ON FIRST-PASSAGE-TIME DENSITIES FOR CERTAIN SYMMETRIC MARKOV CHAINS

A. Di Crescenzo and A. Nastro

Received February 20, 2004

Abstract. The spatial symmetry property of truncated birth-death processes studied in Di Crescenzo [6] is extended to a wider family of continuous-time Markov chains. We show that it yields simple expressions for first-passage-time densities and avoiding transition probabilities, and apply it to a bilateral birth-death process with jumps. It is finally proved that this symmetry property is preserved within the family of strongly similar Markov chains.

1 Introduction A spatial symmetry for the transition probabilities of truncated birth-death processes has been studied in Di Crescenzo [6]. Such a property leads to simple expressions for certain first-passage-time densities and avoiding transition probabilities. In this paper we aim to extend those results to a wider class of continuous-time Markov chains.

Given a set $\{x_n\}$ of positive real numbers and the transition probabilities $p_{k,n}(t)$ of a continuous-time Markov chain whose state-space is $\{0, 1, \dots, N\}$ or $\mathbf{Z}$, in Section 2 we introduce the following spatial symmetry property:

$$p_{N-k,N-n}(t) = \frac{x_n}{x_k}\, p_{k,n}(t). \tag{1}$$

In section 3 we point out some properties of first-passage-time densities and avoiding transition probabilities for Markov chains that are symmetric in the sense of (1). These properties allow one to obtain simple expressions for first-passage-time densities in terms of probability current functions, and for avoiding transition probabilities in terms of the 'free' transition probabilities. In Section 4 we then apply these results to a special bilateral birth-death process with jumps. Finally, in Section 5 we refer to the notion of strong similarity between the transition probabilities of Markov chains, expressed by $\widetilde{p}_{k,n}(t) = (\beta_n/\beta_k)\, p_{k,n}(t)$ (see Pollett [16], and references therein) and show the following preservation result: if $p_{k,n}(t)$ possesses the symmetry property (1), then also $\widetilde{p}_{k,n}(t)$ does it.

2 Symmetric Markov chains Let $\{X(t),\, t \geq 0\}$ be a homogeneous continuous-time Markov chain on a state-space $\mathcal{S}$. We shall assume that $\mathcal{S} = \{0, 1, \dots, N\}$, where N is a fixed positive integer, or $\mathcal{S} = \mathbf{Z} \equiv \{\dots, -1, 0, 1, \dots\}$. Let

$$p_{k,n}(t) = \Pr\{X(\tau + t) = n \,|\, X(\tau) = k\}, \qquad k, n \in \mathcal{S};\;\; t, \tau \geq 0 \tag{2}$$

be the stationary transition probabilities of $X(t)$, satisfying the initial conditions

$$p_{k,n}(0) = \delta_{k,n} = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

2000 *Mathematics Subject Classification.* 60J27, 60J35.

Key words and phrases. Transition probabilities; first-passage-time densities; avoiding probabilities; flow functions; birth-death processes with jumps; strong similarity.

Let Q be the infinitesimal generator of the transition function (2), i.e. the matrix whose (k,n)-th finite entries are:

$$q_{k,n} = \frac{\mathrm{d}}{\mathrm{d}t} p_{k,n}(t)\Big|_{t=0}, \tag{3}$$

satisfying the following relations: *(a)* $q_{k,n} \geq 0$ for all $k, n \in \mathcal{S}$ such that $k \neq n$, *(b)* $q_{n,n} \leq 0$ for all $n \in \mathcal{S}$, and *(c)* $\sum_{n \in \mathcal{S}} q_{k,n} = 0$ for all $k \in \mathcal{S}$.

The spatial symmetry of Markov processes allows one to approach effectively the first-passage-time problem. Indeed, it has been often exploited by various authors to obtain closed-form results for first-passage-time distributions; see Giorno *et al.* [13] and Di Crescenzo *et al.* [8] for one-dimensional diffusion processes, Di Crescenzo *et al.* [7] for two-dimensional diffusion processes, and Di Crescenzo [5] for a class of two-dimensional random walks. Moreover, in Di Crescenzo [6] a symmetry for truncated birth-death processes was expressed as in (1), with x_i suitably depending on the birth and death rates. Such symmetry notion can be extended to the wider class of continuous-time Markov chains considered above. Indeed, for a set of positive real numbers $\{x_n;\ n \in \mathcal{S}\}$ there holds:

$$p_{N-k,N-n}(t) = \frac{x_n}{x_k}\, p_{k,n}(t) \qquad \text{for all } k, n \in \mathcal{S} \text{ and } t \geq 0 \tag{4}$$

if and only if

$$q_{N-k,N-n} = \frac{x_n}{x_k}\, q_{k,n} \qquad \text{for all } k, n \in \mathcal{S}. \tag{5}$$

The proof is similar to that of Theorem 2.1 in Di Crescenzo [6], and thus is omitted.

Eq. (4) focuses on a symmetry with respect to $N/2$, which identifies with the mid point of $\mathcal{S}$ when $\mathcal{S} = \{0, 1, \ldots, N\}$. For each sample-path of $X(t)$ from k to n there is a symmetric path from $N-k$ to $N-n$, and the ratio of their probabilities is time-independent. Hence, in the following we shall say that $X(t)$ possesses a *central symmetry* if relation (4) is satisfied.

Remark 2.1 *If $X(t)$ possesses a central symmetry, then*

$$\frac{x_n}{x_k} = \frac{x_{N-k}}{x_{N-n}} \qquad \textit{for all } k, n \in \mathcal{S}.$$

An example of a Markov chain with finite state-space and a central symmetry is given hereafter.

Example 2.1 Let $X(t)$ be a continuous-time Markov chain with state-space $\mathcal{S} = \{0, 1, 2, 3\}$, with 0 and 3 absorbing states, and infinitesimal generator

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha\,\varrho_0 + \beta & -\alpha(\varrho_0 + \varrho^2) - \beta\,\varrho_0 & \beta\varrho & \alpha\varrho^2 \\ \alpha & \beta & -\alpha(\varrho_0 + \varrho^2) - \beta\,\varrho_0 & (\alpha\,\varrho_0 + \beta)\varrho \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $\alpha, \beta, \varrho > 0$ and $\varrho_0 = 1 + \varrho$. Then, $X(t)$ has a central symmetry, with $p_{N-k,N-n}(t) = \varrho^{k-n}\, p_{k,n}(t)$ for all $k, n \in \mathcal{S}$ and $t \geq 0$, and $q_{N-k,N-n}(t) = \varrho^{k-n}\, q_{k,n}$ for all $k, n \in \mathcal{S}$.

Remark 2.2 *If $X(t)$ has a central symmetry and possesses a stationary distribution $\{\pi_n,\ n \in \mathcal{S}\}$, with $\lim_{t\to+\infty} p_{k,n}(t) = \pi_n > 0$ for all $k, n \in \mathcal{S}$, then the following statements hold:*

(a) Sequence $\{x_n\}$ is constant, so that $p_{N-k,N-n}(t) = p_{k,n}(t)$ for all $k, n \in \mathcal{S}$ and $t \geq 0$.

(b) The stationary distribution is symmetric with respect to $N/2$, i.e.

$$\pi_{N-n} = \pi_n \qquad \textit{for all } n \in \mathcal{S}.$$

(c) Let $X^*(t)$ be the reversed process of $X(t)$, obtained from $X(t)$ when time is reversed, and characterized by rates and transition probabilities

$$q^*_{k,n} = \frac{\pi_n}{\pi_k}\, q_{n,k}, \qquad p^*_{k,n}(t) = \frac{\pi_n}{\pi_k}\, p_{n,k}(t), \qquad k, n \in \mathcal{S}, \quad t \geq 0.$$

Then, also $X^*(t)$ has a central symmetry, with $p^*_{N-k,N-n}(t) = p^*_{k,n}(t)$ for all $k, n \in \mathcal{S}$ and $t \geq 0$.
(d) Let $D = \{d_{k,n}\}$ be the deviation matrix *of $X(t)$, with elements (see Coolen-Schrijner and Van Doorn [2])*

$$d_{k,n} = \int_0^{+\infty} [p_{k,n}(t) - \pi_n]\, \mathrm{d}t, \qquad k, n \in \mathcal{S}.$$

Then, D has a central symmetry, i.e. $d_{N-k,N-n} = d_{k,n}$ for all $k, n \in \mathcal{S}$.

An example of a Markov chain satisfying the assumptions of Remark 2.2 is the birth-death process on $\mathcal{S}$ with birth rate $\lambda_n = \alpha\,(N-n)$ and death rate $\mu_n = \alpha\, n$ (see Giorno *et al.* [11], or Section 4.1 of Di Crescenzo [6]).

3 First-passage-time densities In this section we shall focus on the first-passage-time problem for Markov chains $X(t)$ that have a central symmetry and that satisfy the following assumptions:
(i) $N = 2s$, with s a positive integer;
(ii) $q_{i,j} = q_{j,i} = 0$, $\sum_{i\in\mathcal{S}_-} q_{i,s} > 0$, $\sum_{j\in\mathcal{S}_+} q_{j,s} > 0$, $\sum_{i\in\mathcal{S}_-} q_{s,i} > 0$ and $\sum_{j\in\mathcal{S}_+} q_{s,j} > 0$ for all $i \in \mathcal{S}_-$ and $j \in \mathcal{S}_+$, where

$$S_- = \{n \in \mathcal{S};\ n < s\}, \qquad S_+ = \{n \in \mathcal{S};\ n > s\};$$

(in other words, if states i and j are separated by s then all sample-paths of $X(t)$ from i to j, or from j to i, must cross s);
(iii) the subchains defined on S_- and S_+ are irreducibles.
In addition, we introduce the following non-negative random variables:

$T^+_{i,s} =$ upward first-passage time of $X(t)$ from state $i \in S_-$ to state s,
$T^-_{j,s} =$ downward first-passage time of $X(t)$ from state $j \in S_+$ to state s.

We shall denote by $g^+_{i,s}(t)$ and $g^-_{j,s}(t)$ the corresponding probability density functions. Due to assumptions (i)-(iii), for all $t > 0$ such densities satisfy the following renewal equations:

$$p_{i,j}(t) = \int_0^t g^+_{i,s}(\vartheta)\, p_{s,j}(t-\vartheta)\, \mathrm{d}\vartheta, \qquad i \in S_-,\ \ j \in \{s\} \cup S_+, \tag{6}$$

$$p_{j,i}(t) = \int_0^t g^-_{j,s}(\vartheta)\, p_{s,i}(t-\vartheta)\, \mathrm{d}\vartheta, \qquad i \in S_- \cup \{s\},\ \ j \in S_+. \tag{7}$$

For all $t > 0$ and $k \in \mathcal{S}$ let us now introduce the *probability currents*

$$h^+_{k,s}(t) = \lim_{\tau\downarrow 0} \frac{1}{\tau}\, \mathrm{P}\{X(t+\tau) = s,\ X(t) < s \mid X(0) = k\} = \sum_{i\in\mathcal{S}_-} p_{k,i}(t)\, q_{i,s}, \tag{8}$$

$$h^-_{k,s}(t) = \lim_{\tau\downarrow 0} \frac{1}{\tau}\, \mathrm{P}\{X(t+\tau) = s,\ X(t) > s \mid X(0) = k\} = \sum_{j\in\mathcal{S}_+} p_{k,j}(t)\, q_{j,s}. \tag{9}$$

They represent respectively the upward and downward entrance probability fluxes at state s at time t. Due to assumptions (i)-(iii) and Eqs. (6)-(9), for $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+$ and $t > 0$ they satisfy the following integral equations:

$$h^-_{i,s}(t) = \int_0^t g^+_{i,s}(\vartheta)\, h^-_{s,s}(t-\vartheta)\, \mathrm{d}\vartheta, \tag{10}$$

$$h^+_{j,s}(t) = \int_0^t g^-_{j,s}(\vartheta)\, h^+_{s,s}(t-\vartheta)\, \mathrm{d}\vartheta. \tag{11}$$

Hereafter we extend Proposition 2.2 of Di Crescenzo [6] to the case of Markov chains.

Proposition 3.1 *Under assumptions (i)-(iii), for all* $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+$ *and* $t > 0$ *the following equations hold:*

$$g^+_{i,s}(t) = h^+_{i,s}(t) - \int_0^t g^+_{i,s}(\vartheta)\, h^+_{s,s}(t-\vartheta)\, \mathrm{d}\vartheta, \tag{12}$$

$$g^-_{j,s}(t) = h^-_{j,s}(t) - \int_0^t g^-_{j,s}(\vartheta)\, h^-_{s,s}(t-\vartheta)\, \mathrm{d}\vartheta. \tag{13}$$

Proof. For all $t > 0$ and $i \in \mathcal{S}_-$, making use of assumptions (i)-(iii) and Eq. (8) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{i,s}(t) = \sum_{n\in\mathcal{S}} p_{i,n}(t)\, q_{n,s} = h^+_{i,s}(t) + \sum_{n\in\{s\}\cup S_+} p_{i,n}(t)\, q_{n,s}.$$

Hence, recalling (6) we obtain

$$\begin{aligned} h^+_{i,s}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t g^+_{i,s}(\vartheta)\, p_{s,s}(t-\vartheta)\, \mathrm{d}\vartheta\right] - \sum_{n\in\{s\}\cup S_+}\left[\int_0^t g^+_{i,s}(\vartheta)\, p_{s,n}(t-\vartheta)\, \mathrm{d}\vartheta\right] q_{n,s} \\ &= g^+_{i,s}(t) + \int_0^t g^+_{i,s}(\vartheta)\left[\frac{\partial}{\partial t} p_{s,s}(t-\vartheta) - \sum_{n\in\{s\}\cup S_+} p_{s,n}(t-\vartheta)\, q_{n,s}\right]\mathrm{d}\vartheta, \end{aligned} \tag{14}$$

where use of initial condition $p_{s,s}(0) = 1$ has been made. From Chapman-Kolmogorov forward equation we have

$$\frac{\partial}{\partial t} p_{s,s}(t-\vartheta) - \sum_{n\in\{s\}\cup S_+} p_{s,n}(t-\vartheta)\, q_{n,s} = h^+_{s,s}(t-\vartheta), \qquad t > \vartheta,$$

so that Eq. (14) gives (12). The proof of (13) goes along similar lines. ∎

With reference to a Markov chain that has a central symmetry, we now come to the main result of this paper, expressing the first-passage-time densities through the symmetry state s as difference of probability currents (8) and (9).

Theorem 3.1 *For a Markov chain that has a central symmetry and satisfies assumptions (i)-(iii), for all* $t > 0$ *and* $k \in \mathcal{S}$ *there results:*

$$h^-_{2s-k,s}(t) = \frac{x_s}{x_k}\, h^+_{k,s}(t). \tag{15}$$

Moreover, for all $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+$ *and* $t > 0$ *the upward and downward first-passage-time densities through state* s *are given by*

$$g^+_{i,s}(t) = h^+_{i,s}(t) - h^-_{i,s}(t), \qquad g^-_{j,s}(t) = h^-_{j,s}(t) - h^+_{j,s}(t). \tag{16}$$

Proof. Recalling that $N = 2s$, for $t > 0$ we have

$$\begin{aligned} h^-_{2s-k,s}(t) &= \sum_{j\in\mathcal{S}_+} p_{2s-k,j}(t)\, q_{j,s} && \text{(from (9))} \\ &= \sum_{i\in\mathcal{S}_-} p_{2s-k,2s-i}(t)\, q_{2s-i,s} && \text{(setting } j = 2s - i) \\ &= \frac{x_s}{x_k} \sum_{i\in\mathcal{S}_-} p_{k,i}(t)\, q_{i,s} && \text{(from (4) and (5))} \\ &= \frac{x_s}{x_k}\, h^+_{k,s}(t). && \text{(from (8))} \end{aligned}$$

Eq. (15) then holds. In particular, for $k = s$ it implies that $h^-_{s,s}(t-\vartheta) = h^+_{s,s}(t-\vartheta)$ for all $t > \vartheta$. Hence, relations (16) follow from Eqs. (10)-(13). ■

For a Markov chain $X(t)$ satisfying assumptions (i)-(iii) let us now introduce the *s-avoiding transition probabilities*:

$$p^{\langle s\rangle}_{k,n}(t) = \mathrm{P}\left\{X(t) = n,\ X(\vartheta) \neq s \ \text{ for all } \vartheta \in (0,t) \mid X(0) = k\right\},$$

where $k, n \in \mathcal{S}_- \cup \mathcal{S}_+$. We note that $p^{\langle s\rangle}_{k,n}(t)$ is related to $p_{k,n}(t)$ by

$$p^{\langle s\rangle}_{k,n}(t) = \begin{cases} p_{k,n}(t) - \displaystyle\int_0^t g^+_{k,s}(\vartheta) p_{s,n}(t-\vartheta)\, \mathrm{d}\vartheta, & k,n \in \mathcal{S}_-, \\ p_{k,n}(t) - \displaystyle\int_0^t g^-_{k,s}(\vartheta) p_{s,n}(t-\vartheta)\, \mathrm{d}\vartheta, & k,n \in \mathcal{S}_+. \end{cases} \tag{17}$$

In the following theorem, for symmetric Markov chains two different expressions are given for $p^{\langle s\rangle}_{k,n}(t)$ in terms of $p_{k,n}(t)$. It extends Theorem 2.4 of Di Crescenzo [6]; the proof is similar and therefore is omitted.

Theorem 3.2 *Under the assumptions of Theorem 3.1, for $t > 0$ and for $k, n \in \mathcal{S}_- \cup \mathcal{S}_+$ there holds:*

$$\begin{aligned} p^{\langle s\rangle}_{k,n}(t) &= p_{k,n}(t) - \frac{x_k}{x_s}\, p_{2s-k,n}(t) \\ &= p_{k,n}(t) - \frac{x_s}{x_n}\, p_{k,2s-n}(t). \end{aligned}$$

We conclude this section by pointing out that for a Markov chain having a central symmetry, for all $t > 0$ the following relations hold:

$$g^+_{i,s}(t) = \frac{x_i}{x_s} g^-_{2s-i,s}(t), \qquad g^-_{j,s}(t) = \frac{x_j}{x_s} g^+_{2s-j,s}(t), \qquad i \in \mathcal{S}_-,\ j \in \mathcal{S}_+, \tag{18}$$

$$p^{\langle s\rangle}_{2s-k,2s-n}(t) = \frac{x_n}{x_k}\, p^{\langle s\rangle}_{k,n}(t), \qquad k,n \in \mathcal{S}_- \cup \mathcal{S}_+.$$

4 A bilateral birth-death process with jumps In this section we shall apply the above results to a special symmetric Markov chain $X(t)$ with state-space $\mathbf{Z}$, characterized by the following transitions: (a) from $n \in \mathbf{Z}$ to $n+1$ with rate λ, (b) from $n \in \mathbf{Z}$ to $n-1$ with rate μ, and (c) from $n \in \mathbf{Z} - \{0\}$ to 0 with rate α. Hence, $X(t)$ is a bilateral birth-death process that includes jumps toward state 0. In order to obtain an expression for the transition probabilities $p_{k,n}(t)$, we note that for all $t > 0$ the following system holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{k,n}(t) = -(\lambda+\mu+\alpha)\,p_{k,n}(t) + \lambda\, p_{k,n-1}(t) + \mu\, p_{k,n+1}(t), \qquad n \in \mathbf{Z} - \{0\},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p_{k,0}(t) = -(\lambda+\mu)\,p_{k,0}(t) + \lambda\, p_{k,-1}(t) + \mu\, p_{k,1}(t) + \alpha\sum_{r\neq 0} p_{k,r}(t).$$

The probability generating function

$$H(z,t) = \sum_{n=-\infty}^{+\infty} p_{k,n}(t)\, z^n$$

is thus solution of

$$\frac{\partial}{\partial t}H(z,t) = u(z)\,H(z,t) + \alpha, \tag{19}$$

where $u(z) = -(\lambda+\mu+\alpha) + \lambda\, z + \dfrac{\mu}{z}$, with initial condition $H(z,0) = z^k$. The unique solution of (19) is

$$H(z,t) = H(z,0)\, e^{u(z)\,t} + \alpha\int_0^t e^{u(z)\,\tau}\,\mathrm{d}\tau. \tag{20}$$

Hence, recalling that

$$\exp\left\{\left(\lambda\, z + \frac{\mu}{z}\right)t\right\} = \sum_{n=-\infty}^{+\infty} I_n(\gamma\, t)\,(\beta\, z)^n$$

for $\gamma = 2\sqrt{\lambda\,\mu}$ and $\beta = \sqrt{\lambda/\mu}$, from (20) we obtain:

$$H(s,t) = e^{-(\lambda+\mu+\alpha)t}\sum_{n=-\infty}^{+\infty} I_{n-k}(\gamma t)\,\beta^{n-k}\, s^n + \alpha\int_0^t e^{-(\lambda+\mu+\alpha)\tau}\sum_{n=-\infty}^{+\infty} I_n(\gamma\tau)\,(\beta s)^n\,\mathrm{d}\tau, \tag{21}$$

where $I_n(x)$ denotes the modified Bessel function of the first kind. Equating the coefficients of z^n on both sides of (21) finally yields the transition probabilities

$$p_{k,n}(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{n-k}{2}} I_{n-k}\left(2\sqrt{\lambda\mu}\, t\right)\, e^{-(\lambda+\mu+\alpha)t} + \alpha\left(\frac{\lambda}{\mu}\right)^{\frac{n}{2}}\int_0^t e^{-(\lambda+\mu+\alpha)\tau} I_n(2\sqrt{\lambda\mu}\,\tau)\,\mathrm{d}\tau. \tag{22}$$

Note that (22) can be expressed as

$$p_{k,n}(t) = e^{-\alpha t}\,\widehat{p}_{k,n}(t) + \alpha\int_0^t e^{-\alpha\tau}\,\widehat{p}_{0,n}(\tau)\,\mathrm{d}\tau, \tag{23}$$

where, for all $t \geq 0$ and $k, n \in \mathbf{Z}$,

$$\widehat{p}_{k,n}(t) := \left(\frac{\lambda}{\mu}\right)^{\frac{n-k}{2}} I_{n-k}(2\sqrt{\lambda\mu}\, t)\, e^{-(\lambda+\mu)t}, \tag{24}$$

is the transition probability of the Poisson bilateral birth-death process with birth rate λ and death rate μ (see, for instance, Section 2.1 of Conolly [1]). Assuming that the stationary probabilities $\pi_n = \lim_{t\to+\infty} p_{k,n}(t)$ exist for all $n \in \mathbf{Z}$, from (19) we have

$$\sum_{n=-\infty}^{+\infty} \pi_n z^n = \lim_{t\to+\infty} H(z,t) = -\frac{\alpha}{u(z)} = \frac{\alpha z}{\lambda(z-z_1)(z_2-z)}$$
$$= \frac{\alpha}{\lambda(z_2-z_1)}\left[\sum_{n=-\infty}^{-1}\left(\frac{z}{z_1}\right)^n + \sum_{n=0}^{+\infty}\left(\frac{z}{z_2}\right)^n\right],$$

where

$$z_{1,2} = \frac{\lambda+\mu+\alpha \pm \sqrt{(\lambda+\mu+\alpha)^2 - 4\lambda\mu}}{2\lambda}, \qquad 0 < z_1 < 1 < z_2.$$

Hence,

$$\pi_n = \begin{cases} \dfrac{\alpha\, {z_1}^{-n}}{\lambda(z_2-z_1)} & \text{for } n = -1, -2, \ldots, \\[2ex] \dfrac{\alpha\, {z_2}^{-n}}{\lambda(z_2-z_1)} & \text{for } n = 0, 1, 2, \ldots. \end{cases} \tag{25}$$

It is not hard to see that if $\lambda = \mu$ then $X(t)$ has a central symmetry with respect to state 0, with $x_k = 1$ for all k:

$$p_{-k,-n}(t) = p_{k,n}(t), \qquad q_{-k,-n} = q_{k,n}$$

for all $t > 0$ and $k, n \in \mathbf{Z}$. Note that if $\lambda = \mu$, then z_1 and z_2 are reciprocal zeroes of $u(z)$, so that the stationary distribution (25) is symmetric, i.e. $\pi_n = \pi_{-n}$ for all $n \in \mathbf{Z}$. Since $q_{i,j} = q_{j,i} = 0$, $q_{i,0} > 0$ and $q_{j,0} > 0$ for all $i, j \in \mathbf{Z}$ such that $i < 0 < j$, and $q_{0,-1} > 0$ and $q_{0,1} > 0$, this Markov chain satisfies assumptions (i)-(iii) for which 0 is a symmetry state. In this case the first-passage-time densities through 0 can be obtained via Theorem 3.1. Indeed, if $\lambda = \mu$, making use of (22) and of property $I_n(x) = I_{-n}(x)$, for all $t > 0$ and $k = 1, 2, \ldots$ we have:

$$g^-_{k,0}(t) = h^-_{k,0}(t) - h^+_{k,0}(t) = \sum_{j=1}^{+\infty} p_{k,j}(t)\, q_{j,0} - \sum_{i=-\infty}^{-1} p_{k,i}(t)\, q_{i,0}$$
$$= \lambda\,[p_{k,1}(t) - p_{k,-1}(t)] + \alpha\left[\sum_{j=1}^{+\infty} p_{k,j}(t) - \sum_{i=-\infty}^{-1} p_{k,i}(t)\right]$$
$$= e^{-(2\lambda+\alpha)t}\left\{\lambda\,[I_{k-1}(2\lambda t) - I_{k+1}(2\lambda t)] + \alpha\sum_{j=1}^{+\infty}[I_{k-j}(2\lambda t) - I_{k+j}(2\lambda t)]\right\}. \tag{26}$$

Furthermore, recalling (18), in this special case for all $t > 0$ and $k = 1, 2, \ldots$ there holds:

$$g^+_{-k,0}(t) = g^-_{k,0}(t).$$

In analogy with Theorem 3.2 and by virtue of (22), when $\lambda = \mu$, we have

$$p^{\langle 0\rangle}_{k,n}(t) = p_{k,n}(t) - p_{-k,n}(t)$$
$$= e^{-(2\lambda+\alpha)t}\,[I_{n-k}(2\lambda t) - I_{n+k}(2\lambda t)], \qquad t > 0. \tag{27}$$

Note that

$$p^{\langle 0\rangle}_{k,n}(t) = p^{\langle 0\rangle}_{n,k}(t), \tag{28}$$

$$p^{\langle 0\rangle}_{k,n}(t) = e^{-\alpha t}\, \widehat{p}^{\langle 0\rangle}_{k,n}(t), \tag{29}$$

where $\widehat{p}^{\langle 0\rangle}_{k,n}(t)$ is the transition probability of $\widehat{X}(t)$ when $\lambda=\mu$. Functions (26) and (27) are shown in Figure 1 for some choices of the involved parameters.

We finally remark that Eqs. (23) and (29) are in agreement with similar results for birth-death processes with catastrophes obtained in Di Crescenzo *et al.* [9] and [10].

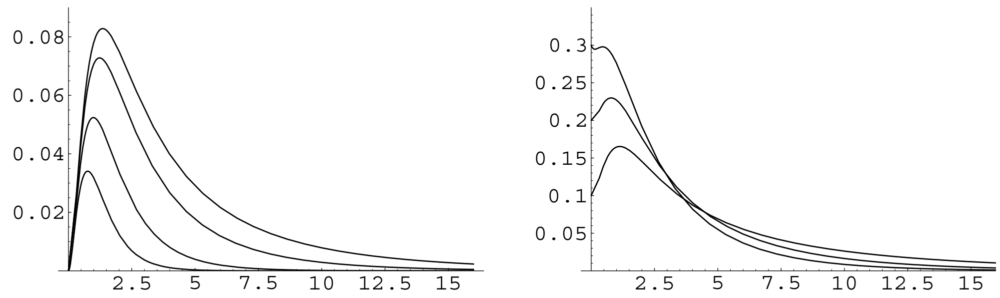

Figure 1: On the left-hand are the plots of the downward first-passage-time density (26) for $k=3$, $\lambda=1$ and $\alpha=0.1,0.2,0.3$, from bottom to top near the origin. On the right the 0-avoiding transition probabilities (27) for $k=3$, $n=1$, $\lambda=1$ and $\alpha=0.1,0.2,0.5,1$ (top to bottom) are indicated.

5 Strong similarity The notion of similarity between stochastic processes has attracted the attention of several authors (see Giorno *et al.* [12], for time-homogeneous diffusion processes, Gutiérrez Jáimez *et al.* [14] for time-nonhomogeneous diffusion processes, Di Crescenzo [3], [4], and Lenin *et al.* [15], for birth-death processes, and Pollett [16], for Markov chains). Two continuous-time Markov chains $X(t)$ and $\widetilde{X}(t)$, with state-space $\mathcal{S}$, are said to be strongly similar if their transition probabilities satisfy

$$\widetilde{p}_{k,n}(t)=\frac{\beta_n}{\beta_k}\,p_{k,n}(t),\qquad \text{for all } t\geq 0 \text{ and } k,n\in\mathcal{S}, \tag{30}$$

where $\{\beta_n,\, n\in\mathcal{S}\}$ is a suitable sequence of real positive numbers (we refer the reader to Pollett [16], for further details). In the following theorem we state that if a Markov chain has a central symmetry, then any of its similar chains has a central symmetry as well.

Theorem 5.1 *Let $X(t)$ and $\widetilde{X}(t)$ be strongly similar continuous-time Markov chains with state-space $\mathcal{S}$; if $X(t)$ has a central symmetry, then for all $t\geq 0$ and $k,n\in\mathcal{S}$ one has:*

$$\widetilde{p}_{N-k,N-n}(t)=\frac{\widetilde{x}_n}{\widetilde{x}_k}\,\widetilde{p}_{k,n}(t)$$

with

$$\widetilde{x}_n=\frac{\beta_{N-n}}{\beta_n}\,x_n,\qquad n\in\mathcal{S}.$$

The proof is an immediate consequence of assumed symmetry and similarity properties.

Hereafter we show an application of Theorem 5.1 to a birth-death process having constant rates and state-space **Z**.

Example 5.1 Let $X(t)$ be the bilateral birth-death process with birth and death rates λ and μ, respectively. From transition probabilities (24) it is not hard to see $X(t)$ has a central symmetry with respect to 0, i.e. for all $t\geq 0$ and $k,n\in\mathbf{Z}$ there results

$$p_{-k,-n}(t)=\frac{x_n}{x_k}\,p_{k,n}(t),\qquad \text{with } x_n=\left(\frac{\lambda}{\mu}\right)^{-n}.$$

The Markov chains that are strongly similar to $X(t)$ constitute a family of bilateral birth-death processes characterized by birth and death rates (see Section 4 of Di Crescenzo [4], and Example 3 of Pollett [16])

$$\widetilde{\lambda}_n = \frac{\beta_{n+1}}{\beta_n}\,\lambda, \qquad \widetilde{\mu}_n = \frac{\beta_{n-1}}{\beta_n}\,\mu, \qquad n \in \mathbf{Z},$$

and by transition probabilities (30), with $p_{k,n}(t)$ given in (24) and

$$\beta_n = 1 + \eta \left(\frac{\lambda}{\mu}\right)^n, \qquad n \in \mathbf{Z},$$

for all $\eta \geq 0$. Due to Theorem 5.1, the family of strongly similar processes has a central symmetry with respect to 0:

$$\widetilde{p}_{-k,-n}(t) = \frac{\widetilde{x}_n}{\widetilde{x}_k}\,\widetilde{p}_{k,n}(t), \qquad \text{with } \widetilde{x}_n = \frac{\beta_{-n}}{\beta_n}\,x_n = \frac{1+\eta\left(\frac{\lambda}{\mu}\right)^{-n}}{1+\eta\left(\frac{\lambda}{\mu}\right)^{n}}\left(\frac{\lambda}{\mu}\right)^{-n}, \qquad n \in \mathbf{Z}.$$

Acknowledgements This work has been partially supported by MIUR (cofin 2003) and INdAM (G.N.I.M.).

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A. Di Crescenzo: Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte Don Melillo, I-84084 Fisciano (SA), Italy

A. Nastro: Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, I-80126 Napoli, Italy