the rest ones follow from Theorem E (2) and (LH) because $0 \leq \frac{\delta}{t} \leq 1$. (2) also follows from Theorem D (2), Theorem E (3) and (LH) since $-1 \leq \frac{\gamma}{t} \leq 0$.

**Remark.** Finally we mention that $A \geq B > 0$ does not imply $B^{-t} \frac{B^p}{\beta_{-\gamma} t} A^p \leq A^\delta$ for $0 \leq t \leq p \leq \beta$, in general. For example, we choose $t = 1$, $p = 2$, $\beta = 6$, clearly $\beta \leq 2p$. Then the inequality

\begin{align*}
B^{-1} \frac{B^2}{\beta_{-\gamma} t} A^2 &\leq A^2 B^2 A^2
\end{align*}

is assured by

\begin{align*}
B^{-1} \frac{B^2}{\beta_{-\gamma} t} A^2 &= A^2 \frac{B^{-1}}{\beta_{-\gamma} t} B^2 = A^2 (A^{-2} \frac{B^2}{\beta_{-\gamma} t} B) A^2 = A^2 (B \frac{A^{-1}}{\beta_{-\gamma} t} A^{-2}) A^2 \\
&= A^2 B (A^{-1} \frac{B}{\beta_{-\gamma} t} A^2) B A^2 \geq A^2 B (A^{-1} \frac{B}{\beta_{-\gamma} t} A^2) B A^2 = A^2 B^2 A^2.
\end{align*}

Now suppose that it holds for every pair $A \geq B > 0$, i.e., $B^{-1} \frac{B^2}{\beta_{-\gamma} t} A^2 \leq A^6$. Then we have $A^6 \geq A^2 B^2 A^2$, which says that $A \geq B > 0$ implies $A^2 \geq B^2$, a contradiction.

**Acknowledgement.** The author would like to express his gratitude to the referee for his kind suggestions.
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