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ON PSEUDO-BCK ALGEBRAS AND PORIMS

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Dedicated to Jan Jakubik on his 80^{th} birthday

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Abstract. In this paper we generalize some of the results from [18] to the noncommutative case. We present reversed left-pseudo-BCK(pP) algebras, reversed left-pseudo-BCK(pRP) algebras and left-X-pseudo-BCK(pR) algebras (i.e porims), left-X-pseudo-BCK(pRP) algebras. We prove they are categorically equivalent.

1 Introduction

The notions of "left" and "right" pseudo-algebras are connected with the left-continuity of a pseudo-t-norm and with the right-continuity of a pseudo-t-conorm on $[0,1]$, respectively, and are discussed in detail in [10]. We can also say that they are connected with the "negative (left)" cone and with the "positive (right)" cone, respectively, of an arbitrary l-group (lattice-ordered group).

At the beginning, pseudo-t-norms and pseudo-t-conorms were defined on $[0,1]$, as follows [10]: a binary operation $\odot$ on the real interval $[0,1]$ is a *pseudo-t-norm* iff it is associative, non-decreasing (isotone) in the first argument and in the second argument, i.e. if $x \geq y$, then $x \odot z \geq y \odot z$ and $z \odot x \geq z \odot y$, for every $x, y, z \in [0,1]$, and it has 1 as neutral element, i.e. $x \odot 1 = x = 1 \odot x$, for every $x \in [0,1]$. A binary operation $\oplus$ on the real interval $[0,1]$ is a *pseudo-t-conorm* iff it is associative, non-decreasing in the first argument and in the second argument and it has 0 as neutral element.

We define, more generally, a *pseudo-t-norm, $\odot$, on a poset* $(A, \geq, 1)$ *with greatest element* 1 iff the above mentioned axioms are fulfilled. We define also, more generally, a *pseudo-t-conorm, $\oplus$, on a poset* $(A, \leq, 0)$ *with smallest element* 0 iff the above mentioned corresponding axioms are fulfilled.

A commutative pseudo-t-norm (pseudo-t-conorm) is a t-norm (t-conorm, respectively).

Recall now the following definition: a *partially ordered, integral left-monoid* (see Definition 3.1 and Remark 3.2) is an algebra $(A, \geq, \odot, 1)$ such that: $(A, \geq, 1)$ is a poset with greatest element 1, $(A, \odot, 1)$ is a left-monoid (i.e. $\odot$ is associative and has 1 as neutral element) and $\odot$ is non-decreasing in the first and in the second argument (or, $\odot$ is compatible with $\geq$); *integral* means that the greatest element of the poset $(A, \geq)$ coincides with the neutral element of the left-monoid.

The inverse notion, the *partially ordered, integral right-monoid*, is an algebra $(A, \leq, \oplus, 0)$ such that: $(A, \leq, 0)$ is a poset with smallest element 0, $(A, \oplus, 0)$ is a right-monoid (i.e. $\oplus$ is associative and has 0 as neutral element) and $\oplus$ is non-decreasing in the first and in the second argument (or, $\oplus$ is compatible with $\leq$).

Remark 1.1 The statement: "$\odot$ is a pseudo-t-norm on the poset $(A, \geq, 1)$ with greatest element 1" is equivalent with the statement: "the algebra $(A, \geq, \odot, 1)$ is a partially ordered,

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integral left-monoid". The statement: "$\oplus$ is a pseudo-t-conorm on the poset $(A, \leq, 0)$ with smallest element 0" is equivalent with the statement "the algebra $(A, \leq, \oplus, 0)$ is a partially ordered, integral right-monoid".

If the algebra is initially defined as "right" (or as "left") algebra, then we shall put the word "right-" (or "left-", respectively) between parenthesis and it can be omitted.

The passage from the "right" algebra to its inverse, the "left" algebra, is made by replacing everywhere the pseudo-t-conorm $\oplus$ by the pseudo-t-norm $\odot$, the pseudo-coresiduum $(\rightarrow_R, \rightsquigarrow_R)$ by the pseudo-residuum $(\rightarrow = \rightarrow_L,\ \rightsquigarrow = \rightsquigarrow_L)$ ("R" comes from "right", "L" comes from "left"), by replacing 0 by 1 (and 1 by 0), by replacing the binary relation $\leq$ by its inverse relation, $\geq$.

The passage from the "left" algebra to its inverse, the "right" algebra, is made by replacing everywhere the pseudo-t-norm $\odot$ by the pseudo-t-conorm $\oplus$, the pseudo-residuum $(\rightarrow = \rightarrow_L,\ \rightsquigarrow = \rightsquigarrow_L)$ by the pseudo-coresiduum $(\rightarrow_R,\ \rightsquigarrow_R)$, by replacing 1 by 0 (and 0 by 1), by replacing the binary relation $\geq$ by its inverse relation, $\leq$.

The motivation for this paper was the curiosity to find out how it looks the generalization to the noncommutative case of some structures and results from [18].

In the following two sections, we discuss about pseudo-BCK algebras and porims and, in connection with them, we introduce and study reversed left-pseudo-BCK(pP), reversed left-pseudo-BCK(pRP) algebras and left-X-pseudo-BCK(pR), left-X-pseudo-BCK(pRP) algebras.

We give an equivalent definition of reversed left-pseudo-BCK algebras (Definition 2.8) in order to be able to define the pseudo-residuum $(\rightarrow, \rightsquigarrow)$ on a poset $(A, \geq, 1)$ and the left-residoid (Definition 2.9).

The most important result of the paper is Theorem 3.12, which is the generalization to the noncommuative case of the basic theorem of [18], namely of Theorem 2.56. Based on this Theorem 3.12, someone can generalize all the other results from [18].

Note that a kind of duality appears between the two "general worlds", of "$\rightarrow, \rightsquigarrow$, 1" and of "$\odot$, 1": the properties of the pseudo-residuum $(\rightarrow, \rightsquigarrow)$ are in correspondence with the properties of the pseudo-t-norm $\odot$. We shall point out from time to time this correspondence in the paper.

We assume the reader is familiar with [18] and [17], but the paper is self-contained as much as possible. The old, already known results are presented without proof.

2 Reversed left-pseudo-BCK(pP) and left-pseudo-BCK(pRP) algebras

In this section we recall the history and the basic facts about pseudo-BCK algebras. Reversed left-pseudo-BCK(pP) algebras are presented in Definition 2.11, while reversed left-pseudo-BCK(pRP) algebras are introduced by Definition 2.12. They are categorically equivalent (Theorem 2.15). Other important results in this section are Theorem 2.7, Definition 2.8, Definition 2.9 and Lemma 2.13.

The notion of pseudo-BCK algebra was introduced in 2001 [14], as a noncommutative generalization of Iséki's (right-) BCK algebras [15], [20].

Definition 2.1 A *(right-) pseudo-BCK algebra* [14] is a structure $\mathcal{A} = (A, \leq, \star, \circ, 0)$, where "$\leq$" is a binary relation on A, "$\star$" and "$\circ$" are binary operations on A and "0" is an element of A, verifying, for all $x, y, z \in A$, the axioms:

(I-R) $(x \star y) \circ (x \star z) \leq z \star y, \quad (x \circ y) \star (x \circ z) \leq z \circ y$,

(II-R) $x \circ (x \star y) \leq y, \quad x \star (x \circ y) \leq y$,

(III-R) $x \leq x$,

(IV-R) $0 \leq x$,

(V-R) $x \leq y,\ y \leq x \Longrightarrow x = y$, (VI-R) $x \leq y \Longleftrightarrow x \star y = 0 \Longleftrightarrow x \circ y = 0$.

We introduce the following defintion, which is the noncommutative generalization of a "BCK algebra with condition (S)" [19]: a (right-) pseudo-BCK algebra $(A, \leq, \star, \circ, 0)$ is *with condition (pS) (pseudo-sum)* if there exist, for all $x, y \in A$, the greatest element of the set $\{z \mid z \star y \leq x\}$ and the greatest element of the set $\{z \mid z \circ x \leq y\}$, if they are equal and are denoted by $x \oplus y$, i.e. if the following condition (pS) holds:

(pS) there exist, for all $x, y \in A$, $x \oplus y \overset{notation}{=} \max\{z \mid z \star y \leq x\} = \max\{z \mid z \circ x \leq y\}$.

The *left-pseudo-BCK algebra* is the "inverse" of the (right-) pseudo-BCK algebra; it is obtained by replacing $\leq$ by the "inverse" order relation, $\geq$, and by replacing 0 by 1:

Definition 2.2 A *left-pseudo-BCK algebra* is a structure $\mathcal{A} = (A, \geq, \Box, \#, 1)$, where "$\geq$" is a binary relation on A, "$\Box$" and "$\#$" are binary operations on A and "1" is an element of A, verifying, for all $x, y, z \in A$, the axioms:
(I-L $(x\Box y)\#(x\Box z) \geq z\Box y$, $\quad (x\#y)\Box(x\#z) \geq z\#y$,
(II-L) $x\#(x\Box y) \geq y$, $\quad x\Box(x\#y) \geq y$,
(III-L) $x \geq x$,
(IV-L) $1 \geq x$,
(V-L) $x \geq y,\ y \geq x \Longrightarrow x = y$,
(VI-L) $x \geq y \Longleftrightarrow x\Box y = 1 \Longleftrightarrow x\#y = 1$.

The *reversed left-pseudo-BCK algebra* is the reversed structure of the left-pseudo-BCK algebra; it is obtained by reversing both operations $\Box$, $\#$, i.e. by replacing $x\Box y$ by $y \to x = y \to_L x$ and $x\#y$ by $y \rightsquigarrow x = y \rightsquigarrow_L x$, for all x, y:

Definition 2.3 A *reversed left-pseudo-BCK algebra* is a structure $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$, where "$\geq$" is a binary relation on A, "$\to$" and "$\rightsquigarrow$" are binary operations on A and "1" is an element of A verifying, for all $x, y, z \in A$, the axioms:
(I) $(z \to x) \rightsquigarrow (y \to x) \geq y \to z$, $\quad (z \rightsquigarrow x) \to (y \rightsquigarrow x) \geq y \rightsquigarrow z$,
(II) $(y \to x) \rightsquigarrow x \geq y$, $\quad (y \rightsquigarrow x) \to x \geq y$,
(III) $x \geq x$,
(IV) $1 \geq x$,
(V) $x \geq y,\ y \geq x \Longrightarrow x = y$,
(VI) $x \geq y \Longleftrightarrow y \to x = 1 \Longleftrightarrow y \rightsquigarrow x = 1$.

Remark that, by (VI), the relation $\geq$ is equationally definable in terms of $\to, \rightsquigarrow$ and 1.
In this section we recall the basic properties of reversed left-pseudo-BCK algebras (the "reversed" of those from [17]) (for the proofs see [14]).
Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$ be a reversed left-pseudo-BCK algebra.

Definition 2.4 We shall say that $\mathcal{A}$ is *commutative* if $x \to y = x \rightsquigarrow y$, for all $x, y \in A$.

Then, we get immediately that

Corollary 2.5 *Any commutative reversed left-pseudo-BCK algebra is a reversed left-BCK algebra.*

¿From now on in this paper we shall work only with reversed left-pseudo-BCK algebras and therefore we shall simply say "left-pseudo-BCK algebras" instead of "reversed left-pseudo-BCK algebras".
We shall freely write $x \geq y$ or $y \leq x$ in the sequel.

Proposition 2.6 *The following properties hold in a left-pseudo-BCK algebra:*

(1) $$x \leq y \Longrightarrow y \to z \leq x \to z \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z,$$

(2) $$x \leq y,\ y \leq z \Longrightarrow x \leq z.$$

(3) $$z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x).$$

(4) $$z \leq y \to x \Longleftrightarrow y \leq z \rightsquigarrow x,$$

(5) $$x \leq y \to x, \quad x \leq y \rightsquigarrow x,$$

(6) $$1 \to x = x = 1 \rightsquigarrow x,$$

(7) $$x \leq y \Longrightarrow z \to x \leq z \to y \text{ and } z \rightsquigarrow x \leq z \rightsquigarrow y.$$

Remark that "$\geq$" is a partial order relation, by (III), (V) and (2) and that $(A, \geq, 1)$ is a poset (partial ordered set) with greatest element 1, by (IV).

Theorem 2.7
i) Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$ *such that:*

(A1) $(A, \geq, 1)$ *is a poset with greatest element* 1 *(i.e.* $1 \geq x$*, for all* $x \in A$*);*

(A2) $(A, \to, \rightsquigarrow, 1)$ *verifies: for all* $x, y, z \in A$,

(R1) $1 \to x = x = 1 \rightsquigarrow x$,

(R2) $(y \to z) \rightsquigarrow [(z \to x) \rightsquigarrow (y \to x)] = 1$, $(y \rightsquigarrow z) \to [(z \rightsquigarrow x) \to (y \rightsquigarrow x)] = 1$*;*

(A3) $x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1 \Longleftrightarrow x \leq y$*, for all* $x, y \in A$*;*

(A4) $x \leq y \Longrightarrow z \to x \leq z \to y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$*, for all* $x, y, z \in A$.

Then, $\mathcal{A}$ *is a left-pseudo-BCK algebra.*
ii) Conversely, every left-pseudo-BCK algebra satisfies (A1) - (A4).

Proof.
(i): (VI) is (A3). (III), (IV), (V) hold by (A1). (I) holds by (R2) and (VI). (II): By (R2) and (VI), for all $x, y, z \in A$, $z \to y \leq (y \to x) \rightsquigarrow (z \to x)$ and $z \rightsquigarrow y \leq (y \rightsquigarrow x) \to (z \rightsquigarrow x)$. Take then $z = 1$ and apply (R1); we get $y \leq (y \to x) \rightsquigarrow x$ and $y \leq (y \rightsquigarrow x) \to x$. Thus, (II) holds.
(ii): (A1) follows by (III),(V) (2) and (IV). (A2): (R1) is (6), (R2) follows by (I) and (VI). (A3) is (VI). (A4) is (7). □

By this theorem we get the following equivalent definition of left-pseudo-BCK algebras:

Definition 2.8 A *left-pseudo-BCK algebra* is an algebra $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$ such that the above (A1) - (A4) hold.

We also get the following definitions:

Definition 2.9 (See the corresponding definitions of a pseudo-t-norm, of a left-monoid and of a partially ordered, integral left-monoid from Introduction)
(i) A *pseudo-residuum* on the poset $(A, \geq, 1)$ with greatest element 1 is an ordered pair of binary operations, $(\to, \rightsquigarrow)$, verifying (A2), (A3), (A4) from Theorem 2.7; $\to$ is *the first component* (left residual [3]) and $\rightsquigarrow$ is *the second component* (right residual [3]) of the pseudo-residuum.
(ii) The algebra $(A, \geq, \to, \rightsquigarrow, 1)$ such that (A2) and (A3) hold is called a *left-residoid.*
(iii) The algebra $(A, \geq, \to, \rightsquigarrow, 1)$ such that (A1) - (A4) hold is called a *partially ordered, integral left-residoid* (i.e. a duplicate name for "left-pseudo-BCK algebra") (*integral* means that the greatest element of the poset $(A, \geq)$ is the element 1 of the left-residoid).

Remarks 2.10

(i) (See Remark 1.1) The statement: "The ordered pair $(\rightarrow, \rightsquigarrow)$ is a pseudo-residuum on a poset $(A, \geq, 1)$ with greatest element 1" is equivalent with the statement: "the algebra $(A, \geq, \rightarrow, \rightsquigarrow, 1)$ is a partially ordered, integral left-residoid, i.e. a left-pseudo-BCK algebra". Note that the places of the two components, $\rightarrow$ and $\rightsquigarrow$, are not commutative.

(ii) A pseudo-residuum is commutative if $\rightarrow = \rightsquigarrow$. It follows that a commutative pseudo-residuum is a residuum, $\rightarrow$ (i.e. we associate to the ordered pair $(\rightarrow, \rightarrow)$ the element $\rightarrow$). An abelian (i.e. commutative) left-residoid is an algebra $(A, \geq, \rightarrow = \rightsquigarrow, 1)$ such that $\rightarrow$ verifies the corresponding axioms (A2) and (A3) [18].

The inverse notion of the "(right-) pseudo-BCK algebra with condition (pS) (pseudo-sum, $\oplus$)" is the "left-pseudo-BCK algebra with condition (pP) (pseudo-product, $\odot$)", the reverse of which is defined as follows.

Definition 2.11 A *(reversed) left-pseudo-BCK algebra with condition (pP) (i.e. with pseudo-product)* or a *left-pseudo-BCK(pP) algebra* for short is an algebra $\mathcal{A} = (A, \geq, \rightarrow, \rightsquigarrow, 1)$ such that:

(I1) $\mathcal{A}$ is a left-pseudo-BCK algebra, i.e. (A1) - (A4) hold,

(I2) for any $x, y \in A$, there exist the smallest (least) element (under $\geq$) of the set $\{z \mid x \leq y \rightarrow z\}$ and the smallest element of the set $\{z \mid y \leq x \rightsquigarrow z\}$, they are equal and are denoted by $x \odot y$, i.e. $\mathcal{A}$ satisfies the following condition (pP):

(pP) there exist, for all $x, y \in A$, $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = \min\{z \mid y \leq x \rightsquigarrow z\}$.

Remark that we could define a left-pseudo-BCK(pP) algebra as an algebra $(A, \geq, \rightarrow, \rightsquigarrow, \odot, 1)$, but we shall not do this because the operation $\odot$ is defined in terms of $\geq$ and $\rightarrow, \rightsquigarrow$.

We denote by r-pBCK(pP) the class of (reversed) left-pseudo-BCK(pP) algebras and by **r-pBCK(pP)** the corresponding category.

Definition 2.12 (See Theorem 2.7)

A *left-pseudo-BCK(pRP) algebra* is an algebra $\mathcal{A} = (A, \geq, \rightarrow, \rightsquigarrow, \odot, 1)$ such that:

(I1-pRP) $\mathcal{A}_{pBCK} = (A, \geq, \rightarrow, \rightsquigarrow, 1)$ satisfies (A1), (A2), (A3),

(I2-pRP) $\odot$ is a binary operation verifying the following condition (pRP):

(pRP) for all $x, y, z \in A$, $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z$.

We denote by r-pBCK(pRP) the class of (reversed) left-pseudo-BCK(pRP) algebras and by **r-pBCK(pRP)** the corresponding category.

An important result is the following:

Lemma 2.13 *Let* $\mathcal{A} = (A, \geq, \rightarrow, \rightsquigarrow, \odot, 1)$ *(or* $\mathcal{A} = (A, \geq, \odot, \rightarrow, \rightsquigarrow, 1)$*) such that:*

(A1) $(A, \geq, 1)$ *is a poset with greatest element* 1,

(pRP) for all $x, y, z \in A$, $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z$.

Then, for all $x, y, z \in A$*, we have:*

$$(y \rightarrow x) \odot y \leq x, \quad y \odot (y \rightsquigarrow x) \leq x, \tag{8}$$

$$y \leq x \rightarrow (y \odot x), \quad x \leq y \rightsquigarrow (y \odot x), \tag{9}$$

$$x \leq y \Longrightarrow z \rightarrow x \leq z \rightarrow y, \quad z \rightsquigarrow x \leq z \rightsquigarrow y, \tag{10}$$

$$x \leq y \Longrightarrow x \odot z \leq y \odot z, \quad z \odot x \leq z \odot y. \tag{11}$$

Proof.

(8): $(y \to x) \odot y \leq x \overset{(pRP)}{\Leftrightarrow} y \to x \leq y \to x$, which is true by (A1). $y \odot (y \rightsquigarrow x) \leq x \overset{(pRP)}{\Leftrightarrow} y \rightsquigarrow x \leq y \rightsquigarrow x$, which is true by (A1).

(9): $y \leq x \to (y \odot x) \overset{(pRP)}{\Leftrightarrow} y \odot x \leq y \odot x$, which is true by (A1). $x \leq y \rightsquigarrow (y \odot x) \overset{(pRP)}{\Leftrightarrow} y \odot x \leq y \odot x$, which is true by (A1).

(10): By (8), $(z \to x) \odot z \leq x$ and since $x \leq y$, it follows that $(z \to x) \odot z \leq y$; hence, $z \to x \leq z \to y$, by (pRP). By (8) also, $z \odot (z \rightsquigarrow x) \leq x$ and since $x \leq y$, it follows that $z \odot (z \rightsquigarrow x) \leq y$; hence, $z \rightsquigarrow x \leq z \rightsquigarrow y$, by (pRP).

(11): By (9), $y \leq z \to (y \odot z)$ and since $x \leq y$, it follows that $x \leq z \to (y \odot z)$; hence, $x \odot z \leq y \odot z$, by (pRP). By (9) also, $y \leq z \rightsquigarrow (z \odot y)$ and since $x \leq y$, it follows that $x \leq z \rightsquigarrow (z \odot y)$; hence, $z \odot x \leq z \odot y$, by (pRP).

□

Remark that (10) is axiom (A4) (while (11) is axiom (X3) from Definition 3.1).

Corollary 2.14 *Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, \odot, 1)$ be a left-pseudo-BCK(pRP) algebra. Then,*
i) for all $x, y, z \in A$: (8), (9), (10) and (11) hold.
ii) $(A, \geq, \to, \rightsquigarrow, 1)$ is a left-pseudo-BCK algebra.

Proof. Obvious, by Lemma 2.13 (we use Definition 2.8 of left-pseudo-BCK algebras). □

Then we have the following

Theorem 2.15 *(We use Definition 2.8 of left-pseudo-BCK algebras)*

1) Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$ be a left-pseudo-BCK(pP) algebra, where for all $x, y \in A$:

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}.$$

Define

$$\pi(\mathcal{A}) = (A, \geq, \to, \rightsquigarrow, \odot, 1).$$

Then, $\pi(\mathcal{A})$ is a left-pseudo-BCK(pRP) algebra.

1') Conversely, let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, \odot, 1)$ be a left-pseudo-BCK(pRP) algebra. Define

$$\pi^*(\mathcal{A}) = (A, \geq, \to, \rightsquigarrow, 1).$$

Then, $\pi^*(\mathcal{A})$ is a left-pseudo-BCK(pP) algebra, where for all $x, y \in A$:

$$\min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\} = x \odot y.$$

2) The above defined mappings are mutually inverse.

Proof.

1): If $x \leq y \to z$, then by (pP), $x \odot y \leq z$. If $x \odot y \leq z$, then, it follows by (A4), that $y \to (x \odot y) \leq y \to z$ and since we also have that $x \leq y \to (x \odot y)$, by (pP), we get $x \leq y \to z$. If $y \leq x \rightsquigarrow z$, then by (pP), $x \odot y \leq z$. If $x \odot y \leq z$, then, it follows by (A4), that $x \rightsquigarrow (x \odot y) \leq x \rightsquigarrow z$ and since we also have that $y \leq x \rightsquigarrow (x \odot y)$, by (pP), we get $y \leq x \rightsquigarrow z$. Thus, (pRP) holds.

1') By Corollary 2.14, $\pi^*(\mathcal{A})$ is a left-pseudo-BCK algebra. It remains to prove that condition (pP) holds. Since $x \odot y \leq x \odot y$, by (pRP) we get that $x \leq y \to (x \odot y)$, i.e. $x \odot y \in \{z \mid x \leq y \to z\}$. If z verifies $x \leq y \to z$, then by (pRP), $x \odot y \leq z$. Thus, $\min\{z \mid x \leq y \to z\} = x \odot y$. Also, since $x \odot y \leq x \odot y$, by (pRP) we get that $y \leq x \rightsquigarrow (x \odot y)$, i.e. $x \odot y \in \{z \mid y \leq x \rightsquigarrow z\}$. If z verifies $y \leq x \rightsquigarrow z$, then by (pRP), $x \odot y \leq z$. Thus, $\min\{z \mid y \leq x \rightsquigarrow z\} = x \odot y$ too.

2) is obvious. □

Lemma 2.16 *Let $\mathcal{A}$ be a left-pseudo-BCK(pP) algebra, where for all $x, y \in A$:*

$$x \odot y \stackrel{notation}{=} \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}.$$

Then, for all $x, y, z \in A$,

$$x \leq y \Longrightarrow x \odot z \leq y \odot z, \quad z \odot x \leq z \odot y. \tag{12}$$

Proof. By Theorem 2.15 and Corollary 2.14. □

Proposition 2.17 *(See [19] for the commutative case)*
Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, \odot, 1)$ be a left-pseudo-BCK(pRP) algebra. Then, the algebra $(A, \odot, 1)$ is a left-monoid.

Proof. By Theorem 2.15, $(A, \geq, \to, \rightsquigarrow, 1)$ is a left-pseudo-BCK(pP) algebra, where for all $x, y \in A$, $\min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\} = x \odot y$.
• associativity:
$(x \odot y) \odot z \leq a \stackrel{(pRP)}{\Leftrightarrow} x \odot y \leq z \to a \stackrel{(pRP)}{\Leftrightarrow} x \leq y \to (z \to a) \stackrel{(4)}{\Leftrightarrow} y \leq x \rightsquigarrow (z \to a) \stackrel{(3)}{\Leftrightarrow} y \leq z \to (x \rightsquigarrow a)$ and
$x \odot (y \odot z) \leq a \stackrel{(pRP)}{\Leftrightarrow} x \leq (y \odot z) \to a \stackrel{(4)}{\Leftrightarrow} y \odot z \leq x \rightsquigarrow a \stackrel{(pRP)}{\Leftrightarrow} y \leq z \to (x \rightsquigarrow a)$. Thus, $(x \odot y) \odot z = x \odot (y \odot z)$.
• $x \odot 1 \leq a \stackrel{(pRP)}{\Leftrightarrow} x \leq 1 \to a \stackrel{(6)}{\Leftrightarrow} x \leq a$; thus, $x \odot 1 = x$. Also, $1 \odot x \leq a \stackrel{(pRP)}{\Leftrightarrow} x \leq 1 \rightsquigarrow a \stackrel{(6)}{\Leftrightarrow} x \leq a$; thus, $1 \odot x = x$. □

Corollary 2.18 *Let $\mathcal{A} = (A, \geq, \to, \rightsquigarrow, 1)$ be a left-pseudo-BCK(pP) algebra, where for all $x, y \in A$:*

$$x \odot y \stackrel{notation}{=} \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}.$$

Then the algebra $(A, \geq, \odot, 1)$ is a partially ordered, integral left-monoid (i.e. a left-X-pseudo-BCK algebra, cf. Definition 3.1), or, equivalently, the operation $\odot$ is a pseudo-t-norm on the poset $(A, \geq, 1)$ with greatest element 1.

Proof. By Remark 1.1, Proposition 2.17 and Lemma 2.16. □

By this corollary, the pseudo-t-norm $\odot$ will be called "the pseudo-t-norm associated with the pseudo-residuum (ordered pair of implications) $(\to, \rightsquigarrow)$".

3 Left-X-pseudo-BCK(pR) and left-X-pseudo-BCK(pRP) algebras

In this section we introduce left-X-pseudo-BCK(pR) algebras (duplicate name for porims) (Definition 3.3) and left-X-pseudo-BCK(pRP) algebras (Definition 3.4); they are categorically equivalent (Theorem 3.6). We prove the equivalence between the category of left-X-pseudo-BCK(pR) algebras and the category of (reversed) left-pseudo-BCK(pP) algebras (Theorem 3.12), which is a fundamental result.

Definition 3.1 A *left-X-pseudo-BCK algebra* is an algebra $\mathcal{A} = (A, \geq, \odot, 1)$, where $\geq$ is a binary relation on A, $\odot$ is a binary operation on A and $1 \in A$, such that :
(A1) $(A, \geq, 1)$ is a poset with greatest element 1,
(X2) $(A, \odot, 1)$ is a left-monoid,
(X3) for every $x, y, z \in A$, $x \geq y \Rightarrow x \odot z \geq y \odot z, \quad z \odot x \geq z \odot y$.

In the sequel we shall freely write $x \geq y$ or $y \leq x$.

Remark 3.2 Note that, in fact, a left-X-pseudo-BCK algebra is a duplicate name for what in the literature is called a "partially ordered, integral left-monoid" (see [3], for example). We shall use in the sequel our terminology, for the sake of the harmony of names. By Remark 1.1, the previous definition says that $\odot$ is a pseudo-t-norm on the poset $(A, \geq, 1)$ with greatest element 1.

Definition 3.3 A left-X-pseudo-BCK algebra *with condition (pR)* (i.e. *with pseudo-residuum,* $(\rightarrow, \rightsquigarrow)$) or a *left-X-pseudo-BCK(pR) algebra* for short is an algebra $\mathcal{A} = (A, \geq, \odot, 1)$ such that:

(X-I1) $\mathcal{A}$ is a left-X-pseudo-BCK algebra, i.e. (A1), (X2), (X3) hold,

(X-I2) for every $y, z \in A$, there exists the greatest (last) element (under $\geq$) of the set $\{x \mid x \odot y \leq z\}$, denoted by $y \rightarrow z$ and for every $x, z \in A$, there exists the greatest element of the set $\{y \mid x \odot y \leq z\}$, denoted by $x \rightsquigarrow z$, i.e. the following condition (pR) holds:

(pR) there exist, for all $x, y, z \in A$, $y \rightarrow z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$ and $x \rightsquigarrow z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}$.

Remark that in fact a left-X-pseudo-BCK(pR) algebra is a duplicate name for what in the literature is called a "porim" (i.e. "partially ordered, residuated, integral left-monoid") (see [3], for example), or, better, a "left-porim". Here also we shall use our terminology for the same reason as above.

Remark also that we could define a left-X-pseudo-BCK(pR) algebra as an algebra $(A, \geq, \odot, \rightarrow, \rightsquigarrow, 1)$, but we shall not do this because the operations (pseudo-residuum) $\rightarrow, \rightsquigarrow$ are defined in terms of $\geq$ and $\odot$.

We denote by X-pBCK(pR) the class of left-X-pseudo-BCK(pR) algebras and by **X-pBCK(pR)** the corresponding category.

Definition 3.4 A *left-X-pseudo-BCK(pRP) algebra* is an algebra $\mathcal{A} = (A, \geq, \odot, \rightarrow, \rightsquigarrow, 1)$ such that:

(X-I1-pRP) $\mathcal{A}_X = (A, \geq, \odot, 1)$ satisfies (A1), (X2),

(X-I2-pRP) $\rightarrow$ and $\rightsquigarrow$ are binary operations verifying condition (pRP):

(pRP) for all $x, y, z \in A$, $x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z \Longleftrightarrow x \odot y \leq z$.

Corollary 3.5 *(See the corresponding Corollary 2.14)*

Let $\mathcal{A} = (A, \geq, \odot, \rightarrow, \rightsquigarrow, 1)$ be a left-X-pseudo-BCK(pRP) algebra. Then,
i) for all $x, y, z \in A$, (8), (9), (10) and (11) hold.
ii) $(A, \geq, \odot, 1)$ is a left-X-pseudo-BCK algebra.

Proof. Obvious, by Lemma 2.13. □

Then, we have:

Theorem 3.6

1) Let $\mathcal{A} = (A, \geq, \odot, 1)$ be a left-X-pseudo-BCK(pR) algebra (porim), where for all $x, y, z \in A$:

$$y \rightarrow z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \rightsquigarrow z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Define

$$\rho(\mathcal{A}) = (A, \geq, \odot, \rightarrow, \rightsquigarrow, 1).$$

Then, $\rho(\mathcal{A})$ is a left-X-pseudo-BCK(pRP) algebra.

1') Conversely, let $\mathcal{A} = (A, \geq, \odot, \to, \rightsquigarrow, 1)$ be a left-X-pseudo-BCK(pRP) algebra. Define

$$\rho^*(\mathcal{A}) = (A, \geq, \odot, 1).$$

Then, $\rho^*(\mathcal{A})$ is a left-X-pseudo-BCK(pR) algebra, where for all $x, y, z \in A$:

$$\max\{x \mid x \odot y \leq z\} = y \to z, \quad \max\{y \mid x \odot y \leq z\} = x \rightsquigarrow z.$$

2) The above defined mappings are mutually inverse.

Proof.

1) If $x \odot y \leq z$, then by (pR), $x \leq y \to z$. If $x \leq y \to z$, it follows, by (X3), that $x \odot y \leq (y \to z) \odot y$, and since we also have by (pR) that $(y \to z) \odot y \leq z$, we get $x \odot y \leq z$. Also, if $x \odot y \leq z$, then by (pR), $y \leq x \rightsquigarrow z$. If $y \leq x \rightsquigarrow z$, it follows, by (X3), that $x \odot y \leq x \odot (x \rightsquigarrow z)$, and since we also have by (pR) that $x \odot (x \rightsquigarrow z) \leq z$, we get $x \odot y \leq z$. Thus, (pRP) holds.

1') By Corollary 3.5, $\rho^*(\mathcal{A})$ is a left-X-pseudo-BCK algebra. It remains to prove that condition (pR) holds. Since $y \to z \leq y \to z$, by (pRP) we get $(y \to z) \odot y \leq z$. If x verifies $x \odot y \leq z$, then, by (pRP), $x \leq y \to z$. Thus, $\max\{x \mid x \odot y \leq z\} = y \to z$. Also, since $x \rightsquigarrow z \leq x \rightsquigarrow z$, by (pRP) we get $x \odot (x \rightsquigarrow z) \leq z$. If y verifies $x \odot y \leq z$, then, by (pRP), $y \leq x \rightsquigarrow z$. Thus, $\max\{y \mid x \odot y \leq z\} = x \rightsquigarrow z$.

2) is obvious. □

Lemma 3.7 *(See the corresponding Lemma 2.16)*

Let $\mathcal{A}$ be a left-X-pseudo-BCK(pR) algebra, where for all $x, y, z \in A$:

$$y \to z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \rightsquigarrow z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Then, for all $x, y, z \in A$,

$$x \leq y \Longrightarrow z \to x \leq z \to y, \quad z \rightsquigarrow x \leq z \rightsquigarrow y.$$

Proof. By Theorem 3.6 and Corollary 3.5(i). □

Lemma 3.8 *Let $\mathcal{A} = (A, \geq, \odot, 1)$ be a left-X-pseudo-BCK(pR) algebra, where for all $x, y, z \in A$:*

$$y \to z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \rightsquigarrow z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Then, for all $x, y, z \in A$, we have:

$$1 \to x = x = 1 \rightsquigarrow x, \tag{13}$$

$$(x \odot y) \to z = x \to (y \to z), \quad (x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z), \tag{14}$$

$$(z \to x) \odot (y \to z) \leq y \to x, \quad (y \rightsquigarrow z) \odot (z \rightsquigarrow x) \leq y \rightsquigarrow x. \tag{15}$$

Proof. By Theorem 3.6, $(A, \geq, \odot, \to, \rightsquigarrow, 1)$ is a left-X-pseudo-BCK(pRP) algebra, i.e. (pRP) holds.

(13): $1 \to x = x \overset{notation}{\Longleftrightarrow} \max\{y \mid y \odot 1 \leq x\} = x \overset{(X2)}{\Longleftrightarrow} \max\{y \mid y \leq x\} = x$, which is true, by (A1).

Also, $1 \rightsquigarrow x = x \overset{notation}{\Longleftrightarrow} \max\{y \mid 1 \odot y \leq x\} = x \overset{(X2)}{\Longleftrightarrow} \max\{y \mid y \leq x\} = x$, which is true, by (A1).

(14): $(x \odot y) \to z \overset{notation}{=} \max\{u \mid u \odot (x \odot y) \leq z\} \overset{(X2)}{=} \max\{u \mid (u \odot x) \odot y \leq z\}$ $\overset{(pRP)}{=} \max\{u \mid u \odot x \leq y \to z\} \overset{notation}{=} x \to (y \to z)$.
Also, $(x \odot y) \rightsquigarrow z \overset{notation}{=} \max\{u \mid (x \odot y) \odot u \leq z\} \overset{(X2)}{=} \max\{u \mid x \odot (y \odot u) \leq z\}$ $\overset{(pRP)}{=} \max\{u \mid y \odot u \leq x \rightsquigarrow z\} \overset{notation}{=} y \rightsquigarrow (x \rightsquigarrow z)$.

(15): $y \to z \overset{notation}{=} \max\{u \mid u \odot y \leq z\}$,
$z \to x \overset{notation}{=} \max\{v \mid v \odot z \leq x\}$,
$y \to x \overset{notation}{=} \max\{w \mid w \odot y \leq x\}$.
Hence, we get that $(y \to z) \odot y \leq z$ and $(z \to x) \odot z \leq x$; then, by (X3),
$(z \to x) \odot [(y \to z) \odot y] \leq (z \to x) \odot z \leq x$, hence, by (X2), $[(z \to x) \odot (y \to z)] \odot y \leq x$.
It follows that $(z \to x) \odot (y \to z) \leq y \to x$.
Also, $y \rightsquigarrow z \overset{notation}{=} \max\{u \mid y \odot u \leq z\}$,
$z \rightsquigarrow x \overset{notation}{=} \max\{v \mid z \odot v \leq x\}$,
$y \rightsquigarrow x \overset{notation}{=} \max\{w \mid y \odot w \leq x\}$.
Hence, we get that $y \odot (y \rightsquigarrow z) \leq z$ and $z \odot (z \rightsquigarrow x) \leq x$; then, by (X3),
$[y \odot (y \rightsquigarrow z)] \odot (z \rightsquigarrow x) \leq z \odot (z \rightsquigarrow x) \leq x$, hence, by (X2), $y \odot [(y \rightsquigarrow z) \odot (z \rightsquigarrow x)] \leq x$.
It follows that $(y \rightsquigarrow z) \odot (z \rightsquigarrow x) \leq y \rightsquigarrow x$.
□

Proposition 3.9 *(See the corresponding Proposition 2.17)*
Let $\mathcal{A} = (A, \geq, \odot, \to, \rightsquigarrow, 1)$ be a left-X-pseudo-BCK(pRP) algebra. Then, the algebra $(A, \geq, \to, \rightsquigarrow, 1)$ is a left-residoid (Definition 2.9), i.e. (A2) ((R1) and (R2)) and (A3) hold.

Proof. (A3): $x \to y = 1 \overset{(A1)}{\Longleftrightarrow} 1 \leq x \to y \overset{(pRP)}{\Longleftrightarrow} 1 \odot x \leq y \overset{(X2)}{\Longleftrightarrow} x \leq y$.
$x \rightsquigarrow y = 1 \overset{(A1)}{\Longleftrightarrow} 1 \leq x \rightsquigarrow y \overset{(pRP)}{\Longleftrightarrow} x \odot 1 \leq y \overset{(X2)}{\Longleftrightarrow} x \leq y$.

(R1): is (13).

(R2): $(y \to z) \rightsquigarrow [(z \to x) \rightsquigarrow (y \to x)] = 1 \overset{(14)}{\Longleftrightarrow} [(z \to x) \odot (y \to z)] \rightsquigarrow (y \to x) = 1 \overset{(A3)}{\Longleftrightarrow} [(z \to x) \odot (y \to z)] \leq y \to x$, which is true by (15).
Also, $(y \rightsquigarrow z) \to [(z \rightsquigarrow x) \to (y \rightsquigarrow x)] = 1 \overset{(14)}{\Longleftrightarrow} [(y \rightsquigarrow z) \odot (z \rightsquigarrow x)] \to (y \rightsquigarrow x) = 1 \overset{(A3)}{\Longleftrightarrow} [(y \rightsquigarrow z) \odot (z \rightsquigarrow x)] \leq y \rightsquigarrow x$, which is true by (15) too. Thus, (A2) holds. □

Corollary 3.10 *(See the corresponding Corollary 2.18)*
Let $\mathcal{A} = (A, \geq, \odot, 1)$ be a left-X-pseudo-BCK(pR) algebra, where for all $x, y, z \in A$:

$$y \to z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \rightsquigarrow z \overset{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Then, the algebra $(A, \geq, \to, \rightsquigarrow, 1)$ is a partially ordered, integral left-residoid (i.e. a left-pseudo-BCK algebra, cf. Definition 2.9), or, equivalently, the ordered pair of operations $(\to, \rightsquigarrow)$ is a pseudo-residuum on the poset $(A, \geq, 1)$ with greatest element 1.

Proof. By Remarks 2.10(i), Proposition 3.9 and Lemma 3.7. □

Theorem 3.11
1) Let $\mathcal{A} = (A, \geq, \odot, \to, \rightsquigarrow, 1)$ be a left-X-pseudo-BCK(pRP) algebra. Define

$$\gamma'(\mathcal{A}) \overset{def}{=} (A, \geq, \to, \rightsquigarrow, \odot, 1).$$

Then, $\gamma'(\mathcal{A})$ is a left-pseudo-BCK(pRP) algebra.

1') Conversely, let $\mathcal{A} = (A, \geq, \rightarrow, \rightsquigarrow, \odot, 1)$ *be a left-pseudo-BCK(pRP) algebra. Define*

$$\delta'(\mathcal{A}) \stackrel{def}{=} (A, \geq, \odot, \rightarrow, \rightsquigarrow, 1).$$

Then, $\delta'(\mathcal{A})$ *is a left-X-pseudo-BCK(pRP) algebra.*

2) The above defined mappings are mutually inverse.

Proof.

1): (A2) holds, since:

• (R1): $1 \rightarrow x \geq 1 \stackrel{(pRP)}{\Longleftrightarrow} 1 \odot x \leq 1 \stackrel{(X2)}{\Longleftrightarrow} x \leq 1$ and $1 \rightsquigarrow x \geq 1 \stackrel{(pRP)}{\Longleftrightarrow} x \odot 1 \leq 1 \stackrel{(X2)}{\Longleftrightarrow} x \leq 1$, which are true by (A1).

• (R2): $(y \rightarrow z) \rightsquigarrow [(z \rightarrow x) \rightsquigarrow (y \rightarrow x)] = 1 \stackrel{(A1)}{\Longleftrightarrow} 1 \leq (y \rightarrow z) \rightsquigarrow [(z \rightarrow x) \rightsquigarrow (y \rightarrow x)]$
$\stackrel{(pRP)}{\Longleftrightarrow} (y \rightarrow z) \odot 1 \leq (z \rightarrow x) \rightsquigarrow (y \rightarrow x) \stackrel{(X2)}{\Longleftrightarrow} y \rightarrow z \leq (z \rightarrow x) \rightsquigarrow (y \rightarrow x)$
$\stackrel{(pRP)}{\Longleftrightarrow} (z \rightarrow x) \odot (y \rightarrow z) \leq y \rightarrow x$
$\stackrel{(pRP)}{\Longleftrightarrow} [(z \rightarrow x) \odot (y \rightarrow z)] \odot y \leq x \stackrel{(X2)}{\Longleftrightarrow} (z \rightarrow x) \odot [(y \rightarrow z) \odot y] \leq x$ which is always true (indeed, by (8), $(y \rightarrow z) \odot y \leq z$; then, by (11), $(z \rightarrow x) \odot [(y \rightarrow z) \odot y)] \leq (z \rightarrow x) \odot z$; but, $(z \rightarrow x) \odot z \leq x$, by (8) again; it follows that $(z \rightarrow x) \odot [(y \rightarrow z) \odot y] \leq x$, by (A1)).
Also, $(y \rightsquigarrow z) \rightarrow [(z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)] = 1 \stackrel{(A1)}{\Longleftrightarrow} 1 \leq (y \rightsquigarrow z) \rightarrow [(z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)]$
$\stackrel{(pRP)}{\Longleftrightarrow} 1 \odot (y \rightsquigarrow z) \leq (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x) \stackrel{(X2)}{\Longleftrightarrow} y \rightsquigarrow z \leq (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)$
$\stackrel{(pRP)}{\Longleftrightarrow} (y \rightsquigarrow z) \odot (z \rightsquigarrow x) \leq y \rightsquigarrow x$
$\stackrel{(pRP)}{\Longleftrightarrow} y \odot [(y \rightsquigarrow z) \odot (z \rightsquigarrow x)] \leq x \stackrel{(X2)}{\Longleftrightarrow} [y \odot (y \rightsquigarrow z)] \odot (z \rightsquigarrow x) \leq x$ which is always true (indeed, by (8), $y \odot (y \rightsquigarrow z) \leq z$; then, by (11), $[y \odot (y \rightsquigarrow z)] \odot (z \rightsquigarrow x) \leq z \odot (z \rightsquigarrow x)$; but, $z \odot (z \rightsquigarrow x) \leq x$, by (8) again; it follows that $[y \odot (y \rightsquigarrow z)] \odot (z \rightsquigarrow x) \leq x$, by (A1)).

(A3) holds: $x \rightarrow y = 1 \stackrel{(A1)}{\Longleftrightarrow} 1 \leq x \rightarrow y \stackrel{(pRP)}{\Longleftrightarrow} 1 \odot x \leq y \stackrel{(X2)}{\Longleftrightarrow} x \leq y$ and $x \rightsquigarrow y = 1 \stackrel{(A1)}{\Longleftrightarrow} 1 \leq x \rightsquigarrow y \stackrel{(pRP)}{\Longleftrightarrow} x \odot 1 \leq y \stackrel{(X2)}{\Longleftrightarrow} x \leq y$.

1'): By Definition 2.12, (A1) and (pRP) hold. By Proposition 2.17, $(A, \odot, 1)$ is a left-monoid, i.e. (X2) holds.

2) is obvious. □

By Theorems 3.6, 3.11 and 2.15, we get the following very important result.

Theorem 3.12

1) Let $\mathcal{A} = (A, \geq, \odot, 1)$ *be a left-X-pseudo-BCK(pR) algebra (porim), where for any* $x, y, z \in A$*:*

$$y \rightarrow z \stackrel{notation}{=} \max\{x \mid x \odot y \leq z\}, \quad x \rightsquigarrow z \stackrel{notation}{=} \max\{y \mid x \odot y \leq z\}.$$

Define

$$\gamma(\mathcal{A}) \stackrel{def}{=} (A, \geq, \rightarrow, \rightsquigarrow, 1).$$

Then, $\gamma(\mathcal{A})$ *is a left-pseudo-BCK(pP) algebra, where for all* $x, y \in A$*:*

$$\min\{z \mid x \leq y \rightarrow z\} = \min\{z \mid y \leq x \rightsquigarrow z\} = x \odot y.$$

1') Conversely, let $\mathcal{A} = (A, \geq, \rightarrow, \rightsquigarrow, 1)$ *be a left-pseudo-BCK(pP) algebra, where for all* $x, y \in A$*:*

$$x \odot y \stackrel{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = \min\{z \mid y \leq x \rightsquigarrow z\}.$$

Define

$$\delta(\mathcal{A}) \stackrel{def}{=} (A, \geq, \odot, 1).$$

Then, $\delta(\mathcal{A})$ is a left-X-pseudo-BCK(pR) algebra, where for all $x, y, z \in A$:

$$\max\{x \mid x \odot y \leq z\} = y \to z, \quad \max\{y \mid x \odot y \leq z\} = x \rightsquigarrow z.$$

2) The above defined mappings are mutually inverse.

□

Remarks 3.13

1) By Remark 3.2 and by Theorem 3.12, we get that the associated pseudo-residuum (ordered pair of implications) $(\to, \rightsquigarrow)$ of a pseudo-t-norm $\odot$ must verify the properties of $\to, \rightsquigarrow$ from the definition of a left-pseudo-BCK algebra (see Definition 2.8). Therefore, we've got Definition 2.9 and Remark 2.10.

2) By the above Theorem 3.12, the category **X-pBCK(pR)** is equivalent with the category **r-pBCK(pP)**, γ and δ being the equivalence functors, where

$$\delta = \rho^* \circ \delta' \circ \pi, \qquad \gamma = \pi^* \circ \gamma' \circ \rho$$

(see Figure 1).

$$\textbf{r-pBCK(pP)} \underset{\pi^*}{\overset{\pi}{\rightleftarrows}} \textbf{r-pBCK(pRP)} \underset{\gamma'}{\overset{\delta'}{\rightleftarrows}} \textbf{X-pBCK(pRP)} \underset{\rho}{\overset{\rho^*}{\rightleftarrows}} \textbf{X-pBCK(pR)} \equiv \text{porims}$$

Figure 1: Equivalent categories

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