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MULTISTAGE THREE-PERSON GAME WITH ARBITRATION

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Abstract. By introducing a specified definition of the equilibrium value of three-person two-choice games, a multistage three-person game with arbitration is formulated and solved. Random offers $\{X_i\}_{i=1}^n$ are presented one-by-one sequentially, and as each offer X_i comes up, each player chooses either to accept (A) or to reject (R) it, with the aim of receiving the most favorable partition of the offer they can get. When the players' choices are different, arbitration comes in and forces the "odd-man" (the "even-men") to receive $pX_i(\bar{p}X_i/2$ each), where $0 \le p = 1 - \bar{p} \le 1$ and the game terminates. It is shown that, in the equilibrium, each player chooses R for small offers (A for large offers), and randomizes between R and A for other offers, if arbitration favors the odd-man side, *i.e.* $p \in \left(\frac{1}{3}, 1\right]$ (the even-men side, *i.e.* $p \in \left[0, \frac{1}{3}\right)$.)

1 Problem. Let $X_i, i = 1, 2, \cdots, n$, be *i.i.d.* random variables each with uniform distribution on $[0, 1]$. As each X_i comes up, each player I, II and III must choose simultaneously and independently of other players' choices, either to accept (A) or to reject (R) it. If all players choose A they get $\frac{1}{3}X_i$ each, and the game terminates. If all players choose R, then X_i is rejected and the next X_{i+1} is presented and the game continues. If players' choices are different, arbitration comes in and forces the "odd-man" (the "even-men") to get $pX_i(\bar{p}X_i/2$ each), where $0 \le p = 1 - \bar{p} \le 1$, and the game terminates. Arbitration is fair if $p = \frac{1}{3}$, and favors the odd-man (even-men) side, if $p > (<)\frac{1}{3}$. If all of the first $n-1$ random variables are rejected, all players must accept the n-th. Each player aims to maximize the expected reward he can get, and the problem is to find a reasonable solution to this three-person n-stage game.

At each stage, each player must think about : (1) He wants to become the odd-man if $p > \frac{1}{3}$, and an even-man if $p < \frac{1}{3}$, especially when he faces large X_i, and (2) Since each X_i is a random variable, he can expect a larger one may come up in the future.

Let (v_n, v_n, v_n) be the eq.values for the game (*c.f.*, the game is symmetric for the players). The Optimality Equation is

$$(v_n, v_n, v_n) = E[\text{eq.val.}\mathbf{M}_n(X)] \qquad (n \ge 1, v_1 = \frac{1}{6}) \tag{1.1}$$

where the payoff matrix $\mathbf{M}_n(X)$ is such that

(1.2)

R by I: $\mathbf{M}_{nR}(x) =$

	R by III	A by III
R by II	$v_{n-1}, v_{n-1}, v_{n-1}$	$*, \; *, \; px$
A by II	$*, \; px, \; *$	$px, \; *, \; *$

A by I: $\mathbf{M}_{nA}(x) =$

	R by III	A by III
R by II	$px, \; *, \; *$	$*, \; px, \; *$
A by II	$*, \; *, \; px$	$x/3, \; x/3, \; x/3$

($*$ stands for $(\bar{p}/2)x$)

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If $p = \frac{1}{3}$, players will evidently coordinate to choose R-R-R repeatedly and switch to A-A-A as soon as $X_i \geq \mu_{n-i}$ appears, where $\{\mu_n\}$ is the Moser's sequence

$$\mu_n = E(X \vee \mu_{n-1}) = \frac{1}{2}(1 + \mu_{n-1}^2) \qquad (n \geq 1, \mu_1 = \frac{1}{2}).$$

So, the CES (common eq.value) is $\frac{1}{3}\mu_n$.

Therefore we are interested in solving the n-stage game for $p \neq \frac{1}{3}$.

The game(1.1)-(1.2) is solved for $p \in (\frac{1}{3}, 1]$ and $[0, \frac{1}{3})$ in Sections 2 and 3, respectively. We need a specified definition of the eq.val.in the optimality equation, as in Assumption A stated in Section 2, since the equilibrium is often undetermined in Nash theory of competitive games.

Two-person best-choice games where arbitration comes in are investigated in Ref.[1, 3, 4]. The present paper is a direct extention to three-person game from the two-person game version Ref.[4]. The game (1.1)-(1.2) reduces to Odd-Man-Wins if $p = 1$, and Odd-Man-Out if $p = 0$, both of which are discussed in Ref.[5]. One of the fundamental and elaborate literature in game theory (including cooperative theory of games) is Petrosjan and Zenkevich [2]. There are a few mathematical literature which discuss three-person competitive games, and two of which are Vorobjev [6] and Sakaguchi [5]. The present paper owes much on Vorobjev's work.

2 Solution to the Game where $\frac{1}{3} < p \leq 1$.

We can rewrite (1.2) as

(2.1) $$\mathbf{M}_{nR}(x) = (\bar{p}/2)x\mathbf{E} + (p - \bar{p}/2)x\mathbf{M}_R(c)|_{c=(x^{-1}v_{n-1}-\bar{p}/2)/(p-\bar{p}/2)},$$

(2.2) $$\mathbf{M}_{nA}(x) = (\bar{p}/2)x\mathbf{E} + (p - \bar{p}/2)x\mathbf{M}_A$$

where

$\mathbf{E} =$

1,	1,	1	1,	1,	1
1,	1,	1	1,	1,	1

(2.3) $\mathbf{M}_R(c) =$

c,	c,	c	0,	0,	1
0,	1,	0	1,	0,	0

and

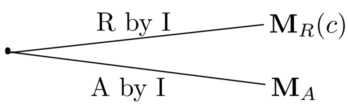
(2.4) $\mathbf{M}_A =$

1,	0,	0	0,	1,	0
0,	0,	1	1/3,	1/3,	1/3

Note that $\mathbf{M}_{nA}(x)$ doesn't involve n. The game continutes to the next stage if and only if R-R-R is chosen. As soon as some one among the players chooses A, the game terminates.

Let $V(c)$ be CEV (common eq.value) of the one-stage game

(2.5)



Then CEV of the n-stage game (2.1)-(2.2) is

(2.6) $$v_n = (\bar{p}/2)Ex + (p - \bar{p}/2)E\left[xV(c)|_{c=(x^{-1}v_{n-1}-\bar{p}/2)/(p-\bar{p}/2)}\right].$$

As is well known in the Nash theory of competitive games the equilibrium is often undetermined, even in three-person two-choice games, which we investigate in the present paper. We prepare the following assumption that is held throughout this paper.

Assumption A *If the equilibrium consists of some corner and or edge and a unique inner* *point, the latter is adopted for the equilibrium. If equilibrium consists of a single point, either a corner or inner point, this is adopted for the equilibrium.*

Lemma 1.1 *The solution to the three-person game* (2.5) *is : If* $c < 1$, *the mixed-strategy triple* $(\alpha_0, \alpha_0, \alpha_0)$, *with* $\alpha_0 = \frac{\sqrt{1-c}}{\sqrt{1-c}+\sqrt{2/3}}$ *is in eq. If* $c \geq 1$, *the pure-strategy triple* R-R-R *is in eq. The* CEV *is*

$$V(c) = \begin{cases} (1-c/3)/(\sqrt{1-c}+\sqrt{2/3})^2, & \text{if } c < 1, \\ c, & \text{if } c \geq 1. \end{cases} \tag{2.7}$$

$V(c)$ is convex and increasing with values

$c =$	$-1/3$	0	$1/3$	$1/2$	1
$V(c) =$	$5(1-2\sqrt{2}/3) \approx 0.2859$	$3(5-2\sqrt{6}) \approx 0.3031$	$1/3$	$5(7-4\sqrt{3}) \approx 0.3590$	1

For the proof, see Sakaguchi [5 ; Theorem 1].

Now recalling that $c = (x^{-1}v_{n-1} - \bar{p}/2)/(p - \bar{p}/2)$ in (2.1), and rewriting v_{n-1} simply by v we find that

$$c = (2x^{-1}v - \bar{p})/(3p-1), \qquad 1 - c = 2(p - x^{-1}v)/(3p-1),$$

$$1 - c/3 = \frac{2}{3} \cdot \frac{4p-1-x^{-1}v}{3p-1}, \quad \alpha_0 = \frac{\sqrt{1-c}}{\sqrt{1-c}+\sqrt{2/3}} = \frac{\sqrt{px-v}}{\sqrt{px-v}+\sqrt{(p-1/3)x}},$$

and hence, by Lemma 1.1, we obtain

$$V(c)|_{c=(x^{-1}v-\bar{p}/2)/(p-\bar{p}/2)}$$

$$= \begin{cases} c = (2x^{-1}v - \bar{p})/(3p-1), & \text{if } x < p^{-1}v \\ (1-c/3) \Big/ \left(\sqrt{1-c}+\sqrt{2/3}\right)^2 & \\ \quad = \frac{1}{3}\{(4p-1)x - v\}\left\{\sqrt{px-v}+\sqrt{(p-1/3)x}\right\}^{-2}. & \text{if } x > p^{-1}v \end{cases}$$

Thus the CEV of the n-stage game is, by (2.6),

$$v_n = E\left[(\bar{p}/2)x + \frac{1}{2}(3p-1)xV(c)\,\Big|_{c=(2x^{-1}v-\bar{p})/(3p-1)}\right] \tag{2.8}$$

$$= \frac{\bar{p}}{4} + \frac{1}{2}(3p-1)E\left[\frac{2v-\bar{p}x}{3p-1}I(x < p^{-1}v) + \frac{x\{(4p-1)x-v\}}{3\left\{\sqrt{px-v}+\sqrt{(p-1/3)x}\right\}^2}I(x > p^{-1}v)\right]$$

$$= \frac{\bar{p}}{4} + \frac{5p-1}{4p^2}v^2 + \left(\frac{3p-1}{6}\right)\int_{p^{-1}v}^{1} \frac{x\{(4p-1)x-v\}}{\left\{\sqrt{px-v}+\sqrt{(p-1/3)x}\right\}^2}dx$$

if $p^{-1}v_{n-1} < 1$; and $v_n = v_{n-1}$, if $p^{-1}v_{n-1} > 1$.

Before we state Theorem 1, we give two more lemmas. Consider the function

$$T(v|p) \equiv \frac{\bar{p}}{4} + \left(\frac{5p-1}{4p^2}\right)v^2 + \left(\frac{3p-1}{6}\right)\int_{p^{-1}v}^{1} \frac{x\{(4p-1)x-v\}}{\left\{\sqrt{(p-1/3)x}+\sqrt{px-v}\right\}^2}dx. \tag{2.9}$$

for $0 \leq v \leq p$; and v, if $p \leq v \leq 1$.

Lemma 1.2 *$T(v|p)$ is positive and increasing in $v \in [\frac{1}{6}, p]$.*

Proof. Since the integral part in the r.h.s.of (2.9) is positive for $\forall p \in (\frac{1}{3}, 1]$, we have $T(v|p) > 0$, for $\forall 0 \leq v < p$. And

$$T'(v) = \left(\frac{5p-1}{2p^2}\right)v + \left(\frac{3p-1}{6}\right)\frac{\partial}{\partial v}\int_{p^{-1}v}^{1}\frac{x\{(4p-1)x-v\}}{\left\{\sqrt{(p-1/3)x}+\sqrt{px-v}\right\}^2}dx.$$

The derivative part in the r.h.s.is equal to

$$-3p^{-1}u + \int_u^1 \frac{x\left\{(3p-1)x - \sqrt{p(p-\frac{1}{3})x(x-u)}\right\}}{\sqrt{p(x-u)}(G(x,u))^3}dx$$

where we have set $u = p^{-1}v$ and $G(x,u) = \sqrt{(p-1/3)x}+\sqrt{p(x-u)}$ (Note that the integral in the r.h.s.doesn't diverge, since $\int_u^1 \frac{dx}{\sqrt{x-u}}$ converges to $2\sqrt{1-u}$). Therefore it follows that

$$T'(v) = u + \frac{3p-1}{6}\int_u^1 \frac{x\left\{(3p-1)x - \sqrt{p(p-1/3)x(x-u)}\right\}}{\sqrt{p(x-u)}(G(x,u))^3}dx, \tag{2.10}$$

where $u = p^{-1}v$, in the r.h.s.

Now, since $\frac{1}{3} < p \leq 1$, we have

$$(3p-1)x - \sqrt{p(p-1/3)x(x-u)} \geq (3p-1)x - \frac{1}{2}\left\{p(x-u) + (p-\frac{1}{3})x\right\} = \left(2p-\frac{5}{6}\right)x + \frac{1}{2}pu$$

and hence $T'(v) > 0$, if $\frac{5}{12} < p \leq 1$.

For $\frac{1}{3} < p < \frac{5}{12}$, we obtain from (2.10)

$$\begin{aligned}
T'(v) &> u - \left(\frac{3p-1}{6}\right)\int_u^1 x^{3/2}\sqrt{p-1/3}(G(x,u))^{-3}dx \\
&> u - \left(\frac{3p-1}{6}\right)\int_u^1 \left(p-\frac{1}{3}\right)^{-1}dx \qquad \left[cf.\ G(x,u) \leq (p-\frac{1}{3})^{-1/2}x^{-1/2}\right] \\
&= \frac{1}{2}(3u-1) \geq \frac{1}{2}\left(\frac{6}{5}-1\right) = \frac{1}{10} > 0 \qquad \left[cf.\ u = p^{-1}v \geq \frac{12}{5}\cdot\frac{1}{6}\right]
\end{aligned}$$

The lemma is proven. □

Note that, from (2.9) and (2.10),

$$T(p|p) = 1, \qquad T(0|p) = \frac{1}{4}\bar{p} + \frac{(3p-1)(4p-1)}{12} \bigg/ \left(\sqrt{p-\frac{1}{3}}+\sqrt{p}\right)^2 > 0, \tag{2.11}$$

$$T'(p|p) = 1, \tag{2.12}$$

and

$$T'(0|p) = \frac{(p-1/3)^{3/2}}{2\sqrt{p}}\frac{3\sqrt{p-1/3}-\sqrt{p}}{\sqrt{p-1/3}+\sqrt{p}}\left\{\begin{array}{c} < \\ > \end{array}\right\}0, \text{if } p \in \left\{\begin{array}{c} (1/3, 3/8) \\ (3/8, 1] \end{array}\right\}.$$

Lemma 1.3 $\Delta(p) \equiv T(\frac{1}{6}|p) - \frac{1}{6} \geq 0$, *for* $p \in (\frac{1}{3}, 1)$.

Proof. Eq.(2.8) gives

$$\Delta(p) = \frac{1-3p}{12} + \frac{5p-1}{144p^2} + \left(\frac{3p-1}{6}\right)\int_{(6p)^{-1}}^{1} \frac{x\left\{(4p-1)x-\frac{1}{6}\right\}}{\left\{\sqrt{(p-\frac{1}{3})x}+\sqrt{px-\frac{1}{6}}\right\}^2}dx.$$

The third term in the r.h.s.is

$$\begin{aligned} &\geq \left(\frac{3p-1}{12}\right)\int_{(6p)^{-1}}^{1} \frac{x\left\{(4p-1)x-\frac{1}{6}\right\}}{(2p-\frac{1}{3})x-\frac{1}{6}}dx \quad \left[cf.\ (\sqrt{A}+\sqrt{B})^2 \leq 2(A+B)\right] \\ &\geq \frac{(3p-1)(4p-1)}{4(6p-1)}\int_{(6p)^{-1}}^{1} x dx. \quad \left[cf.\ \frac{(4p-1)x-1/6}{(2p-1/3)x-1/6} \geq 1+\frac{2(3p-1)}{6p-1} = \frac{3(4p-1)}{6p-1}\right] \end{aligned}$$

Therefore

$$\begin{aligned} \Delta(p) &\geq \frac{1-3p}{12} + \frac{5p-1}{144p^2} + \frac{(3p-1)(4p-1)}{4(6p-1)}\cdot\frac{1}{2}\left(1-\frac{1}{36p^2}\right) \\ &= -\frac{3p-1}{24(6p-1)} + \frac{1}{144p^2}\cdot\frac{48p^2-15p+1}{2(6p-1)} \\ &= \frac{-18p^3+30p^2-(15/2)p+1/2}{144p^2(6p-1)}, \end{aligned}$$

in which the numerator is easily shown to be positive for $\forall p \in [1/3, 1]$. □

Theorem 1 *The solution to the three-person game* (1.1)-(1.2) *for* $\frac{1}{3} < p \leq 1$, *is as follows;*
The CES *in state* (n, x) *is*
Choose R, *if* $x < p^{-1}v_{n-1}$,
Employ the mixed strategy $\langle\bar{\alpha}_0(x), \alpha_0(x)\rangle$, *where*

$$\alpha_0(x) = \frac{\sqrt{px-v_{n-1}}}{\sqrt{px-v_{n-1}}+\sqrt{(p-1/3)x}}, \quad \textit{if } x > p^{-1}v_{n-1}.$$

The CEV v_n *satisfies the recursion*

$$v_n = T(v_{n-1}), \quad (n \geq 2, v_1 = 1/6)$$

where $T(v) = T(v|p)$ *is given by* (2.8). *And as* $n \to \infty, v_n \uparrow v_\infty(p) \equiv \sup\{v \in (0,p)|T(v'|p) > v', \forall v' \in (0,v)\}$, *for* $\forall p \in (\frac{1}{3}, 1]$.

Proof. By Lemma 1.2

$$v_{n-1} < v_n \implies v_n = T(v_{n-1}) < T(v_n) = v_{n+1}$$

and by Lemma 1.3,

$$v_2 = T(v_1) = T(1/6) \geq v_1.$$

Therefore induction gives $v_n \uparrow v_\infty(p)$.

By Lemma 1.2, together with (2.11) and (2.12), $v_\infty(p)$ is equal to the stated one. This completes the proof of the theorem. □

Remark 1 Some special case ; For $p = 1$ (*i.e.*, Odd-Man-Wins), Eq.(2.8)-(2.9) becomes $v_n = T(v_{n-1})$, where

$$T(v) = v^2 + \int_v^1 \frac{x(x-v/3)}{\left(\sqrt{x-v}+\sqrt{2x/3}\right)^2}dx.$$

See Sakaguchi [5 ; Theorem 2], in which $v_n \uparrow v_\infty(1) \approx 0.2057$ as $n \to \infty$ is proven.

For $p = \frac{1}{3}+0$, Eq.(2.9) gives $T(v|\frac{1}{3}+0) = \frac{1}{6}+\frac{3}{2}v^2$ and $v_n = \frac{1}{6}+\frac{3}{2}v_{n-1}^2 (n \geq 2, v_1 = \frac{1}{6})$, and it is easily shown that $v_n \uparrow \frac{1}{3}$ as $n \to \infty$.

3 Solution to the Game where $\mathbf{0 \le p < \frac{1}{3}}$. For $0 \le p < \frac{1}{3}$, we can rewrite (1.2) as

$$\mathbf{M}_{nR}(x) = px\mathbf{E} + (1-3p)x\mathbf{Q}_R(h)|_{h=(x^{-1}v_{n-1}-p)/(1-3p)}, \tag{3.1}$$

$$\mathbf{M}_{nA}(x) = px\mathbf{E} + (1-3p)x\mathbf{Q}_A, \tag{3.2}$$

where

$$\mathbf{E} = \begin{array}{|ccc|ccc|} \hline 1, & 1, & 1 & 1, & 1, & 1 \\ \hline 1, & 1, & 1 & 1, & 1, & 1 \\ \hline \end{array}$$

$$\mathbf{Q}_R(h) = \begin{array}{|ccc|ccc|} \hline \text{h}, & \text{h}, & \text{h} & 1/2, & 1/2, & 0 \\ \hline 1/2, & 0, & 1/2 & 0, & 1/2, & 1/2 \\ \hline \end{array} \tag{3.3}$$

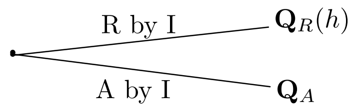
and

$$\mathbf{Q}_A = \begin{array}{|ccc|ccc|} \hline 0, & 1/2, & 1/2 & 1/2, & 0, & 1/2 \\ \hline 1/2, & 1/2, & 0 & 1/3, & 1/3, & 1/3 \\ \hline \end{array} \tag{3.4}$$

where h is a given constant. Note that $\mathbf{M}_{nA}(x)$ doesn't involve n by the same reason as in Section 2.

Let $W(h)$ be the CEV of the one-stage game

(3.5)



Then the CEV of the n-stage game (3.1)-(3.2) is

$$w_n = pEx + E\left[(1-3p)xW(h)|_{h=(x^{-1}w_{n-1}-p)/(1-3p)}\right] \tag{3.6}$$

We refer to a result in Sakaguchi [5 ; Theorem 3].

Lemma 2.1 *The solution to the three-person game* (3.5) *is as follows : For $h \le 0$, the pure-strategy triple* A-A-A *is in eq. For $h > 0$, the mixed-strategy triple $(\alpha_0, \alpha_0, \alpha_0)$, with $\alpha_0 = \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}}$ is in eq. The* CEV *is*

$$W(h) = \begin{cases} 1/3, & \text{if } h \le 0, \\ \frac{\sqrt{h/3}+h/3}{\left(\sqrt{h}+\sqrt{1/3}\right)^2}, & \text{if } h > 0. \end{cases} \tag{3.7}$$

The function is increasing and convex-concave for $0 < h < 3$, attains maximum at $h = 3$, and decreasing and concave-convex for $h > 3$. The two points of inflexion are $h = \frac{1}{3}(9 \pm 4\sqrt{5}) (\approx 0.018, 5.981)$. Computation gives

$h =$	0	1/3	1/2	1	2	3	12	∞
$W(h) =$	0	1/3	$3\sqrt{6}-7 \approx 0.3485$	$(\sqrt{3}-1)/2 \approx 0.3660$	0.3739	3/8	18/49	1/3

Hereafter we sometimes write w_{n-1} simply by w, omitting the subscript.
Since $h = (x^{-1}w - p)/(1-3p)$, we obtain, from (3.7),

$$W(h)|_{h=(x^{-1}w-p)/(1-3p)} = \begin{cases} \frac{w-px+\sqrt{3(1-3p)x(w-px)}}{\left\{\sqrt{3(w-px)}+\sqrt{(1-3p)x}\right\}^2}, & \text{if } x < p^{-1}w \\ 1/3, & \text{if } x > p^{-1}w, \end{cases}$$

and

$$\alpha_0(x) = \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}} = \frac{\sqrt{w-px}}{\sqrt{w-px}+\sqrt{(1/3-p)x}} \quad (\text{if } x > p^{-1}w)$$

Then the CEV of the n-stage game is, by (3.6),

$$
\begin{aligned}
(3.8)\qquad w_n &= E\left[px + (1-3p)xW(h)|_{h=(x^{-1}w-p)/(1-3p)}\right] \\
&= \frac{p}{2} + (1-3p)E\left[J(x,w)I(x < p^{-1}w) + \frac{x}{3}I(x > p^{-1}w)\right] \\
&= \begin{cases} \dfrac{1}{6} - \dfrac{1-3p}{6p^2}w^2 + (1-3p)\displaystyle\int_0^{p^{-1}w} J(x,w)dx, & \text{if } 0 \le w \le p, \\ \dfrac{p}{2} + (1-3p)\displaystyle\int_0^1 J(x,w)dx, & \text{if } p \le w \le 1, \end{cases}
\end{aligned}
$$

where $J(x,w) = \frac{x\left\{w-px+\sqrt{3(1-3p)x(w-px)}\right\}}{\left\{\sqrt{3(w-px)}+\sqrt{(1-3p)x}\right\}^2}$.

So we arrive at

Theorem 2 *The solution to the three-person game* (1.1)-(1.2), *for* $0 \le p < \frac{1}{3}$, *is as follows;*
The CES *in state* (n,x) *is*
Employ the mixed-strategy triple $\langle \bar{\alpha}_0(x), \alpha_0(x)\rangle$, *with*

$$\alpha_0(x) = \frac{\sqrt{w_{n-1}-px}}{\sqrt{w_{n-1}-px} + \sqrt{(1/3-p)x}}, \quad \text{if } x < p^{-1}w_{n-1}, \text{ and}$$

Choose A, *if* $x > p^{-1}w_{n-1}$
The CEV *satisfies the recursion*

$$(3.9)\qquad w_n = U(w_{n-1}) \quad (n \ge 2, w_1 = 1/6)$$

where $U(w) = U(w|p)$ *is given by the r.h.s.of* (3.8).

We find from (3.8) that

$$U(0|p) = \frac{p}{2} + (1-3p)E(x/3) = \frac{1}{6},$$

and

$$U(1|p) = \frac{p}{2} + (1-3p)\int_0^1 \frac{x\left\{1-px+\sqrt{3(1-3p)x(1-px)}\right\}}{\left\{\sqrt{3(1-px)}+\sqrt{(1-3p)x}\right\}^2}dx.$$

Differentiation gives

$$(3.10)\qquad \frac{\partial}{\partial w}U(w|p) = \left(\frac{1-3p}{6}\right)\int_0^{(p^{-1}w)\wedge 1} \frac{(1-3p)\sqrt{x/(w-px)} - \sqrt{1/3-p}}{\left\{\sqrt{x^{-1}w-p}+\sqrt{1/3-p}\right\}^3}dx - \left(\frac{1-3p}{3p^2}\right)wI(p^{-1}w < 1).$$

It seems difficult to know further about $U(w|p)$, as well as its increasingness and monotonicity of w_n.

Remark 2 Some special cases ; For $p = 0$ (*i.e.*, Odd-Man-Out), we obtain from (3.8) and (3.10) that

$$U(w|0) = \int_0^1 \frac{\sqrt{wx/3}+w/3}{\left(\sqrt{w/x}+\sqrt{1/3}\right)^2}dx.$$

And

$$\frac{\partial}{\partial w}U(w|0) = \frac{1}{6}\int_0^1 \frac{\sqrt{x/w}-\sqrt{1/3}}{\left(\sqrt{w/x}+\sqrt{1/3}\right)^3}dx.$$

These are identical to the result in Sakaguchi [5 ; Theorem 4]. Furthermore it is shown that w_n converges to

$$w_\infty \equiv \inf\left\{w \in \left(0, \frac{1}{6}\right) \;\middle|\; U(w'|0) < w', \forall w' \in \left(w, \frac{1}{6}\right)\right\} \approx 0.1601.$$

For $p = 1/3 - 0$, Eq.(3.6) gives $U(w|\frac{1}{3}-0) = \frac{1}{6}$ and $w_n \equiv \frac{1}{6}, \forall n \geq 1$.

Remark 3 The game (1.1)-(1.2) has quite different solutions for the two cases $\frac{1}{3} < p \leq 1$ and $0 \leq p < \frac{1}{3}$, as observed in Theorems 1 and 2, although they seemingly look similar in (1.1)-(1.2). Furthermore the particular cases $p = \frac{1}{3} \pm 0$ give somewhat abnormal phase to the solution of the problem, as is mentioned in Remarks 1 and 2.

Remark 4 Theorems 1 and 2 show that, in the equilibrium, each player chooses R for small offers (A for large offers), and randomize R and A, for other offers, if arbitration favors the odd-man side (even-men side). This optimal behavior is almost the same as in the two-player game version investigated in Ref.[4].

References

[1] Mazalov, V. V., Sakaguchi, M., and Zabelin, A. A., *Multistage arbitration game with random offers*, Game Th. Appl., **8**(2002), 95-106.

[2] Petrosjan, L. A. and Zenkevich, N. A., *Game Theory,* World Scientific, Singapore, 1996 .

[3] Sakaguchi, M., *Best-choice games where arbitration comes in,* to appear in Game Th. Appl., **9**(2003).

[4] Sakaguchi, M., *Multistage non-zero-sum arbitration game*, Sci. Math. Japonicae,**58**(2003), 183-189.

[5] Sakaguchi, M., *Three-person games of odd-man-wins*, to appear in Game Th. Appl.,**10**(2004).

[6] Vorobjiev, N. N., *Fundamentals of Game Theory*, Radio & Sviaz, New York, 1984.

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