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THE STRUCTURE AND PROPERTIES OF WEAKLY REGULAR ALGEBRAS

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Abstract. Abstract In this paper, we introduce the concept of quasi dimension of weakly regular algebras and study its some properties. Moreover, we provide characterizations between regular and weakly regular algebras.

1. Introduction and Preliminaries

All algebras in this paper are associative with unit ,and all modules are unital.

An algebra A is called regular,if for each $a \in A$, there exists $b \in A$ such that $a = aba$. An algebra A is weakly regular,if for each $a \in A$, there exist $b, c \in A$ such that $a = abac$. It is clear that every regular algebra is weakly regular. We know that an algebra A is regular if and only if for every right A-module is flat and an algebra A is regular if and only if weakly dimension $w.dimA$ of A, $w.dimA = 0$. In ([6]),Y.F.Xiao introduce the concept of semiflat module and obtain that an algebra A is weakly regular if and only if for every right A-module is semiflat.In this paper, we introduce the concept of quasi dimension of weakly regular algebras and study its some properties. Moreover, we provide characterizations between regular and regular algebras.

In this paper,we use $w.dim, q.dim$ and $f.dim$ to denote the weakly dimension, quasi dimension and flat dimension respectively.

2. The properties of weakly regular algebras

Definition 2.1 ([6]) A right A-module M is called semiflat if for every $J \leqslant_A A_A$, the sequence $0 \to M \otimes_A J \to M \otimes_A A$ is left exact.

Lemma 2.2 Let B be a right A-module and I an ideal of A, then there exists a unique epic $\theta : B \otimes_A I \to BI$ such that $\theta(b \otimes x) = bx$ for every $b \in B$ and $x \in I$. If B is semiflat, then θ is an isomorphism.

Proof Suppose $f : B \times I \to BI$ such that $(b, x) \mapsto bx$, then f is an A-biadditive function, there exists a unique homomorphism θ making the diagram commute:

$$\begin{array}{ccc} B \times I & \xrightarrow{h} & B \otimes_A I \\ & \searrow f \qquad \swarrow \theta & \\ & BI & \end{array}$$

where $h(b, x) = b \otimes x$ and θ is epic and unique.

If B is semiflat,then $1 \otimes i : B \otimes_A I \to B \otimes_A A$ is monic. Since $\phi : B \otimes_A A \to B$ is an isomorphism, $\phi(1 \times i)(b \otimes x) = \theta(b \otimes x)$, i.e., $\theta = \phi(1 \otimes i)$, then θ is monic, but θ is epic, θ is an isomorphism.

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Theorem 2.3 Let B be a right A-module and $0 \to K \xrightarrow{i} F \xrightarrow{g} B \to 0$ be an exact sequence of right A-module, where F is semiflat. The following are equivalent:

(i) B is semiflat;

(ii) $K \cap FI = KI$ for every ideal I of A.

Proof Note that $KI \subseteq K \cap FI$ for every ideal I of A. By Lemma 2.2, there exists a unique epic $\theta : B \otimes_A I \to BI$ such that $\theta(x \otimes y) = xy$ for every $x \in B$ and $y \in I$.

We can show that θ is an isomorphism if and only if $K \cap FI = KI$.

In fact,tensoring the original exact sequence by an ideal I give exactness of

$$K \otimes_A I \xrightarrow{i \otimes 1} F \otimes_A I \xrightarrow{g \otimes 1} B \otimes_A I \to 0$$

then(1) $FI/KI \cong B \otimes_A I$

In fact, F is semiflat, by Lemma 2.2, there exists a unique isomorphism $\phi : F \otimes_A I \to FI$ such that $\phi(x \otimes y) = xy$, then $(g \otimes 1)\phi^{-1} : FI \to B \otimes_A I$ is epic. Since $ker((g \times 1)\phi^{-1}) = \phi(ker(g \otimes 1)) = \phi(Im(i \otimes 1)) = KI$. Hence $\gamma : FI/KI \to B \otimes_A I$ is an isomorphism and $\gamma(xy + KI) = g(x) \otimes y$.

(2) $BI \cong FI/(K \cap FI)$

In fact, $g(xy) = g(x)y \in BI$ for every $x \in F$, and $y \in I$,then $g(FI) \subseteq B$. Conversely, let $x \in B$ and $y \in I$, since g is epic,there exists $x' \in F$ such that $x = g(x')$,then $xy = g(x'y) \in g(FI)$. Hence $g(FI) = BI$. Write $g' = g|_{FI}$, then there exists an epic $g' : FI \to BI$ and $kerg' = K \cap FI$, and thus $\delta : BI \to FI/(K \cap FI)$ is an isomorphism and $\delta(g(x)y) = xy + K \cap FI$.

Let $\sigma = \delta\theta\gamma : FI/KI \to KI/(K \cap FI)$. Moreover, $KI \subseteq K \cap FI$, then σ is an isomorphism if and only if θ is an isomorphism, then θ is an isomorphism if and only if $K \cap FI = KI$.

(i)⇒(ii) If B is semiflat , by Lemma 2.2, $\theta : B \otimes_A I \to BI$ is an isomorphism for every ideal I of A. Thus $K \cap FI = FI$.

(ii)⇒(i) Suppose that $K \cap FI = KI$ for every ideal I, then $\theta : B \otimes_A I \to BI$ is an isomorphism. Moreover, $\tau : B \otimes_A A \to B$ is an isomorphism,then it makes the following diagram commute:

$$\begin{array}{ccc} B \otimes_A I & \xrightarrow{1 \otimes \rho} & B \otimes_A A \\ \theta \big\downarrow & & \tau \big\downarrow \\ BI & \xrightarrow[i]{} & B \end{array}$$

Hence $1 \otimes \rho$ is monic,and B is semiflat.

Proposition 2.4 A flat module is semiflat.

Proof Let M be a flat right A-module,then $0 \to M \otimes_A I \to M \otimes_A A$ is left exact for every left ideal of A. Certainly, $0 \to M \otimes_A I \to M \otimes_A A$ is left exact for every ideal of A. By Definition 2.1, M is semiflat.

By Proposition 2.4, every module M has a semiflat resolution, i.e., there exists an exact sequence:

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

which every F_i is semiflat.

Now, we introduce the following concept:

Definition 2.5 If M is a right A-module, then $s.dimM \leqslant n$ ($s.dim$ abbreviates semiflat dimension) if there is a semiflat resolution:

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

If no such finite resolution exists, define $s.dimM = \infty$; otherwise,if n is the least such integer, define $s.dimM = n$.

Theorem 2.6 An A-module M is semiflat if and only if $s.dimM = 0$.

Proof Let M be a semiflat module, then exists a semiflat resolution:

$$\cdots \to 0 \to 0 \to F_0 \xrightarrow{\varepsilon} M \to 0$$

where $F_0 = M; F_i = 0, i \geqslant 1; \varepsilon = 1_M$. Hence $s.dimM = 0$.
Conversely, if $s.dimM = 0$. then there exists a semiflat resolution:

$$\cdots \to 0 \to 0 \to F_0 \to M \to 0$$

where $F_i = 0, i \geqslant 1$, then $M \cong F_0$. Hence M is semiflat.
The following concept is necessary:

Definition 2.7 $q.dimA = sup\{s.dimM | \in \mathcal{M}_A\}$ is called quasi dimension of A.

Lemma 2.8 ([6]) An algebra A is weakly regular if and only if for every A-module is semiflat.

Theorem 2.9 An algebra A is weakly regular if and only if $q.dimA = 0$.

Proof Let M be a right A-module. By Lemma 2.8 , A is weakly regular if and only if every M is semiflat. By Theorem 2.6, M is flat if and only if $s.dimM = 0$. Hence A is weakly regular if and only if for every M , $s.dimM = 0$. By Definition 2.7, A is weakly regular if and only if $q.dimA = 0$.

Lemma 2.10 M is a semiflat right A-module A-module if and only if $Tor_1(M, A/J) = 0$ for every ideal J of A.

Proof Since A is a flat right A-module , then $Tor_1(M, A/J) = 0$. Moreover, $0 \to J \xrightarrow{i} A/J \to 0$ is exact. By long exact Theorem, $0 = Tor_1(M, A) \to Tor_1(M, A/J) \to M \otimes_A J \xrightarrow{1\otimes i} M \otimes_A A$ is also exact.If M is semiflat, then $1 \otimes i$ is monic,and $Tor_1(M, A/J) = 0$.
Conversely, if $Tor_1(M, A/J) = 0$, then $1 \otimes i$ is monic, and thus M is semiflat.

Lemma 2.11 For any algebra A, the following are equivalent:
(i) A is weakly regular;
(ii)$_A(A/J)$ is flat for every ideal of A.
In fact, it is induced by Lemma 2.10.

Theorem 2.12 For any algebra A, the following are equivalent:
(i) A is weakly regular;
(ii) For every right A-module M and every ideal J of A, $Tor_1(M, A/J) = 0$.
In fact, it follows from Lemma 2.10 and Lemma 2.11.

Theorem 2.13 For any algebra $A, q.dimA \leqslant w.dimA$.

Proof Let M be a right A-module, then $s.dimM \leqslant f.dimM$. In fact, suppose $f.dimM \leqslant n$, then there exists a flat resolution: $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$. Since every flat module is semiflat, this is a semiflat resolution of A exhibiting $s.dimM \leqslant n$. Hence $s.dimM \leqslant f.dimM$ for every module M, and thus $q.dimA \leqslant w.dimA$.

Definition 2.14 An algebra A is called left (right) duo algebra, if for every left (right) ideal of A is an ideal.

Lemma 2.15 Let A be a left duo algebra and M a right A-module,the following are equivalent:
(i) M is flat;
(ii) M is semiflat.
It's obvious.

Theorem 2.16 For any left duo algebra A, $q.dimA = w.dimA$.
In fact, it follows from Lemma 2.15.

3. The structure of weakly regular algebras

Definition 3.1 An algebra A is called $MRLT(MERT)$ if for every maximal essential left(right) ideal of A is an ideal.

Theorem 3.2 Let A be an $MELT$ algebra, then the following are equivalent:
(i) A is regular;
(ii) A is weakly regular.
Now,we give the following Lemmas:

Lemma 1([3]**)** For any algebra A, I is an ideal of A, then A is weakly regular if and only if A/I and I are weakly.

Lemma 2 An algebra A is weakly regular if and only if A is fully right idempotent.

Proof If I is an ideal of A and A is weakly regular, then for each $a \in I \subseteq A$, there exist a', a'' such that $a = aa'aa'' \in II = I^2$,i.e., $I \subseteq I^2$, but $I^2 \subseteq I$, and thus $I = I^2$.

Conversely, let $I = aA$, then A is a right ideal of A,and thus $aA = aAaA$. If $a \in A$, then $a \in aA = aAaA$. Hence A is weakly regular.

Lemma 3 ([8]**)** Let A be an $MELT$ algebra. If I is an ideal of A, so is A/I.

Lemma 4 ([8]**)** Let A be an $MELT$ algebra, then A/S is left quasi-duo algebra,where S is socle of A .

Proof of Theorem 3.2:
(i)⇒(ii) It's obvious.
(ii)⇒(i) Let P be a prime ideal of A . Write $B = A/P$, A/P is weakly regular by Lemma 1, and thus B is fully right idempotent by Lemma 2. Moreover B is a prime algebra(Theorem 3.7.6 in (4)), then B is semiprime and B has no non-zero nilpotent ideal. Hence $S = Soc(_BB) = Soc(B_B)$ (Proposition 5 in (2)). Putting $H = B/S$, by Lemma 3, H is an $MELT$ algebra, and thus H is left quasi-duo algebra by Lemma 4. Hence H is strongly regular (Lemma 7 in (8)) and B is regular(Theorem 1 in (7)). Therefore A is regular(Corollary 1.18 in (1)).

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