THE ARENS REGULARITY OF WEIGHTED SEMIGROUP ALGEBRAS

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Abstract. In [4], Baker and Rejali are concerned with the regularity of $\ell_1(S, w)$, where S is a discrete semigroup. We study the Arens regularity of the weighted semigroup algebra $M_b(S, w)$, for non-discrete S. We show that $\ell_1(S, w)$ is regular if and only if $M_b(S, w)$ is regular.

We obtain conditions for the regularity of $M_a^\ell(S, w)$, analogous to the weighted group algebra $L^1(G, w)$. It is shown that $L^1(G, w)$ is regular if and only if G is finite or G is discrte and Ω is $0-$cluster, where $\Omega(x, y) = w(xy)/w(x)w(y)$ for $x, y \in G$. We obtain that $M_a^\ell(S, w)$ is regular whenever $M_a^\ell(S)$ is. Furthermore, $M_a^\ell(S, w)$ is regular if and only if $\ell_1(F_a(S), w)$ is regular, where $F_a(S)$ is the foundation semigroup of $M_a^\ell(S, w)$. For a foundation semigroup S, the regularity of $M_b(S, w), M_a^\ell(S, w)$ and $\ell_1(S, w)$ are equivalent.

Introduction

Let A b a Banach algebra. In [1, 2], R. Arens showed how to construct two multiplications on A^{**} which make A^{**} into a Banach algebra and extend the product on A. The two multiplications are, in general, distinct. According to Arens, A is called regular if they coincide.

In [5], Crow and Young obtained a partial answer for the regularity of the weighted semigroup algebra $\ell_1(S, w)$. Baker and Rejali [4] obtained some new criteria for the regularity of $\ell_1(S, w)$, for any semigroup S.

In [15], the weighted semigroup algebra $M_b(S, w)$, for non-discrete S, and $M_a^\ell(S, w)$, analogous to the group algebra $L^1(G, w)$ were introduced, where S is a completely regular semitopological semigroup and w is a Borel-measurable weight function for which w^{-1} is bounded on compact subsets of S.

Crow and Young [5] showed that there exists a weight function w for which $L^1(G, w)$ is regular if and only if G is discrete and countable. Following [4], we denote by Ω the map $(x, y) \to w(xy)/w(x)w(y)$ of $S \times S$ to $(0, 1]$, where $w : S \to (0, \infty)$ is a weight function; that is $w(st) \leq w(s)w(t)$ for all $s, t \in S$. A bounded real-valued function f on $S \times S$ is called a cluster [respectively 0-cluster], if for each pair of sequences (x_n) and (y_m) of distinct elements of S there exist subsequences (x'_n) and (y'_m), respectively, such that

$$\lim_n \lim_m f(x'_n, y'_m) = \lim_m \lim_n f(x'_n, y'_m)$$

[respectively, both limits equal zero], whenever both limits exist.

We will show that $L^1(G, w)$ is regular if and only if G is finite or G is discrte and Ω is 0-cluster. We obtain conditions for the regularity of the convolution measure algebras $M_b(S, w)$ and $M_a^\ell(S, w)$. If $M_a^\ell(S)$ [respectively, $M_b(S)$] is regular, then $M_a^\ell(S, w)$ [respectively, $M_b(S, w)$] is regular. If S is a foundation semigroup, then $M_a^\ell(S, w)$ is regular if and only if $\ell_1(S, w)$ is regular.

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In [9], Duncan and Hosseiniun asked for which semigroup S, the convolution measure algebra $\ell_1(S)$ is regular. Young [17] obtained some equivalent conditions for the regularity of $\ell_1(S)$. We establish necessary and sufficient conditons for which $M_a^\ell(S,w)$ is regular.

The Arens regularity of weighted semigroup algebras

Throughout this paper S is a locally compact topological semigroup and w is a Borel-measurable weight function on S for which w^{-1} is bounded on compacta and Ω is a separately continuous function.

In [16], Wong applied generalized functions to study invariant means on locally compact semigroups. Lashkarizadeh, in [12], used it for the representation theory. We apply weighted generalized functions for the study of regularity of weighted semigroup algbras.

For an introduction to the theorey of semigroups, we refer the reader to [6] and [11].

Following [15], let $M_b^+(S,w)$ be the set of all positive Radon measures μ on S, i.e., μ is inner-regular and finite on compacta, such that $\mu w \in M_b^+(S)$ where

$$\mu w(E) = \int_E w d\mu, \text{for Borel measurable } E \subseteq S.$$

Let $C_0(S,w) = \{f : S \to \mathbb{R} : \frac{f}{w} \in C_0(S)\}$, where $C_0(S)$ is the Banach space of all continuous (real-valued) functions on S which are zero at infinity.

Let $\varphi : M_b^+(S,w) \times M_b^+(S,w) \to C_0(S,w)^*$ be defined by $(\mu,\nu) \longmapsto I_\mu - I_\nu$, where

$$I_\mu - I_\nu(f) = \int_S f d\mu - \int_S f d\nu, \text{for } f \in C_0(S,w).$$

In general, φ need not be injective. Let "$\sim$" be the equivalence relation on $M_b^+(S,w) \times M_b^+(S,w)$ defined by

$$(\mu,\nu) \sim (\mu',\nu') \text{ if and only if } \mu + \nu' = \mu' + \nu$$

and let $[\mu,\nu]$ be the equivalence class of (μ,ν). Then we define

$$M_b(S,w) = \{[\mu,\nu] : \mu,\nu \in M_b^+(S,w)\}.$$

Let $[\mu,\nu]$ and $[\mu',\nu']$ be in $M_b(S,w)$. Then $M_b(S,w)$ with the following norm and multiplication is a convolution measure algebra, see [15, Th. (2.2)].

$$\|[\mu,\nu]\|_w = \|\mu w - \nu w\|$$
$$[\mu,\nu] * [\mu',\nu'] = [\mu * \mu' + \nu * \nu', \mu * \nu' + \mu' * \nu]$$

where

$$\mu * \nu(E) = \int_G \int_G \chi_E(xy) d\mu(x) d\nu(y)$$

for all Borel measurable $E \subseteq G$ and $\mu,\nu \in M_b^+(S,w)$. Let $\eta = [\mu,\nu] \in M_b(S,w)$. Then $\eta^+ = (\mu w - \nu w)^+ w^{-1}$ and $\eta^- = (\mu w - \nu w)^- w^{-1}$, so $|\eta| = \eta^+ + \eta^- \in M_b^+(S,w)$ [For w^{-1} is bounded on compacta], see [10, p.180].

It is routine to show that the map $[\mu,\nu] \to I_\mu - I_\nu$ from $M_b(S,w)$ onto $C_0(S,w)^*$ is an isometric isomorphism such that, for all $f \in C_0(S,w)$,

$$I_\mu - I_\nu(f) = \int_S f d\mu - \int_S f d\nu.$$

Example 1: Let $G = (\mathbb{R}, +)$ and $w_0(x) = 1/(1 + |x|)^3, w(x) = |x| + 1$ for $x \in \mathbb{R}$. Let $\mu = w_0\lambda|_{\mathbb{R}^+}$ and $\nu = w_0\lambda|_{\mathbb{R}^-}$, where λ is the Lebesgue measure on $\mathbb{R}$. Then $\mu, \nu \in M_b^+(G, w)$, but there is no finite signed measure η on $\mathbb{R}$ such that

$$I(f) = \int_{\mathbb{R}} f d\eta = \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\nu, \text{ for } f \in C_0(\mathbb{R}, w)$$

[For there exist $f_m, g_m \in C_{oo}(\mathbb{R})_1^+$ such that $I(f_m) = m$ and $I(g_m) = -m$ for all $m \in \mathbb{N}$].

Following [12, p. 3], let $\mu \in M_b(S, w)$ and let $L_\infty(|\mu|, w)$ be the Banach space of all $w-$bounded Borel-measurable (real-valued) functions on S with essential supremum norm,

$$\|f\|_{w,\mu} = \|f/w\|_\mu = \inf\left\{\alpha \geq 0 : \{x \in S : \frac{f}{w}(x) > \alpha\} \text{ is } |\mu| - null\right\}.$$

Consider the product linear space $\Pi\{L_\infty(|\mu|, w) : \mu \in M_b(S, w)\}$. An element $f = (f_\mu)$ in this product is called a $w-$generalized function on S if the following conditions are satisfied:

(i) $\|f\|_{w,\infty} = \sup\{\|f_\mu\|_{w,\mu} : \mu \in M_b(S, w)\}$ is finite,

(ii) if $\mu, \nu \in M_b(S, w)$ and $|\mu| \ll |\nu|$, then $f_\mu = f_\nu, |\mu| - a.e.$

Let $GL(S, w)$ denote the linear subspace of all $w-$generalized functions on S. Then $M_b(S, w)^*$ is isometrically linear-isomorphic to the space $GL(S, w)$, see [12]. Furthermore, let $T : GL(S, w) \to M_b(S, w)^*$ be defined by

$$Tf[\mu, \nu] = \int_S f_\mu d\mu - \int_S f_\nu d\nu,$$

for all $f \in GL(S, w)$ and $[\mu, \nu] \in M_b(S, w)$. Then T is an isometric isomorphism such that

$$Tf(\nu) = \int_S f_\mu d\nu, \quad \text{for all } \nu \ll \mu \text{ and } \mu, \nu \in M_b^+(S, w).$$

It is easy to show that $\{(s_\mu) \in GL(S) : s_\mu$ is a simple function$\}$ is norm-dense in $GL(S)$ and $(\chi_{B_\mu}) \in GL(S)$ if and only if $\mu(B_\mu) = \mu(B_\nu)$, for all $\mu \ll \nu$, where B_ν and B_μ are Borel-measurable subsets of S. In particular $(\chi_B) \in GL(S)$, for all Borel-measurable $B \subseteq S$.

Let A be a convolution measure algebra in $M_b(S, w)$, i.e. a normed-closed solid subalgebra of $M_b(S, w)$. In other words, let $\mu \in M_b(S, w)$ and $\nu \in A$ such that $|\mu| \ll |\nu|$. Then $\mu \in A$. Analogous to the above arguments, one can define, the linear subspace $GL(A)$ of $\Pi\{L_\infty(|\mu|, w) : \mu \in A\}$ consisting of all $w-$generalised functions on S. Let $f, g \in GL(A)$ and $f.g := (f_\mu g_\mu/w)$. Then $GL(A)$ is a commutative C^*-algebra with identity. Also for each $F \in A^*$ there exists a unique generalized function $f = (f_\nu) \in GL(A)$ such that $F(\nu) = \int_S f_\nu d\nu$ for all $\nu \in A^+$. Define ${}_\mu f \in GL(A)$, by $({}_\mu f)_\nu := f_{\mu*\nu}$ for $\nu \in A$. Then ${}_\mu F$ is the linear functional corresponding to ${}_\mu f$. Since $A^{**} = GL(A)^*$, by using the Eberlein-Smulian Theorem, $\{{}_\mu F : \mu \in A_1\}$ is relatively weakly compact (r.w.c.) in A^* if and only if $\{{}_\mu F : \mu \in A_1\}$ is r.w.c. in the product space $\Pi\{L^\infty(|\nu|, w) : \nu \in A\}$ if and only if $\{f_{\mu*\nu} : \mu \in A_1\}$ is r.w.c. in $L^\infty(|\nu|, w)$ for each $\nu \in A$, by using the Kelley-Namioka Theorem stating that weak topology for a product of linear topological spaces is the product of the weak topologies for the coordinate spaces. Therefore $F \in Wap(A)$ if and only if $f_\nu \in Wap_\nu(S, w)$ for each $\nu \in A$, that is $f \in Wap_A(S, w)$, where $Wap_\nu(S, w)$ [resp. $Wap_A(S, w)$] is the set of all $w-$bounded Borel-measurable functions g on S [resp. $f \in G(A)$] such that $\{\frac{{}_s g}{w(s)} : s \in Supp(|\nu|)\}$ [resp. $\{\mu f : \mu \in A_1\}$] is relatively weakly compact in $L_\infty(|\nu|, w)$ [resp $GL(A)$].

Let g be a $w-$bounded Borel-measureable function on S and $P(S, w)$ be denoted the set of all $w-$probability η measures on S. Define

$${}_\eta g(s) := \int_S g(ts) d\eta(t) \quad \text{for } s \in S.$$

Since $P(S,w) = \overline{co}^{w^*}\{Ext(P(S,w))\}$ and $Ext(P(S,w)) = \{\frac{\bar{s}}{w(s)} | s \in S\}$ where $\bar{s} = \delta_s$ is the point measure at the point s, by Krein-Milman Theorem. In addition $\eta_\alpha \xrightarrow{w^*} \eta$, for some $\eta_\alpha \in co\{\frac{\bar{s}}{w(s)} : s \in \mathrm{Supp}(\eta)\}$, so ${}_{\eta_\alpha}g \xrightarrow{w} \eta^g$ [because, $\mu({}_\eta f) = \eta(f_\mu)$ for all $\mu \in M_b^+(S,w)$ and $f \in C_0(S,w)$; also Ω is a separately continuous functions, and $f_\mu/w(x) = \int_S (f/w)(xy)\Omega(x,y)d\mu w(y)$, so $f_\mu \in C_0(S,w)$]. It is routine to show that

$$\{\frac{{}_s g}{w(s)} : s \in \mathrm{Supp}(|\nu|)\} \subseteq \{{}_\eta g : \eta \ll |\nu|\} \text{ and}$$
$$\{{}_\eta g : \eta \ll |\nu|\} \subseteq \overline{co}^w\{\frac{{}_s g}{w(s)} : s \in Supp(|\nu|)\}.$$

Hence the map $(x,y) \to g/w(xy)\Omega(x,y)$ is cluster on $\mathrm{Supp}(|\nu|) \times Supp(|\nu|)$ if and only if $g \in Wap_\nu(S,w)$ if and only if the map $(\eta,\zeta) \to \int_S\int_S g(xy)d\eta(x)d\zeta(y)$ is cluster on $L^1(|\nu|,w)_1 \times L^1(|\nu|,w)_1$, see [18,§4.1, Ex.(2)].

Lemma 2. *The following statements are equivalent.*

(i) $M_b(S,w)$ *is regular.*

(ii) *For all* $(k_\eta) \in GL(S)$ *, the map* $(x,y) \to k_\eta(xy)\Omega(x,y)$ *is cluster on* $Supp(|\eta|) \times Supp(|\eta|)$.

(iii) *For all* $(\chi_{A_\eta}) \in GL(S)$, *the map* $(x,y) \to \chi_{A_\eta}(xy)\Omega(x,y)$ *is cluster on* $Supp(|\eta|) \times Supp(|\eta|)$.

(iv) $\ell_1(S,w)$ *is regular.*

Proof. Let $F \in M_b(S,w)^*$. Then there is a unique $f \in GL(S,w)$ such that $F[\mu,\nu] = \int_S f_\mu d\mu - \int_S f_\nu d\nu$, for all $[\mu,\nu] \in M_b(S,w)$. Let $(\mu_n),(\nu_m)$ be in $M_b^+(S,w)_1$. Then

$$\begin{aligned} F(\mu_n * \nu_m) &= \int_S f_{\mu_n*\nu_m} d\mu_n * \nu_m = \int_S\int_S f_{\mu_n*\nu_m}(xy)d\mu_n(x)d\nu_m(y) \\ &= \int_S\int_S f_\eta(xy)d\mu_n(x)d\nu_m(y) \\ &= \int_S\int_S f_\eta/w(xy)\Omega(x,y)d\mu_n w(x)d\nu_m w(y), \end{aligned}$$

where $\eta = \sum_{n=1}^{\infty} \frac{1}{2^n}\xi^n$ and $\xi = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}(\mu_k+\nu_k)$. Clearly $\eta \in M_b^+(S,w)$ and $\mu_n, \nu_m, \mu_n*\nu_m \ll \eta$, for all $n,m \in \mathbb{N}$.

By applying [18, §4.1, Ex.(2)], (i) and (ii) are equivalent. Since $\{(s_\mu) \in GL(S) : s_\mu$ is a simple function $\}$ is norm-dense in $GL(S)$ and the uniform limit of bounded cluster generalized functions is cluster. The equivalence of (ii) and (iii) follows. Clearly (iv) implies (iii), by [4, Th (3.2)]. Also $\ell_1(S,w)$ is a closed subalgebra of $M_b(S,w)$, so (iv) follows from (i), by [9, p.312].□

Furthermore, $M_b(S,w)$ is regular whenever $M_b(S)$ is, by [4, Cor. (3.4)].

We can now study the Arens regularity of weighted semigroup algebras $M_a^\ell(S,w)$. In [3], A.C. Baker and J.W. Baker studied the subalgebra $M_a^\ell(S)$, analogous to the group algebra. In [15], we studied the weighted semigroup algebra $M_a^\ell(S)$ inside $M_b(S)$, for completely regular semitopological semigroups S. We answered in the affirmative a question raised by

Baker and Dzinotyiweyi in [8, p. 9], whether $M_a^\ell(S)$ is a convolution measure algebra for semitopological semigroup.

Let $\mu \in M_b(S)$ such that the map $x \to |\mu|(x^{-1}K)$ is continuous on S, for all compact $K \subseteq S$. Then μ is called an absolutely continuous measure and the set of all such measures is denoted by $M_a^\ell(S)$. We define

$$M_a^\ell(S,w) = \{[\mu,\nu] \in M_b(S,w) : \mu w - \nu w \in M_a^\ell(S)\}.$$

The subspace $M_a^\ell(S,w)$ is a Banach algebra, left ideal and solid in $M_b(S,w)$, by [15, Th(3.2)].

Following [12, p.4], one can define a weighted absolutely continuous generalized function $f = (f_\mu)$ that satisfies the following conditions:

(i) $\|f\|_w = \sup\{\|f_\mu\|_{w,\mu} : \mu \in M_a^\ell(S,w)\}$ is finite,

(ii) If $\mu,\nu \in M_a^\ell(S,w)$ and $|\mu| \ll |\nu|$, then $f_\mu = f_\nu, |\mu| - a.e.$

Let $GL_a(S,w)$ be the set of all such functions. Since $M_a^\ell(S,w)$ is solid, in $M_b(S,w)$ by an argument similar to the one used in [16, Th(2.2)], one can show that $M_a^\ell(S,w)^*$ is isometrically linear-isomorphic to the space $GL_a(S,w)$, see [12, Th.(1.2)]. Let $F \in M_a^\ell(S,w)^*$. Then there exists $f \in GL_a(S,w)$ such that

$$F[\mu,\nu] = \int_S f_\mu d\mu - \int_S f_\nu d\nu,$$

for all $[\mu,\nu] \in M_a^\ell(S,w)$.

We define

$$WAP_a(S,w) := \{(f_\mu) \in GL_a(S,w) : f_\mu \in Wap_\mu(S,w), \text{ for all } \mu \in M_a^\ell(S,w)\},$$

where $g \in Wap_\mu(S,w)$ if and only if the map $(x,y) \to \frac{g(xy)}{w(x)w(y)}$ is cluster on $S_\mu \times S_\mu$, where $S_\mu := \mathrm{Supp}(|\mu|)$. Let f be a $w-$bounded Borel-measurable function on S. Then $F(\eta) := \int_S f d\eta^+ - \int_S f d\eta^-$ defines an element of $M_b(S,w)^*$.

Theorem 3. *The following statements are equivalent.*

(i) *$M_a^\ell(S,w)$ is regular.*

(ii) *$WAP_a(S,w) = GL_a(S,w)$.*

(iii) *For all $(k_\eta) \in GL_a(S)$, the map $(x,y) \to k_\eta(xy)\Omega(x,y)$ is cluster on $S_\eta \times S_\eta$.*

(iv) *For all $(\chi_{B_\eta}) \in GL_a(S)$, the map $(x,y) \to \chi_{B_\eta}(xy)\Omega(x,y)$ is cluster on $S_\eta \times S_\eta$.*

(v) *For all $(\chi_{B_\eta}) \in GL_a(S)$, the map $(x,y) \to \chi_{B_\eta}(xy)\Omega(x,y)$ has a separately continuous extension to a map from $\beta S_\eta \times \beta S_\eta$ into $[0,1]$.*

(vi) *For all $(\chi_{B_\eta}) \in GL_a(S)$ and for each pair of sequences $(x_n),(y_m)$ in S_η,*

$$\{\chi_{B_\eta}(x_n y_m)\Omega(x_n,y_m) : n < m\}^- \cap \{\chi_{B_\eta}(x_n y_m)\Omega(x_n,y_m) : n > m\}^- \neq \varnothing$$

Proof. Let $F \in M_a^\ell(S,w)^*$. Then there exists a unique $f \in GL_a(S,w)$ such that, for all $[\mu,\nu] \in M_a^\ell(S,w)$,

$$F[\mu,\nu] = \int_S f_\mu d\mu - \int_S f_\nu d\nu.$$

Let $(\mu_n),(\nu_m)$ be in $M_a^\ell(S,w)_1$. Then there is $\eta \in M_a^\ell(S,w)_1$ such that $\mu_n, \nu_m, \mu_n * \nu_m \ll \eta$, for all $n,m \in \mathbb{N}$. Thus,

$$\begin{aligned} F(\mu_n * \nu_m) &= \int_S f_{\mu_n * \nu_m} d\mu_n d\nu_m \\ &= \int_S \int_S f_\eta(xy) d\mu_n(x) d\nu_m(y) \\ &= \int_S \int_S k_\eta(xy)\Omega(x,y) d\mu_n w(x) d\nu_m w(y) \end{aligned}$$

where $k_\eta = f_\eta/w \in L_\infty(\eta)$.

Hence the map $(\mu,\nu) \to F(\mu * \nu)$ is cluster on $M_a^\ell(S,w)_1 \times M_a^\ell(S,w)_1$ if and only if the map $(x,y) \to k_\eta(xy)\Omega(x,y)$ is cluster on $S_\eta \times S_\eta$, that is $f_\eta \in Wap_\eta(S,w)$, see [18,§4.1, Ex.(2)]. Thus (i), (ii) and (iii) are equivalent. By an argument similar to the one used in Lem. (2), the equivalence of (iii), (iv) is immediate. The remaining equivalence are routine, see [4, Th.(3.2)].□

Corollary 4. *Let S be a subsemigroup of a locally compact group with positive Haar-measure. Then the following statements are equivalent.*

(i) $M_a^\ell(S,w)$ *is regular*

(ii) $WAP(S,w) = L_\infty(S,w)$

(iii) S *is finite or* Ω *is* $0-$*cluster.*

Proof. Since $M_a^\ell(S,w) = L^1(S,w)$, by [15, Th.(3.6)], we have $f_\mu = f_\lambda$ where λ is the Haar measure, $|\mu| - a.e$ for all $\mu \in M_a^\ell(S,w)$. Thus $GL_a(S,w) = L_\infty(S,w)$ and $WAP_a(S,w) = WAP(S,w)$, as Banach spaces.

Suppose $M_a^\ell(S,w)$ is regular and S is infinite. Let $(x_n),(y_m)$ be sequences of distinct elements in S. Since S is cancellative semigroup, so there are subsequences $(x'_n),(y'_m)$ of $(x_n),(y_m)$, respectively, such that

$$\{x'_n y'_m : n < m\} \cap \{x'_n y'_m : n > m\} = \varnothing.$$

Let $B = \{x'_n y'_m : n < m\}$. Then

$$\begin{aligned}\lim_n \lim_m \Omega(x'_n, y'_m) &= \lim_n \lim_m \chi_B(x'_n y'_m)\Omega(x'_n, y'_m)\\ &= \lim_m \lim_n \chi_B(x'_n y'_m)\Omega(x'_n, y'_m) = 0.\end{aligned}$$

Also Ω is cluster, by choosing $B = S$. Thus Ω is $0-$cluster.□

Remark 5. Let G be a locally compact topological group. Then a lot more is known about the Arens regularity of $L^1(G,w)$. For example G is discrte, whenever $L^1(G,w)$ is regular, by [7, Th.(1.22)]. Also the refree mentioned that $L^1(G,w)$ is strongly Arens irregular whenver the function $s \to w(s)w(s^{-1})$ is bounded on a set which cannot be covered by less than $k(S)$ many compacta, where $k(S)$ is the least cardinality of a compact covering of S; see [7, Th. (11.3)] and [13, Th.(1.2)].

The following example shows that, in general, $M_a^\ell(S,w) \neq L^1(S,w)$ and (Cor. 4) does not hold for any semigroup.

Example 6.

(i) Let $S = ([0,1],\cdot)$ where $x \cdot y = \min\{x+y, 1\}$ for $x,y \in S$, and let w be a weight function on S for which w^{-1} is bounded. If S has the Euclidean topology, then

$$M_a^\ell(S,w) = L^1(S,w) \oplus \{\lambda\overline{1} : \lambda \in \mathbb{R}\}.$$

(ii) If S has the usual multiplication, then $M_a^\ell(S,w) = \{\lambda\overline{0} : \lambda \in \mathbb{R}\}$ is regular and S is not discrte. But $M_b(S,w)$ is irregular, for some w, where $\overline{0}$ is the point mass measure. Also $f = (f_\alpha) \in GL_a(S,w)$ if and only if $f_\alpha(0) = f_\beta(0)$, for all $\alpha,\beta \in \mathbb{R}$. Thus $M_a^\ell(S,w)^* = \{F_f : f \in GL_a(S,w)\}$, where $F_f(\alpha\overline{0}) = \alpha f_\alpha(0)$ for $\alpha \in \mathbb{R}$.

(iii) If $S = (\mathbb{R}, \cdot)$ where $x \cdot y = x$ for $x, y \in S$, then $M_a^\ell(S) = \{0\}$, [for, $\mu(x^{-1}K) = \mu(S)\chi_K(x)$ for $x \in S$]. If $x \cdot y = y$, for $x, y \in S$, then $M_a^\ell(S, w) = M_b(S, w)$ is regular, but S is neihter finite nor Ω is $0-$cluster, where $w(x) = 1$ for $x \in \mathbb{R}$.

Let $F_a(S) := c\ell\{\bigcup \mathrm{Supp}(|\mu|) : \mu \in M_a^\ell(S, w)\}$ and $F_a(S) = S$. Then S is called a foundation semigroup, see [3]. If the map $(x, y) \to f(xy)\Omega(x, y)$ is cluster on $\mathrm{Supp}(|\mu|) \times \mathrm{Supp}(|\mu|)$ for all $\mu \in M_a^\ell(S, w)$, then $(x, y) \to f(xy)\Omega(x, y)$ is cluster on $F_a(S) \times F_a(S)$, for each bounded Borel-measurable function f on S, see (Th.7). We can now state the main results of this paper.

Theorem 7. *The following statements are equivalent.*

(i) $M_a^\ell(S, w)$ *is regular.*

(ii) $\ell_1(F_a(S), w)$ *is regular.*

(iii) $M_b(F_a(S), w)$ *is regular.*

Proof. Since $A = M_a^\ell(S, w)$ is a closed left ideal in $M_b(S, w)$, $F_a(S)$ is a closed left ideal in S. [For, let $x \in F_a(S)$ and $y \in S$. Then $x_\alpha \to x$, for some $x_\alpha \in \mathrm{Supp}(|\mu_\alpha|)$ and $\mu_\alpha \in A$. Since $yx_\alpha \in \mathrm{Supp}(\bar{y} * \mu_\alpha)$ and $\bar{y} * \mu_\alpha \in M_a^\ell(S, w)$, so $yx \in F_a(S)$]. Hence $F_a(S)$ is a locally compact topological semigroup. Let $x \in F_a(S)$. Then $\frac{\bar{x}}{w(x)} \in \overline{A_1}^{w^*}$. [For, $\frac{f}{w}(x_\alpha) = \frac{\bar{x}_\alpha}{w(x_\alpha)}(f) \to \frac{f}{w}(x) = \frac{\bar{x}}{w(x)}(f)$, for some $x_\alpha \in \mathrm{Supp}(\mu_\alpha)$ with $x_\alpha \to x$ and $\mu_\alpha \in A$, for each $f \in C_0(S, w)$]. Let $F \in A^*$. Then,

$$\{_\mu F : \mu \in \overline{A_1}^{w^*}\} \subseteq \overline{co}^w\{_\mu F : \mu \in A_1\},$$

see [12, p. 223].

Suppose A is regular, so $\{_\mu F : \mu \in \overline{A_1}^{w^*}\}$ is relatively weakly compact, by Krein-Smulian Theorem. Let f be a $w-$bounded Borel-measurable function on S and $G(\mu, \nu) := \int_S \int_S f(xy) d\mu(x) d\nu(y)$ for $\mu, \nu \in \overline{A_1}^{w^*}$. Then

$$\{G_\nu : \nu \in \overline{A_1}^{w^*}\} \subseteq \overline{co}^w\{G_\nu : \nu \in A_1\},$$

where $G_\nu(\mu) =_\mu F(\nu)$ for $\mu \in \overline{A_1}^{w^*}$ and $\nu \in A_1$. Let (μ_n) be in $\overline{A_1}^{w^*}$ and (ν_m) be in A_1. Then,

$$\begin{aligned}\lim_n \lim_m G_{\nu_m}(\mu_n) &= \lim_n \lim_m \mu_n F(\nu_m) \\ &= \lim_m \lim_n \mu_n F(\nu_m) = \lim_m \lim_n G_{\nu_m}(\mu_n)\end{aligned}$$

by Gröthendieck's Theorem. Therefore $\{G_\nu : \nu \in A_1\}$ is a relatively weakly compact set and so $\{G_\nu : \nu \in \overline{A_1}^{w^*}\}$. Hence the map $(\mu, \nu) \to G(\mu, \nu)$ is cluster on $\overline{A_1}^{w^*} \times \overline{A_1}^{w^*}$. Let $(x_n), (y_m)$ be sequences of distinct elements in $F_a(S)$ and $\mu_n = \frac{\overline{x_n}}{w(x_n)}, \nu_m = \frac{\overline{y_m}}{w(y_m)}$. Then $\mu_n, \nu_m \in \overline{A_1}^{w^*}$, for all $n, m \in \mathbb{N}$. Thus,

$$\begin{aligned}\lim_n \lim_m f/w(x_n y_m)\Omega(x_n, y_m) &= \lim_n \lim_m G(\mu_n, \nu_m) \\ &= \lim_m \lim_n G(\mu_n, \nu_m) = \lim_m \lim_n f/w(x_n y_m)\Omega(x_n, y_m)\end{aligned}$$

Therefore the map $(x, y) \to f/w(xy)\Omega(x, y)$ is cluster on $F_a(S) \times F_a(S)$, so $\ell_1(F_a(S), w)$ is regular, by [4, (3.1)]. The rest of the proof is immediate from Lem. (2) and [9, p. 312]. □

Corollary 8. *Let S be a foundation semigroup. Then the following statements are equivalent.*

(i) $M_a^\ell(S,w)$ *is regular.*

(ii) $M_b(S,w)$ *is regular.*

(iii) $\ell_1(S,w)$ *is regular.*

Since $M_a^\ell(S)$ [resp. $M_b(S)$] is regular if and only if $\ell_1(F_a(S))$ [resp. $\ell_1(S)$] is regualr, so by using (Th.7) and [4, Cor.(3.4)] the following statement is immediate.

Corollary 9.

(i) If $M_a^\ell(S)$ [resp. $M_b(S)$] is regualr, then so is $M_a^\ell(S,w)$ [resp. $M_b(S,w)$].

(ii) If Ω is $0-$cluster, then $M_a^\ell(S,w)$ is regular.

Remarks 10.

(i) Let S be either a locally compact or a complete metric semitopological seigroup. Then $M_a^\ell(S,w)$ and $M_b(S,w)$ are Banach algebras, see [15, Th.(2.2), (3.2)]. Furthermore, the results from Lem. (2) to Cor. (9) hold for such S.

(ii) By a similar argument as used in this paper, all results can be extended to a convolution measure algebra A in $M_b(S,w)$. For example, let $F_A(S) = c\ell\{\cup Supp(|\mu|) : \mu \in A\}$. Then A is regular if and only if $\ell_1(F_A(S),w)$ is regular.

The following example shows that $WAP(S,w) = \ell_\infty(S,w)$, for some non-compact foundation topological semigroups, see [12, Th.(3.12)].

Example 11. Let $S = \mathbb{Z}$, the integer numbers with addition and the discrete topology. Then S is a foundation semigroup such that $C^{-1}D, CD^{-1}$ are compact, for all compact sets $C, D \subseteq S$. Also $M_a^\ell(S,w) = \ell_1(S,w)$, where $w(n) = |n| + 1$ for $n \in \mathbb{Z}$, is regular [for, Ω is $0-$cluster]. Thus $Wap(S,w) = \ell_\infty(S,w)$, by Cor.(4), but S is not finite.

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