ON GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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ABSTRACT. We prove the boundedness of the generalized fractional integral operators and their modified versions on Morrey spaces and on Campanato spaces respectively. Our approach involves the Hardy-Littlewood maximal function and Young functions.

1. Introduction

Let \( \mathbb{R}^+ := (0, \infty) \). Associated to a function \( \rho : \mathbb{R}^+ \to \mathbb{R}^+ \), we define the mapping \( f \mapsto T_\rho f \) by

\[
T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x - y|)}{|x - y|^n} dy,
\]

for any suitable function \( f \) on \( \mathbb{R}^n \). We also define its modified version \( \tilde{T}_\rho \) by

\[
\tilde{T}_\rho f(x) := \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,
\]

where \( B_0 \) is the unit ball around the origin and \( \chi_{B_0} \) is the characteristic function of \( B_0 \). For example, if \( \rho(t) = t^\alpha, 0 < \alpha < n \), then \( T_\rho = I_\alpha \) — the fractional integral operator or the Riesz potential. Hence \( T_\rho \) may be viewed as a generalization of the fractional integral operator.

Next, for \( 1 \leq p < \infty \) and a suitable function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \), we define the generalized Morrey space \( \mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbb{R}^n) \) to be the set of all functions \( f \in L^p_{loc}(\mathbb{R}^n) \) for which

\[
\|f\|_{\mathcal{M}_{p,\phi}} := \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty,
\]

and the generalized Campanato space \( \mathcal{L}_{p,\phi} = \mathcal{L}_{p,\phi}(\mathbb{R}^n) \) to be the set of all functions \( f \in L^p_{loc}(\mathbb{R}^n) \) for which

\[
\|f\|_{\mathcal{L}_{p,\phi}} := \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{1/p} < \infty.
\]

Here the supremums are taken over all open balls \( B = B(a,r) \) in \( \mathbb{R}^n \), \( |B| \) denotes the Lebesgue measure of \( B \) in \( \mathbb{R}^n \), \( \phi(B) = \phi(r) \), and \( f_B := \frac{1}{|B|} \int_B f(y) dy \). For \( \mathcal{M}_{p,\phi} \), the function \( \phi(r) \) is usually required to be nonincreasing and \( r^{\alpha} \phi(r) \) to be nondecreasing. For \( \mathcal{L}_{p,\phi} \), it is \( \frac{\phi(r)}{r} \) that is required to be nonincreasing.

One may observe that \( f \) belongs to \( \mathcal{L}_{p,\phi} \) if there exist a constant \( C < \infty \) and, for every ball \( B \), a constant \( c_B < \infty \) such that

\[
\frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p} < C,
\]

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for we then have \( \|f\|_L_{p, \phi} < 2C \). Accordingly, \( M_{p, \phi} \subseteq L_{p, \phi} \). Further, if \( 1 \leq p \leq q < \infty \),
then \( M_{p, \phi} \supseteq M_{q, \phi} \) and \( L_{p, \phi} \supseteq L_{q, \phi} \). Unlike BMO (the space of Bounded Mean Oscillation functions), the Campanato space \( L_{p, \phi} \) is generally dependent of the exponent \( p \) (see [1] or [14]). For certain functions \( \phi \), \( M_{p, \phi} \) and \( L_{p, \phi} \) reduce to some classical spaces. For a brief history of these spaces, see [10], where further references are listed. For recent applications, see e.g. [6].

In [8, 9], Nakai showed that \( T_\rho \) is bounded from \( M_{1, \phi} \) to \( M_{1, \psi} \), while \( \tilde{T}_\rho \) is bounded from \( \mathcal{L}_{1, \phi} \) to \( \mathcal{L}_{1, \psi} \), under some appropriate conditions on \( \rho, \phi \) and \( \psi \). In [3], Eridani showed that, for \( 1 < p < \infty \), \( T_\rho \) is bounded from \( M_{p, \phi} \) to \( M_{p, \psi} \), while \( \tilde{T}_\rho \) is bounded from \( M_{p, \phi} \) to \( \mathcal{L}_{p, \psi} \), under similar conditions on \( \rho, \phi \) and \( \psi \). In this paper, we prove that, under some other conditions on \( \rho, \phi \) and \( \psi \), the operator \( T_\rho \) is bounded from \( M_{p, \phi} \) to \( M_{q, \psi} \), while \( \tilde{T}_\rho \) is bounded from \( \mathcal{L}_{p, \phi} \) to \( \mathcal{L}_{q, \psi} \), for \( 1 < p < q < \infty \).

Related results may be found in a recent work of Sugano and Tanaka [12].

2. Basic assumptions and facts

Let us begin with a few assumptions, particularly on the associated function \( \rho \), and some relevant facts that follow. Hereafter, \( C, C_1, C_2 \) and \( C_{p, q} \) denote positive constants, which are not necessarily the same from line to line.

In the definition of \( T_\rho \), we always assume that \( \rho \) satisfies the following conditions:

\begin{align}
(2.1) & \quad \int_0^1 \frac{\rho(t)}{t} \, dt < \infty; \\
(2.2) & \quad \frac{1}{2} \leq \frac{\rho(s)}{s} \leq \frac{1}{C} \leq \frac{\rho(r)}{r} \leq C_1.
\end{align}

For \( \tilde{T}_\rho \), we assume that \( \rho \) also satisfies two additional conditions, namely:

\begin{align}
(2.3) & \quad \int_r^\infty \frac{\rho(t)}{t^2} \, dt \leq C_2 \frac{\rho(r)}{r^2} \quad \text{for all } r > 0; \\
(2.4) & \quad \frac{1}{2} \leq \frac{\rho(s)}{s} \leq 2 \Rightarrow \frac{\rho(s)}{s} - \frac{s^{\alpha}}{\alpha} \leq C_3 |r^\alpha - s^{\alpha}|.
\end{align}

For example, the function \( \rho(r) = r^\alpha \), \( 0 < \alpha < n \), satisfies (2.1), (2.2) and (2.4). If \( 0 < \alpha < 1 \), then \( \rho(r) = r^\alpha \) also satisfies (2.3).

A function \( \rho \) satisfying (2.2) is said to satisfy the doubling condition (with a doubling constant \( C_1 \)). If \( \rho \) satisfies the doubling condition, then for every integer \( k \) and \( r > 0 \) we have

\[ \int_{2^kr}^{2^{k+1}r} \frac{\rho(t)}{t} \, dt \sim \rho(2^{k}r). \]

Further, it follows from the doubling condition that

\[ \rho(r) \leq C \int_0^r \frac{\rho(t)}{t} \, dt, \]

for every \( r > 0 \). Next, if \( \rho \) satisfies (2.1)–(2.4), then we have Nakai’s lemma which states that

\[ \int_{\mathbb{R}^n} \left( \frac{\rho(\sqrt{x_1-y})}{|x_1-y|^n} - \frac{\rho(\sqrt{x_2-y})}{|x_2-y|^n} \right) dy = 0 \]

for every choice of \( x_1 \) and \( x_2 \) (see [8]). For such a function \( \rho \), the operator \( \tilde{T}_\rho \) maps a constant to a constant, and hence it is well-defined from one generalized Campanato space to another.
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In the next section, we shall involve the so-called Hardy-Littlewood maximal operator \( M \), which is defined by
\[
Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy.
\]
A classical result for \( M \) is that it is bounded on \( L^p \) for \( 1 < p \leq \infty \) (see e.g. [13]). Now, if \( \phi \) satisfies the doubling condition and
\[
\int_r^\infty \frac{\varphi(t)\mu}{t} \, dt \leq C \phi(r)^p \quad \text{for all} \quad r > 0,
\]
for some \( 1 < p < \infty \), then there exists \( C_p > 0 \) such that
\[
\|Mf\|_{\mathcal{M}_{p,\alpha}} \leq C_p \|f\|_{\mathcal{M}_{p,\alpha}},
\]
that is, \( M \) is bounded on \( \mathcal{M}_{p,\alpha} \) (see [7]).

We shall also involve Young functions and Orlicz spaces in our discussion. A function \( \Phi : [0, \infty] \to [0, \infty] \) is called a Young function if \( \Phi \) is convex, \( \lim_{r \to 0^+} \Phi(r) = \Phi(0) = 0 \) and \( \lim_{r \to \infty} \Phi(r) = \Phi(\infty) = \infty \). A Young function is always nondecreasing. For a Young function \( \Phi \), we define \( \Phi^{-1}(r) = \inf\{s : \Phi(s) > r\} \) (with \( \inf\emptyset = \infty \)). If \( \Phi \) is continuous and bijective, then \( \Phi^{-1} \) is the usual inverse function. If a Young function \( \Phi \) satisfies
\[
0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty,
\]
then \( \Phi \) is continuous and bijective from \([0, \infty)\) to itself. In this case, the inverse function \( \Phi^{-1} \) is increasing, continuous and concave, and hence satisfies the doubling condition.

For a Young function \( \Phi \), we define the Orlicz space \( L^\Phi = L^\Phi(\mathbb{R}^n) \) to be the set of all locally integrable function \( f \) on \( \mathbb{R}^n \) for which \( \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\epsilon}\right) \, dx < \infty \) for some \( \epsilon > 0 \). We equip \( L^\Phi \) with the norm
\[
\|f\|_{L^\Phi} := \inf\left\{\epsilon > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\epsilon}\right) \, dx \leq 1\right\}.
\]
Note that for \( \Phi(r) = r^p \), \( 1 \leq p < \infty \), we have \( L^\Phi = L^p \). For further properties of Young functions and Orlicz spaces, see [8, 9]. For their relevance with our subject, see \[8, 9\].

One more terminology. A function \( \theta : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be *almost decreasing* if there exists a constant \( C > 0 \) such that \( \theta(r) \geq C \theta(s) \) for \( r \leq s \). *Almost increasing* functions can be defined analogously.

3. THE BOUNDEDNESS OF \( T_\rho \) ON MORREY SPACES

We shall here consider the generalized fractional integral operator \( T_\rho \). For \( 1 < p < q < \infty \), it is well-known that the fractional integral operator \( I_\alpha \) is bounded from \( L^p \) to \( L^q \) provided that \( \alpha/n = 1/p - 1/q \) (see e.g. [13], p. 354). More generally, \( I_\alpha \) is bounded from the Morrey space \( L^{p,\lambda} \) to \( L^{q,\mu} \) where \( \alpha/n = 1/p - 1/q, \) \( 0 \leq \lambda < n - \alpha p \) and \( p \mu = q \lambda \). (In our notation, \( L^{p,\lambda} = \mathcal{M}_{p,\lambda} \) with \( \phi(r) = r^{(\lambda-n)/p} \).) This result is due to Spanne (see [10], Theorem 5.4) and is reproved by Chiarenza and Frasca [2]. (Actually, Chiarenza and Frasca obtained a stronger result stating that \( I_\alpha \) is bounded from \( L^{p,\lambda} \) to \( L^{q,\lambda} \) where \( \alpha/(n-\lambda) = 1/p - 1/q \) and \( 0 < \lambda < n - \alpha p \), from which Spanne’s result follows as a corollary. Their proofs are valid for the case \( \lambda = 0 \).) The classical result can be recovered from Spanne’s by taking \( \lambda = 0 \) (because \( L^{0,0} = L^p \)). A further generalization of the above result is obtained by Nakai [7], who showed that \( I_\alpha \) is bounded from \( \mathcal{M}_{p,\lambda} \) to \( \mathcal{M}_{q,\mu} \) for appropriate functions \( \phi \) and \( \psi(r) = r^{\alpha/n}\phi(r) \). Here Spanne’s result can be recovered from Nakai’s by taking \( \phi(r) = r^{(\lambda-n)/p} \) with \( 0 \leq \lambda < n - \alpha p \) and \( \alpha/n = 1/p - 1/q \).

For \( T_\rho \), we have the results of Nakai [9] and Eridani [3] mentioned earlier. While \( T_\rho \) is a generalization of \( I_\alpha \), these results for \( T_\rho \) cannot, unfortunately, be viewed as a natural
generalization of those for $I_0$ (in the sense that we cannot recover the $L^p - L^q$ boundedness of $I_0$ from them). Recently, Eridani and Gunawan [4] obtains a generalization of Chiarenza-Frasca’s result, which has been reformulated by Gunawan [5] as follows. Notice that Chiarenza-Frasca’s result can be recovered by taking $\rho(r) = r^\alpha$ and $\phi(r) = r^{(\lambda-n)/p}$ with $0 \leq \lambda < n$ and $\alpha/(n - \lambda) = 1/p - 1/q$.

**Theorem 3.1.** [4, 5] Suppose that $\rho$ and $\phi$ satisfies the doubling condition. Suppose also that $\phi$ is surjective and satisfies the inequality (2.5) and

$$
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C\phi(r)^{p/q}, \quad \text{for all } r > 0,
$$

for $1 < p < q < \infty$. Then there exists $C_{p,q} > 0$ such that

$$
\|T_\rho f\|_{M_{ \rho, \phi}^{p,q}} \leq C_{p,q} \|f\|_{M_{ \rho, \phi}}
$$

that is, $T_\rho$ is bounded from $M_{ \rho, \phi}$ to $M_{q,\psi}^{p,q}$.

**Sketch of Proof.** The idea is to split the integral into two parts, namely

$$
T_\rho f(x) = \int_{|x-y|<R} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy + \int_{|x-y|\geq R} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy = I_1(x) + I_2(x).
$$

Then we estimate each part, by decomposing the integral further, diadically. For $I_1(x)$, we use the hypotheses on $\rho$ and $\phi$ and the property of the Hardy-Littlewood maximal operator $M$ to get

$$
|I_1(x)| \leq CM f(x) \phi(R)^{(p-q)/q}.
$$

For $I_2(x)$, we use the hypotheses on $\rho$ and $\phi$ and the fact that $f \in M_{ \rho, \phi}$ to obtain

$$
|I_2(x)| \leq C \|f\|_{M_{ \rho, \phi}} \phi(R)^{p/q}.
$$

By the surjectivity of $\phi$, we can choose $R > 0$ such that $\phi(R) = Mf(x) \|f\|_{M_{q,\psi}}^{-1}$, assuming that $f$ is not identically 0 and that $Mf(x) < \infty$ for every $x \in \mathbb{R}^n$. With this value of $\phi(R)$, the two estimates equal and hence, for every $x \in \mathbb{R}^n$, we have

$$
|T_\rho f(x)|^q \leq CM f(x)^p \|f\|_{M_{q,\psi}}^{q-p}.
$$

The desired inequality then follows from this and the fact that the maximal operator $M$ is bounded on $M_{q,\psi}$. (QED)

Our new result for $T_\rho$ is the following theorem, which may be considered as a generalization of Spanne’s result.

**Theorem 3.2.** Suppose that $\rho$ satisfies (2.1) and (2.2). Suppose further that $\frac{\rho(t)}{t}$ and $r^{-n/p} \int_0^r \frac{\rho(t)}{t} dt$ are almost decreasing, $\int_r^\infty \frac{\rho(t)t^{-n/p}}{t} dt \leq Cr^{-n/p} \int_0^r \frac{\rho(t)}{t} dt$, and there exist Young functions $\Phi_1$ satisfying (2.6) and $\Phi_2$ such that

$$
r^{-n/p} \int_0^r \frac{\rho(t)}{t} dt \sim \Phi_1^{-1}(r^{-n}) \quad \text{and} \quad \Phi_1^{-1}(r^{-n})\Phi_2^{-1}(r^{-n}) \sim r^{-n/q}
$$

for $1 < p \leq q < \infty$. If $\phi$ satisfies the doubling condition and

$$
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C\psi(r), \quad \text{for all } r > 0,
$$

then $T_\rho$ is bounded from $M_{ \rho, \phi}$ to $M_{q,\psi}$.
Proof. Let $B = B(a, r)$ be any ball in $\mathbb{R}^n$ and $\bar{B} = B(a, 2r)$. For every $x \in \mathbb{R}^n$, write
\[
T_p f(x) = \int_B f(y) \frac{\rho(|x - y|)}{|x - y|^n} dy + \int_{B^c} f(y) \frac{\rho(|x - y|)}{|x - y|^n} dy = I^1_B(x) + I^2_B(x).
\]
To estimate $I^1_B$, we set $\tilde{f} = f \chi_{\bar{B}}$. Then we have
\[
\left(\int_B |I^1_B(x)|^q dx \right)^{1/q} = \left(\int_{\mathbb{R}^n} |T_p \tilde{f}(x)\chi_B(x)|^q dx \right)^{1/q} \leq C \|T_p \tilde{f}\|_{L^{\Phi_1}} \|\chi_B\|_{L^{\Phi_2}}
\]
(see [11]). But $T_p$ is bounded from $L^p$ to $L^{\Phi_1}$ (see Corollary 3.2 of [8]) and $\|\chi_B\|_{L^{\Phi_2}} \leq (\Phi_2^{-1}(|B|^{-1}))^{-1}$. Hence we obtain
\[
\left(\int_B |I^1_B(x)|^q dx \right)^{1/q} \leq C \|\tilde{f}\|_{L^p} (\Phi_2^{-1}(|B|^{-1}))^{-1} \leq C r^{n/p} \rho(r) \|f\|_{\mathcal{M}_{p,a}} (\Phi_1^{-1}(r^{-n}) r^{n/q}) \leq C r^{n/q} \rho(r) \|f\|_{\mathcal{M}_{p,a}} \int_0^r \frac{\rho(t)}{t} dt \leq C r^{n/q} \rho(r) \|f\|_{\mathcal{M}_{p,a}}.
\]
Now we estimate $I^2_B$. Observe that for every $x \in B$ we have
\[
|I^2_B(x)| \leq \int_{|x-y| \geq r} |f(y)| \frac{\rho(|x-y|)}{|x-y|^n} dy.
\]
Hence, as in [3], we obtain
\[
|I^2_B(x)| \leq C \|f\|_{\mathcal{M}_{p,a}} \int_r^\infty \frac{\rho(t) \phi(t)}{t} dt \leq C \psi(r) \|f\|_{\mathcal{M}_{p,a}},
\]
whence
\[
\left(\int_B |I^2_B(x)|^q dx \right)^{1/q} \leq C r^{n/q} \psi(r) \|f\|_{\mathcal{M}_{p,a}}.
\]
Combining the two estimates, we get the desired inequality for $T_p$. (QED)

4. The boundedness of $\tilde{T}_p$ on Campanato spaces

We now turn to the modified fractional integral operator $\tilde{T}_p$. For $\rho(r) = r^\alpha$, the operator $\tilde{T}_p = \tilde{I}_\alpha$ is well-defined for $0 < \alpha < n + 1$ and is known to be bounded from $L^p$ to BMO when $p > 1$ and $\alpha = n/p$, from $L^p$ to $\text{Lip}_\beta$ when $p > 1$ and $0 < \alpha - n/p = \beta < 1$, from BMO to $\text{Lip}_\alpha$ when $0 < \alpha < 1$, and from $\text{Lip}_\beta$ to $\text{Lip}_\gamma$ when $0 < \alpha + \beta < 1$.

For a general function $\rho$, Nakai [8, 9] proved that $\tilde{T}_p$ is bounded from $\mathcal{L}_{1,\phi}$ to $\mathcal{L}_{1,\psi}$ for appropriate functions $\phi$ and $\psi$. For $\phi(r) = r^\beta$ with $0 \leq \beta \leq 1$, the space $\mathcal{L}_{1,\phi}$ reduces to BMO (when $\beta = 0$) or $\text{Lip}_\beta$ (when $0 < \beta \leq 1$). In this case, Nakai’s result covers the BMO–$\text{Lip}_\alpha$ and $\text{Lip}_\beta$–$\text{Lip}_\gamma$ results for $\tilde{I}_\alpha$. For $\phi(r) = r^\beta$ with $-n/p \leq \beta < 0$, $1 < p < \infty$, we have Eridani’s result [3] which covers the other results for $\tilde{I}_\alpha$. The following theorem is an extension of Eridani’s.

Theorem 4.1. Suppose that $\rho$ satisfies (2.1)–(2.4), and that $\phi$ satisfies the doubling condition and $\int_1^\infty \frac{\phi(t)}{t} dt < \infty$. If
\[
\int_r^\infty \frac{\phi(t)}{t} dt \int_0^r \frac{\rho(t)}{t} dt + r \int_r^\infty \frac{\rho(t) \phi(t)}{t^2} dt \leq C \psi(r) \quad \text{for all } r > 0,
\]

then $\tilde{T}_p$ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$ for $1 < p < \infty$.

Proof. Let $f \in \mathcal{L}_{p,\phi}$. For any ball $B = B(a, r)$ in $\mathbb{R}^n$, let $\tilde{B} = B(a, 2r)$ and, for every $x \in B$, write

$$I_B(x) := \int_{\mathbb{R}^n} (f(y) - f_B) \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|a - y|)(1 - \chi_{\tilde{B}}(y))}{|a - y|^n} \right) dy$$

$$I_B'(x) := \int_{\tilde{B}} (f(y) - f_B) \frac{\rho(|x - y|)}{|x - y|^n} dy$$

$$I_B''(x) := \int_{B^c} (f(y) - f_B) \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|a - y|)(1 - \chi_{\tilde{B}}(y))}{|a - y|^n} \right) dy$$

$$C_B^1 := \int_{\mathbb{R}^n} (f(y) - f_B) \left( \frac{\rho(|x - y|)(1 - \chi_{\tilde{B}}(y))}{|a - y|^n} - \frac{\rho(|y|)(1 - \chi_{\tilde{B}}(y))}{|y|^n} \right) dy$$

$$C_B^2 := \int_{\mathbb{R}^n} f_B \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y|)(1 - \chi_{\tilde{B}}(y))}{|y|^n} \right) dy.$$ 

Then clearly

$$\tilde{T}_p f(x) = (C_B^1 + C_B^2) = I_B(x) = I_B'(x) + I_B''(x),$$

and one may observe that $C_B^1$ and $C_B^2$ are well-defined constants (see [8]).

To estimate $I_B'$, write $f := (f - f_B)\chi_{\tilde{B}}$ and $\phi(r) := \int_r^{2r} \frac{dt}{t}$. Then, as in [3], we have

$$|I_B'(x)| \leq \int_{\tilde{B}} |\tilde{f}(y)||\frac{\rho(|x - y|)}{|x - y|^n}| dy \leq C M \tilde{f}(x) \int_0^r \frac{\rho(t)}{t} dt \leq C \frac{\rho(r)}{\phi(r)} M \tilde{f}(x).$$

By $L^p$ boundedness of $M$ and Fact 6.2 (see Appendices), we obtain

$$\frac{1}{\psi(r)} \left( \frac{1}{|B|} \int_{B} |I_B'(x)|^p dx \right)^{1/p} \leq \frac{C}{\phi(r)|B|^{1/p}} \left( \int_{B} [M\tilde{f}(x)]^p dx \right)^{1/p} \leq \frac{C_p}{\phi(r)|B|^{1/p}} \|\tilde{f}\|_{L^p}$$

$$\leq \frac{C_p}{\phi(r)|B|^{1/p}} \left( \|f - \sigma(f)\|_{L^p} + |\tilde{B}|^{1/p} |f_B - \sigma(f)| \right) \leq C_p \|f - \sigma(f)\|_{M_{p, \psi}} + \|f\|_{L^p} \leq C_p\|f\|_{L^p},$$

where $\sigma(f) = \lim_{r \to \infty} f_{B(0, r)}$.

It then remains to estimate $I_B''$. By (2.2) and (2.4), we have

$$|I_B''(x)| \leq \int_{B^c} |f(y) - f_B| \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y - a|)}{|y - a|^n} \right) dy$$

$$\leq C|x - a| \int_{|y - a| \geq 2r} |f(y) - f_B| \frac{\rho(|y - a|)}{|y - a|^n} dy$$

$$= C|x - a| \sum_{k=2}^{\infty} \int_{2^{k-1}r \leq |y - a| < 2^kr} |f(y) - f_B| \frac{\rho(|y - a|)}{|y - a|^n} dy$$

$$\leq C|x - a| \sum_{k=2}^{\infty} \frac{\rho(2^kr)}{2^kr^n} \int_{|y - a| < 2^kr} |f(y) - f_B| dy$$

$$\leq C|x - a| \sum_{k=2}^{\infty} \frac{\rho(2^kr)}{2^kr^n} \left( \frac{1}{(2^kr)^n} \int_{|y - a| < 2^kr} |f(y) - f_B|^p dy \right)^{1/p}.$$
But, for each \( k \geq 2 \), we have
\[
\left( \frac{1}{|B(a,2^k r)|} \int_{B(a,2^k r)} |f(y) - f_B| \, dy \right)^{1/p} \leq C \|f\|_{L_{p,\phi}} \int_{2r}^{2^{k+1} r} \frac{\phi(s)}{s} \, ds
\]
(see Fact 6.1 in Appendices). Hence, by (2.2), (2.3), and our assumption on \( \phi \) and \( \psi \), we obtain
\[
|I_B^2(x)| \leq C \|x - a\|f\|_{L_{p,\phi}} \sum_{k=2}^{\infty} \frac{\rho(2^k r)}{2^k r} \int_{2r}^{2^{k+1} r} \frac{\phi(s)}{s} \, ds \leq C \|x - a\|f\|_{L_{p,\phi}} \int_{2r}^{\infty} \frac{\rho(t)}{t^2} \left( \int_{2r}^{t} \frac{\phi(s)}{s} \, ds \right) \, dt
\]
\[
= C \|x - a\|f\|_{L_{p,\phi}} \int_{2r}^{\infty} \frac{\rho(t)}{t^2} \left( \int_{2r}^{t} \frac{\phi(s)}{s} \, ds \right) \, dt
\]
\[
\leq C r \|f\|_{L_{p,\phi}} \int_{2r}^{\infty} \frac{\rho(s)\phi(s)}{s^2} \, ds \leq C \psi(r) \|f\|_{L_{p,\phi}},
\]
whence
\[
\frac{1}{\psi(r)} \left( \frac{1}{|B|} \int_B |I_B^2(x)|^p \, dx \right)^{1/p} \leq C \|f\|_{L_{p,\phi}}.
\]
This completes the proof. (QED)

The results for \( T_\rho \) indicate that the modified fractional integral operator \( \widetilde{T}_\rho \) must also be bounded from \( L_{p,\phi} \) to \( L_{q,\psi} \) for appropriate functions \( \phi \) and \( \psi \). Indeed, we have the following analog of Theorem 3.2 for \( \widetilde{T}_\rho \).

**Theorem 4.2.** Suppose that \( \rho \) satisfies (2.1) – (2.4). Suppose further that \( \frac{\rho(t)}{t^p} \) and \( r^{-n/p} \int_0^r \frac{\rho(t)}{t} \, dt \) are almost decreasing, \( \int_0^\infty \frac{\rho(t)^{n/p}}{t} \, dt \leq C r^{-n/p} \int_0^r \frac{\rho(t)}{t} \, dt \), and there exist Young functions \( \phi_1 \) satisfying (2.6) and \( \phi_2 \) such that
\[
r^{-n/p} \int_0^r \frac{\rho(t)}{t} \, dt \sim \Phi_1^{-1}(r^{-n}) \quad \text{and} \quad \Phi_1^{-1}(r^{-n})\Phi_2^{-1}(r^{-n}) \sim r^{-n/q}
\]
for \( 1 < p \leq q < \infty \). If \( \phi \) satisfies the doubling condition and
\[
\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt \leq C \psi(r), \quad \text{for all } r > 0,
\]
then \( \widetilde{T}_\rho \) is bounded from \( L_{p,\phi} \) to \( L_{q,\psi} \).

**Proof.** Let \( f \in L_{p,\phi} \). For any ball \( B = B(a,r) \) in \( \mathbb{R}^n \), let \( \widetilde{B} = B(a,2r) \) and define \( I_B, I_{B_1}, I_{B_2}, C_B^1 \) and \( C_B^2 \) as in the proof of Theorem 4.1.

To estimate \( I_B^1 \), write \( \tilde{f} := (f - f_B)1_{\widetilde{B}} \) as before. Then \( I_B^1 = T_\rho \tilde{f} \) (it is \( T_\rho \), not \( \widetilde{T}_\rho \), and hence (as in the proof of Theorem 3.2) we have
\[
\left( \int_B |I_B^1(x)|^q \, dx \right)^{1/q} \leq C r^{n/q} \psi(r) \|f\|_{L_{p,\phi}}.
\]
Meanwhile, we have the same estimate for $I_2^B$ as in the proof of Theorem 4.1, whence

$$\left( \int_B \left| I_2^B(x) \right|^q \, dx \right)^{1/q} \leq C r^{n/q} \psi(r) \|f\|_{L^p,\psi}.$$

The desired inequality for $\tilde{T}_\rho$ follows immediately from these two estimates. (QED)

5. Examples

Let $\ell$ be a continuous function on $(0, \infty)$ such that

$$\ell(r) = \begin{cases} 1/(\log 1/r) & \text{for small } r > 0, \\ \log r & \text{for large } r > 0. \end{cases}$$

We assume that $\ell$ is Lipschitz continuous on every closed and bounded interval contained in $(0, \infty)$. Then $\ell(r) \sim \ell(r^n) \sim 1/(1/r^n)$. Let

$$(5.1) \quad 1 < p < \infty, \ 0 < \alpha < n/p, \ \beta \geq 0 \quad \text{and} \quad \rho(r) = r^{\alpha} \ell^{\beta}(r).$$

Then $\rho$ satisfies the assumption in Theorem 3.2. Moreover, if $0 < \alpha < 1$, then $\rho$ satisfies the assumption in Theorem 4.2. In particular, one may observe that

$$\int_0^r \frac{\rho(t)}{t} \, dt \sim r^{\alpha} \ell^{\beta}(r).$$

Example 5.1. Take $\phi(r) = r^{\alpha} \ell^{\beta}(r)$ where $1/q = 1/p - \alpha/n$. Then $\phi(r)^{(p-q)/q} = \rho(r)$ and

$$\int_0^\infty \frac{\rho(t) \phi(t)}{t} \, dt \sim \phi(r)^{p/q}.$$

From Theorem 3.1 it follows that $T_\rho$ is bounded from $M_{p,\phi}$ to $M_{q,\phi^{p/q}}$.

Now, for $\beta > 0$, let $\Phi_i$ ($i = 1, 2$) be Young functions and

$$\Phi_1(s) \sim s^{\beta}(s) \quad \text{for } s > 0, \quad \frac{1}{q} = 1/p - \alpha/n,$$

$$\Phi_2(s) = \begin{cases} 1/\exp(1/s^{1/\beta}) & \text{for small } s > 0, \\ \exp(s^{1/\beta}) & \text{for large } s > 0. \end{cases}$$

Then

$$\Phi_1^{-1}(r) \sim r^{1/q}/\ell^{\beta}(r), \quad \Phi_2^{-1}(r) \sim \ell^{\beta}(r),$$

and

$$r^{-n/p} \int_0^r \frac{\rho(t)}{t} \, dt \sim \Phi_1^{-1} \left( \frac{1}{t^{\alpha}} \right) \sim \left( \frac{1}{t^n} \right)^{1/q} \ell^{\beta}(r),$$

$$\Phi_1^{-1}(r) \Phi_2^{-1}(r) \sim r^{1/q}.$$

For $\beta = 0$, let

$$\Phi_1(s) = s^q, \quad \frac{1}{q} = 1/p - \alpha/n, \quad \Phi_2(s) = \begin{cases} 0 & \text{for } 0 \leq s < 1, \\ +\infty & \text{for } s \geq 1. \end{cases}$$

Then

$$\Phi_1^{-1}(r) \sim r^{1/q}, \quad \Phi_2^{-1}(r) \equiv 1.$$
Example 5.2. Under the condition (5.1), let \( \phi(r) r^{n/p} \) be almost increasing and \( \phi(r) r^{\alpha + \epsilon} \) be almost decreasing for some \( \epsilon > 0 \). Then

\[
\int_r^\infty \frac{\rho(t)\phi(t)}{t} \, dt \sim \rho(r)\phi(r).
\]

From Theorem 3.2 it follows that \( T_\rho \) is bounded from \( M_{p,\phi} \) to \( M_{q,\psi} \) for \( \psi(r) = \rho(r)\phi(r) \).

(In the case \( \beta = 0 \), this boundedness also follows from Theorem 3 in [7].)

Example 5.3. Under the condition (5.1) and \( 0 < \alpha < 1 \), let \( \phi(r) r^{n/p} \) be almost increasing and \( \phi(r) r^{\alpha - 1 + \epsilon} \) be almost decreasing for some \( \epsilon > 0 \). Then

\[
\int_r^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt \sim \rho(r)\phi(r).
\]

From Theorem 4.2 it follows that \( \tilde{T}_\rho \) is bounded from \( L_{p,\phi} \) to \( L_{q,\psi} \) for \( \psi(r) = \rho(r)\phi(r) \). (If \( \psi(r) \) is almost increasing, then this boundedness also follows from Theorem 3.6 in [9] since \( L_{p,\phi} \subset L_{1,\phi} \) and \( L_{q,\psi} = L_{1,\psi} \).)

Let us now consider the case where

\[
(5.2) \quad 1 < p = q < \infty, \quad \beta > 0 \quad \text{and} \quad \rho(r) = \begin{cases} (\log 1/r)^{-\beta - 1} & \text{for small } r > 0, \\ (\log r)^{\beta - 1} & \text{for large } r > 0. \end{cases}
\]

We assume that \( \rho \) is Lipschitz continuous on every closed and bounded interval contained in \((0, \infty)\). Then \( \int_0^r \frac{\rho(t)}{t} \, dt \sim r^{-\beta}(r) \) and \( \rho \) satisfies the assumptions in Theorem 3.2 and in Theorem 4.2. Now let \( \Phi_i \) (\( i = 1, 2 \)) be Young functions and

\[
\Phi_1(s) \sim s^\rho \ell^\beta(s), \quad \Phi_2(s) = \begin{cases} 1/\exp(1/s^{1/\beta}) & \text{for small } s > 0, \\ \exp(s^{1/\beta}) & \text{for large } s > 0. \end{cases}
\]

Then

\[
\Phi_1^{-1}(r) \sim r^{1/p} \ell^{-\beta}(r), \quad \Phi_2^{-1}(r) \sim \ell^\beta(r),
\]

and

\[
r^{-n/p} \int_0^r \frac{\rho(t)\phi(t)}{t} \, dt \sim \Phi_1^{-1}\left(\frac{1}{r^n}\right) \sim \left(\frac{1}{r^n}\right)^{1/p} \ell^\beta(r),
\]

\[
\Phi_1^{-1}(r)\Phi_2^{-1}(r) \sim r^{1/p}.
\]

Example 5.4. Under the condition (5.2), let \( \phi(r) = r^\delta \ell^\gamma(r) \), for \( -n/p < \delta < 0 \) and \(-\infty < \gamma < +\infty \), or for \( \delta = -n/p \) and \( 0 \leq \gamma < +\infty \). Then

\[
\int_r^\infty \frac{\rho(t)\phi(t)}{t} \, dt \sim \rho(r)\phi(r) \leq C\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt \sim r^{\delta + \gamma}(r).
\]

From Theorem 3.2 it follows that \( T_\rho \) is bounded from \( M_{p,\phi} \) to \( M_{p,\psi} \) for \( \psi(r) = r^{\delta + \gamma}(r) \). (This boundedness also follows from Theorem 1 in [3].)

Example 5.5. Under the condition (5.2), let \( \phi(r) = r^\delta \ell^\gamma(r) \), for \( -n/p < \delta < 1 \) and \(-\infty < \gamma < +\infty \), or for \( \delta = -n/p \) and \( 0 \leq \gamma < +\infty \). Then

\[
\int_r^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt \sim \rho(r)\phi(r) \leq C\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt \sim r^{\delta + \gamma}(r).
\]

From Theorem 4.2 it follows that \( \tilde{T}_\rho \) is bounded from \( L_{p,\phi} \) to \( L_{p,\psi} \) for \( \psi(r) = r^{\delta + \gamma}(r) \). (If \( \delta > 0 \), then \( \int_r^\infty \frac{\phi(t)}{t} \, dt \sim \phi(r) \) and so this boundedness also follows from Theorem 4.1. If \( \delta > 0 \), or if \( \delta = 0 \) and \( \beta + \gamma \geq 0 \), then this boundedness also follows from Theorem 3.6 in [9] since \( L_{p,\phi} \subset L_{1,\phi} \) and \( L_{p,\psi} = L_{1,\psi} \).)
Fact 6.1. If \( f \in \mathcal{L}_{p,\phi} \) for some \( 1 \leq p < \infty \) and \( \phi \) satisfies the doubling condition, then for any ball \( B = B(a, r) \) in \( \mathbb{R}^n \) and \( k = 1, 2, 3, \ldots \), we have
\[
\left( \frac{1}{|B(a, 2^kr)|} \int_{B(a, 2^kr)} |f(y) - f_B|^p dy \right)^{1/p} \leq C \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2^kr} \frac{\phi(t)}{t} dt,
\]
where \( C > 0 \) is dependent only on \( n \) and the doubling constant of \( \phi \).

Proof. By Minkowski’s inequality, we have
\[
\left( \frac{1}{|B(a, 2^kr)|} \int_{B(a, 2^kr)} |f(y) - f_B|^p dy \right)^{1/p} \leq \left( \frac{1}{|B(a, 2^kr)|} \int_{B(a, 2^kr)} |f(y) - f_{B(a, 2^j r)}|^p dy \right)^{1/p} + \sum_{j=0}^{k-1} |f_{B(a, 2^j r)} - f_{B(a, 2^{j+1} r)}|.
\]
But, for each \( j = 0, \ldots, k - 1 \), one may observe that
\[
|f_{B(a, 2^j r)} - f_{B(a, 2^{j+1} r)}| \leq \frac{1}{|B(a, 2^{j+1} r)|} \int_{B(a, 2^{j+1} r)} |f(y) - f_{B(a, 2^{j+1} r)}| dy \leq 2^n \left( \frac{1}{|B(a, 2^{j+1} r)|} \int_{B(a, 2^{j+1} r)} |f(y) - f_{B(a, 2^{j+1} r)}|^p dy \right)^{1/p} \leq C \phi(2^{j+1} r) \|f\|_{\mathcal{L}_{p,\phi}}.
\]
Summing up, we get
\[
\frac{1}{|B(a, 2^kr)|} \int_{B(a, 2^kr)} |f(y) - f_B| dy \leq C \|f\|_{\mathcal{L}_{p,\phi}} \sum_{j=0}^{k-1} \phi(2^{j+1} r)
\leq C \|f\|_{\mathcal{L}_{p,\phi}} \sum_{j=0}^{k-1} \int_{2^j r}^{2^{j+1} r} \frac{\phi(t)}{t} dt = C \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2^kr} \frac{\phi(t)}{t} dt,
\]
since \( \phi \) satisfies the doubling condition. (QED)

Fact 6.1 can actually be generalized as follows.

Fact 6.1’. Let \( f \in \mathcal{L}_{p,\phi} \) for some \( 1 \leq p < \infty \) and \( \phi \) satisfy the doubling condition. If \( B(a, r) \subset B(b, s) \) in \( \mathbb{R}^n \), then
\[
|f_{B(a, r)} - f_{B(b, s)}| \leq C \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2s} \frac{\phi(t)}{t} dt,
\]
where \( C > 0 \) is dependent only on \( n \) and the doubling constant of \( \phi \).
Proof. Indeed, if $2^{-k-1}s \leq r < 2^{-k}s$, then, choosing balls $B_j$ ($j = 0, 1, \ldots, k$) so that the radius of $B_j$ is $2^{-j}s$ and $B(a, r) \subset B_k \subset B_{k-1} \subset \cdots \subset B_0$, we have

$$|f_{B(a, r)} - f_{B(b, r)}| \leq |f_{B(a, r)} - f_{B_k}| + \sum_{j=0}^{k-1} |f_{B_{j+1}} - f_{B_{j+1}}|$$

$$\leq C\|f\|_{L_{p, \phi}} \sum_{j=0}^{k} \phi(2^{-j}s) \leq C\|f\|_{L_{p, \phi}} \int_{2^{-k}s}^{2s} \frac{\phi(t)}{t} dt. \quad \text{(QED)}$$

**Fact 6.2.** Let $1 \leq p < \infty$, $\phi$ satisfy the doubling condition and $\int_1^{\infty} \frac{\phi(t)}{t} dt < \infty$. If $f \in L_{p, \phi}$, then $f_{B(a, r)}$ converges as $r \to \infty$ and

$$\|f - \lim_{r \to \infty} f_{B(a, r)}\|_{M_{p, \phi}} \leq C\|f\|_{L_{p, \phi}},$$

where $\tilde{\phi}(r) = \int_{r}^{\infty} \frac{\phi(t)}{t} dt$ and $C > 0$ is dependent only on $n$ and the doubling constant of $\phi$.

**Proof.** From Fact 6.1 it follows that there exists a constant $\sigma(f)$, independent of $a \in \mathbb{R}^n$, such that

$$\lim_{r \to \infty} f_{B(a, r)} = \sigma(f),$$

and

$$|f_{B(a, r)} - \sigma(f)| \leq C\|f\|_{L_{p, \phi}} \int_{r}^{\infty} \frac{\phi(t)}{t} dt.$$

Hence we have, for all $B = B(a, r)$,

$$\left(\frac{1}{|B|} \int_{B} |f(x) - \sigma(f)|^p dx\right)^{1/p} \leq \left(\frac{1}{|B|} \int_{B} |f(x) - f_{B}|^p dx\right)^{1/p} + |f_{B} - \sigma(f)| \leq \|f\|_{L_{p, \phi}} \tilde{\phi}(r) + C\|f\|_{L_{p, \phi}} \tilde{\phi}(r) \leq C\|f\|_{L_{p, \phi}} \tilde{\phi}(r). \quad \text{(QED)}$$

**References**


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