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FUZZY CLASSES OF COOPERATIVE GAMES WITH TRANSFERABLE UTILITY*

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Abstract. The present work is concerned with cooperative games with transferable utility the characteristic function of which may have fuzzy values. Main attention is devoted to notions of superadditivity, convexity, core, and Shapley value. It turns out that the proposed fuzzy counterparts of these notions eliminate difficulties and disappointing features connected with previous approaches.

1. Introduction

Since the basic concepts of cooperative game theory and their elementary properties were established by von Neumann and Morgenstern [10], much work has been done in developing and analyzing solution concepts of various types of cooperative games. Most of this literature deals with cooperative games in characteristic function form where the characteristic function of a game is a mapping that assigns to each subset (coalition) of the set of players a real number, worth of the coalition or payoff to the coalition.

To take into account uncertainty about the degree of participation of players in coalitions, Aubin [1] and Butnariu [2] extend the domain of definition of the characteristic function from subsets to fuzzy subsets (fuzzy coalitions) of the set of players, that is, the characteristic function assigns to each fuzzy coalition again a real number.

In contrast, Mareš [4] – see also Nishizaki and Sakawa [8], and Mareš and Vlach [5, 6] – is concerned with the uncertainty in the values of characteristic functions. In these models, the domain of the characteristic function of a game remains to be the system of deterministic coalitions but the values assigned to coalitions are fuzzy quantities.

The present work continues the study of the Mareš model but introduces other kind of fuzzy counterparts of conventional notions of the deterministic theory. It turns out that the presented alternative way of fuzzification eliminates the difficulties connected with the original approach pointed out in Mareš [4]. We focus our attention on the notions of superadditivity, convexity, core, and Shapley value of games with transferable utility. For a similar treatment of games with non-transferable utility, we refer to Mareš and Vlach [7].

Throughout the paper, letter I denotes a non-empty finite set and $\mathcal{P}(I)$ stands for the power set of I, that is, the set of all subsets of I. The elements of I are called players and subsets of I are called coalitions. For convenience and without loss of generality, we assume that $I = \{1, 2, \dots, n\}$ where n is a given positive integer. The n-dimensional Euclidean space of ordered n-tuples of real numbers is denoted by $\mathbb{R}^n$. If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are points in $\mathbb{R}^n$ such that $x_i \leq y_i$ for each $i \in I$, then we briefly write $x \leq y$ or $y \geq x$.

In the next two sections we briefly recall some concepts and facts concerning fuzzy quantitites and cooperative games in characteristic function form with transferable utility

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(TU games). Then we introduce the approach to cooperative games with fuzzy payoffs proposed and analyzed in Mareš [4]. In the remaining sections, an alternative approach is presented. Special attention is devoted to fuzzy counterparts of supperadditivity, convexity, core, and Shapley value of such fuzzy games.

2. Fuzzy Quantities We use the term fuzzy quantity for any fuzzy subset A of the set $I\!R$ of all real numbers whose membership function $\mu_A : I\!R \to [0,1]$ satisfies the following requirements: There are real numbers a, b, c such that $a < b < c$, $\mu_A(b) = 1$ and $\mu_A(x) = 0$ for each $x \notin [a,c]$. The set of all fuzzy quantities is denoted by $\mathcal{F}(I\!R)$. For a fuzzy quantity A, we define $-A$ as the fuzzy subset of $I\!R$ whose membership function μ_{-A} is given by $\mu_{-A}(x) = \mu_A(-x)$ for each $x \in I\!R$. It can easily be seen that $-A$ is also a fuzzy quantity. The fuzzy quantity whose membership function is defined by $\mu(r) = 1, \mu(x) = 0$ for $x \neq r$ will be denoted by $\langle r \rangle$.

The arithmetic operations with real numbers are extended to operations with fuzzy quantities by means of the extension principle proposed by Zadeh [12], see Mareš [3] for a detailed explanation. Here we need to recall only addition of fuzzy quantities and multiplication of a fuzzy quantity by a real number. Let A and B be fuzzy quantities, and let r be a real number. The sum of A and B, denoted by $A \oplus B$, and the multiplication of A by r, denoted by rA, are defined as follows:

$$\mu_{A \oplus B}(x) = \sup_{y \in I\!R} \left[\min(\mu_A(y), \mu_B(x-y)) \right],$$

$$\mu_{rA}(x) = \begin{cases} 1 & \text{for} \quad r = 0 \text{ and } x = 0, \\ \mu_A\left(\dfrac{x}{r}\right) & \text{for} \quad r \neq 0, \\ 0 & \text{for} \quad r = 0 \text{ and } x \neq 0. \end{cases}$$

In addition to the arithmetic operations with fuzzy quantities we shall need the fuzzy ordering relation on the set of fuzzy quantities used in Mareš [4] to fuzzify TU cooperative games. Accepting the paradigm that relation between fuzzy quantities is to be fuzzy, we identify the fuzzy ordering relations over $\mathcal{F}(I\!R)$ with fuzzy subsets of $\mathcal{F}(I\!R) \times \mathcal{F}(I\!R)$. The special fuzzy ordering relation $\succeq$ used in [4] is then the fuzzy subset of $\mathcal{F}(I\!R) \times \mathcal{F}(I\!R)$ whose membership function is denoted by $\nu_{\succeq}$. For any ordered pair of fuzzy quantities A, B the value $\nu_{\succeq}(A,B)$ determines the possibility that $A \succeq B$, where

$$\nu_{\succeq}(A,B) = \sup_{\substack{x,y \in I\!R \\ x \geq y}} \min \left[\mu_A(x), \mu_B(x) \right].$$

In some publications, the general concept of fuzzy quantity is reduced to the concept of (triangular) fuzzy number with one modal value of the membership function, i.e., $x_0 \in I\!R$ such that $\mu_A(x_0) = 1$, and linearly monotonous segments of $\mu_A(x_)$ for $x < x_0$ and $x > x_0$. Even if the following concepts are formulated (and correct - see [4]) for fuzzy quantities, it can be easier, at least in the first reading, to interpret them as fuzzy numbers.

3. Cooperative Games with Transferable Utility A deterministic cooperative game with transferable utility and player set I is a mapping $v : \mathcal{P}(I) \to I\!R$ such that $v(\emptyset) = 0$. When there is no danger of misunderstanding, we use brief expressions like a "TU game v" or just a "game v"; from time to time v is also called the characteristic function of a game. The set of all TU games with player set I is denoted by $V(I)$.

A game $v : \mathcal{P}(I) \to I\!R$ is said to be superadditive, if

$$v(K) + v(L) \leq v(K \cup L)$$

for each pair of disjoint coalitions K and L. If a game $v : \mathcal{P}(I) \to I\!R$ satisfies the inequality

$$v(K) + v(L) \leq v(K \cup L) + v(K \cap L)$$

for each pair of coalitions K and L, then we say that v is convex.

A vector $x \in I\!R^n$ is said to be accessible for coalition K if

$$\sum_{i \in K} x_i \leq v(K).$$

Informally, the accessibility of x for K can be interpreted as the possibility to distribute the total payoff $v(K)$ among the members of coalition K in such a manner that each member $i \in K$ receives at least x_i.

To define the core of a game, one of the basic solution concepts of a TU game, we first introduce a relation of dominance between vectors of $I\!R^n$ with respect to a coalition. If $x, y \in I\!R^n$ and $K \subset I$ are such that $x_i \geq y_i$ for each $i \in K$, and $x_j > y_j$ for at least one $j \in K$, then we say that x dominates y via K and write $x \operatorname{dom}_K y$. The core of a TU game $v : \mathcal{P}(I) \to I\!R$ is the subset of $I\!R^n$ satisfying the following two requirements: First, each vector of the core is accessible for the coalition I of all players; second, for each coalition K, no vector from the core is dominated via K by a vector accessible for K. The core of a game v is denoted $C(v)$.

Since the core is the solution set of a finite system of nonstrict linear inequalities, we know that the core is a closed convex subset of $I\!R^n$. The fact that the core of some games is empty and the core of some games consists of uncountably many points motivated the search for single-point solution concepts. One of the earliest such concepts is the Shapley value [9]. To recall the definition of Shapley value of a game we need some preparatory definitions.

Let $v : \mathcal{P}(I) \to I\!R$ be a TU game with player set I. A carrier of v is a coalition K such that $v(L) = v(L \cap K)$ for each coalition L. For a permutation π of I, we define $\pi v : \mathcal{P}(I) \to I\!R$ by requiring that, for each coalition $\{i_1, i_2, \ldots, i_k\}$ of k players, $1 \leq k \leq n$, the value of πv is defined by

$$\begin{aligned} (\pi v)\ (\{\pi(i_1), \ldots, \pi(i_k)\}) &= v(\{i_1, \ldots, i_k\}), \\ (\pi v)\,(\emptyset) &= 0. \end{aligned}$$

Finally, if u and v are TU games with player set I, then the sum $u + v$ of these games is defined as the standard sum of functions, that is, $(u + v)\,(K) = u(K) + v(K)$ for each coalition K. Obviously, if u and v are games, so are πv and $u + v$.

Now we are ready to recall the definition of Shapley value.

The Shapley value of a game $v : \mathcal{P}(I) \to I\!R$ is an n-vector $T(v) = (T_1(v), T_2(v), \ldots, T_n(v))$ satisfying the following three conditions:

(a) If K is a carrier of v, then $\sum_{i \in K} T_i(v) = v(K)$.

(b) If π is a permutation of I, then $T_{\pi(i)}(\pi v) = T_i(v)$ for each $i \in I$.

(c) $T_i(u + v) = T_i(u) + T_i(v)$ for each $i \in I$.

It turns out that each TU game v has exactly one Shapley value and, for each $i \in I$,

$$T_i(v) = \sum_{i \in K} \frac{(n-1)!(n-k)!}{n!} \left[v(K) - v(K \setminus \{i\})\right]$$

where k is the number of players in coalition K and the summation is meant over all coalitions K containing player i.

4. Cooperative Fuzzy Games with Transferable Utility In this section we recall some concepts and facts concerning cooperative fuzzy games considered by Mareš [4], see also Mareš and Vlach [5, 6]. As mentioned in the Introduction, in this type of games, the characteristic function is defined on the set of all deterministic coalitions and has values in the set of fuzzy quantities. Formally, we define such games as follows.

A fuzzy game with player set I and transferable utility, briefly a TU fuzzy game or a fuzzy game, is a mapping $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ such that $w(\emptyset) = \langle 0 \rangle$. The set of all fuzzy games with player set I is denoted by $W(I)$. For the simplicity, we suggest to limit attention to the cases in which the fuzzy values of $w(K)$ are fuzzy numbers in the traditional sense.

A fuzzy game $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ is said to be a fuzzy extension of a game $v : \mathcal{P}(I) \to \mathbb{R}$ if, for each $K \subset I$, $\mu^w_K(v(K)) = 1$ where μ^w_K is the membership function of $w(K)$.

The fuzzy subset of $W(I)$ the membership function of which is defined by

$$w \mapsto \min_{K, L \subset I,\, K \cap L = \emptyset} \nu_{\succeq}(w(K \cup L), w(K) \oplus w(L))$$

is introduced by Mareš as a fuzzy counterpart of the superadditivity of deterministic games. The value of this membership function indicates the degree to which a fuzzy game w is superadditive.

Analogously, a fuzzy counterpart of the convexity of deterministic games is defined as the fuzzy subset of $W(I)$ given by the membership function

$$w \mapsto \min_{K, L \subset I} \nu_{\succeq}(w(K \cup L) \oplus w(K \cap L), w(K) \oplus w(L)).$$

Furthermore, the core of a fuzzy game $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ is the fuzzy subset of $\mathbb{R}^n$ the membership function of which is defined by

$$x \mapsto \min \left\{ \begin{array}{c} \nu_{\succeq}\left(w(I), \left\langle \sum_{i \in I} x_i \right\rangle \right) \\ \min_{K \subset I} \nu_{\succeq}\left(\left\langle \sum_{i \in K} x_i \right\rangle, w(K) \right) \end{array} \right\},$$

and the Shapley value of a fuzzy game $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ is the vector $T(w) = (T_1(w), T_2(w), \ldots, T_n(w))$ of fuzzy quantities defined by

$$T_i(w) = \textstyle\sum^{\oplus}_{K \subset I} \left\langle \frac{(k-1)!(n-k)!}{n!} \right\rangle [w(K) \oplus (-w(K \setminus \{i\}))]$$

where k stands for the number of players in coalition K and $\sum^{\oplus}$ denotes the summation of fuzzy quantities defined in the section on fuzzy quantities.

For a detailed analysis, we refer to Mareš [4] where also some disappointing features of this type of fuzzification of superadditivity, convexity and other standard concepts are discussed. In what follows, we present an alternative way of introducing fuzzy counterparts of superadditivity, convexity, core and Shapley value by which the difficulties pointed out by Mareš can be avoided.

5. Fuzzy Subsets Generated by Fuzzy Games Let $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ be a TU fuzzy game, and let $\rho_w : V(I) \to [0, 1]$ be defined by

$$\rho_w(v) = \min_{K \subset I} \mu^w_K(v(K)),$$

where $V(I)$ is the set of all games with player set I. In this way, each fuzzy game w generates a fuzzy subset of $V(I)$, namely, the fuzzy subset the membership function of which is ρ_w.

It is natural to ask whether the fuzzy subsets generated by different fuzzy games are also different. The following example shows that in general this is not so.

Example. Let w_1 and w_2 be TU fuzzy games with player set $I = \{1,2\}$ defined as follows.

$$\mu^{w_1}_{\{1\}}(a) = \frac{1}{2}, \quad \mu^{w_1}_{\{2\}}(b) = 1, \quad \mu^{w_1}_{\{1,2\}}(c) = \frac{1}{10},$$
$$\mu^{w_2}_{\{1\}}(a) = 1, \quad \mu^{w_2}_{\{2\}}(b) = \frac{1}{2}, \quad \mu^{w_2}_{\{1,2\}}(c) = \frac{1}{10},$$

for some real numbers a, b, c, and

$$\mu^{w_1}_K(x) = \mu^{w_2}_K(x) = 0$$

for each nonempty coalition K and each $x \notin \{a,b,c\}$.

Easy calculations show that, for the deterministic TU game u defined by

$$u(K) = \begin{cases} a & \text{for} \quad K = \{1\}, \\ b & \text{for} \quad K = \{2\}, \\ c & \text{for} \quad K = \{1,2\}, \end{cases}$$

we have

$$\rho_{w_1}(u) = \rho_{w_2}(u) = \frac{1}{10},$$

and, for each deterministic game v different from u, we have $\rho_{w_1}(v) = \rho_{w_2}(v) = 0$. Therefore the different fuzzy games w_1 and w_2 generate the same fuzzy subset of $V(I)$. $\square$

Observe that the games w_1 and w_2 of the previous example are fuzzy extensions of no deterministic TU game of two players. It turns out that such an example cannot be constructed for fuzzy extensions of a deterministic game.

Lemma 1. If $w : \mathcal{P}(I) \to \mathcal{F}(I\!R)$ is a fuzzy extension of a game $v : \mathcal{P}(I) \to I\!R$, then $\rho_w(v) = 1$.

Proof. According to the definition of a fuzzy extension, we have $\mu^w_K(v(K)) = 1$ for each $K \subset I$. Therefore

$$\rho_w(v) = \min_{K \subset I} \mu^w_K(v(K)) = 1. \qquad \square$$

Lemma 2. Different fuzzy extensions of a deterministic game with player set I generate different fuzzy subsets of $V(I)$.

Proof. Let w_1 and w_2 be different fuzzy extensions of a game $v_0 \in V(I)$. Since $w_1 \neq w_2$, there is a coalition K_0 and a real number x_0 such that $\mu^{w_1}_{K_0}(x_0) \neq \mu^{w_2}_{K_0}(x_0)$. Let v be the deterministic game defined by

$$v(K) = \begin{cases} v_0(K) & \text{for} \quad K \neq K_0, \\ x_0 & \text{for} \quad K = K_0. \end{cases}$$

Since w_1 is a fuzzy extension of v_0, we have

$$\min_{K \neq K_0} \mu^{w_1}_K(v_0(K)) = 1.$$

It follows that

$$\begin{aligned} \rho_{w_1}(v) &= \min_{K \subset I} \mu^{w_1}_K(v(K)) \\ &= \min \left\{ \mu^{w_1}_{K_0}(v(K_0)), \min_{K \neq K_0} \mu^{w_1}_K(v(K)) \right\} \\ &= \min \left\{ \mu^{w_1}_{K_0}(x_0), 1 \right\} = \mu^{w_1}_{K_0}(x_0). \end{aligned}$$

Analogously we obtain that $\rho_{w_2}(v) = \mu_{K_0}^{w_2}(x_0)$. Since $\mu_{K_0}^{w_1}(x_0)$ and $\mu_{K_0}^{w_2}(x_0)$ are different, we have $\rho_{w_1}(v) \neq \rho_{w_2}(v)$; in other words, the fuzzy subsets generated by w_1 and w_2 are different. □

Superadditivity As explained in the previous section, Mareš [4] defines the degree of superadditivity of a fuzzy game as the minimum of $\nu_{\succeq}(w(K \cup L), w(K) \oplus w(L))$ taken over all pairs of disjoint coalitions K and L. Here we introduce an alternative notion of the degree of superadditivity.

Let $w : \mathcal{P}(I) \to \mathcal{F}(\mathbb{R})$ be a fuzzy game. Using the membership function ρ_w of the fuzzy subset of $V(I)$ generated by w, we define the degree of superadditivity of w as the value of the function $\sigma : W(I) \to [0, 1]$ given by

$$\sigma(w) = \sup \rho_W(v) = \sup \min_{K \subset I} \mu_K^w(v(K))$$

where the supremum is taken over all superadditive games $v \in V(I)$.

The function σ has all formal properties of a membership function with the domain $W(I)$. In this sense the superadditive TU fuzzy games form a fuzzy subset of the universum of all TU fuzzy games $W(I)$.

Theorem 1. If w is a fuzzy extension of a superadditive deterministic game, then $\sigma(w) = 1$.

Proof. Let w be a fuzzy extension of a superadditive game $v : \mathcal{P}(I) \to \mathbb{R}$. From the definition of fuzzy extensions, we know that $\mu_K^w(v(K)) = 1$ for each $K \subset I$. It follows that $\rho_w(v) = 1$, and therefore $\sigma(w) = 1$. □

Theorem 2. If w_1 and w_2 are fuzzy games with player set I such that $\mu_K^{w_1}(x) \geq \mu_K^{w_2}(x)$ for each $K \subset I$ and each $x \in \mathbb{R}$, then $\sigma(w_1) \geq \sigma(w_2)$.

Proof. It can easily be seen that this is an immediate consequence of the definition of σ. □

Convexity Similarly to the previous section, we introduce an alternative concept of the degree of convexity based on the membership functions of the fuzzy subsets generated by fuzzy games. Namely, the degree of convexity of a fuzzy game $w \in W(I)$ is the value of the function $\delta : W(I) \to [0, 1]$ given by

$$\delta(w) = \sup \rho_w(v)$$

where the supremum is taken over all convex games $v \in V(I)$.

Also the convexity of TU fuzzy games introduced here may be interpreted as fuzzy subset of $W(I)$ with membership function δ, quite analogously to the property of superadditivity.

Theorem 3. If w is a fuzzy extension of a convex deterministic TU game, then $\delta(w) = 1$.

Proof. Let w be a fuzzy extension of a convex game $v \in V(I)$. Since w is a fuzzy extension of v, we have $\mu_K^w(v(K)) = 1$ for each $K \subset I$. It follows that $\rho_w(v) = 1$, and $\sigma(w) = 1$ because v is convex. □

Theorem 4. If w_1 and w_2 are fuzzy games from $W(I)$ and such that $\mu_K^{w_1}(x) \geq \mu_K^{w_2}(x)$ for each $K \subset I$ and $x \in \mathbb{R}$, then $\delta(w_1) \geq \delta(w_2)$.

Proof. This is an easy consequence of the definition of δ. □

Core To introduce the core of a fuzzy game we follow the basic paradigm that it is a fuzzy subset of $\mathbb{R}^n$. However, in contrast to the core defined in Section 4, we define the core with the help of the fuzzy subset of $V(I)$ generated by w. The core $C(w)$ of a fuzzy game $w \in W(I)$ is the fuzzy subset of $\mathbb{R}^n$ with the membership function $\gamma_w : \mathbb{R}^n \to [0,1]$ defined by

$$\gamma_w(x) = \sup_{x \in C(v)} \rho_w(v)$$

where the supremum is taken over all $v \in V(I)$ such that x belongs to the core of v. Again in this case, the core is a fuzzy subset of $\mathbb{R}^n$, like in Section 4. Nevertheless, its construction is different. Whereas in Section 4, the core is constructed from the values $w(K)$ by a procedure identical with that used in the crisp games, here the membership function of the core is derived from the possibility that there exists a (crisp) game for which the referred imputation belongs to its core. The relation between both approaches is considered below. Notice that, for each $x \in \mathbb{R}^n$, there exists a game $v \in V(I)$ such that $x \in C(v)$. For example, for an arbitrarily given $x = (x_1, x_2, \ldots, x_n)$, we can take v defined by

$$v(K) = \begin{cases} 0 & \text{for} \quad K = \emptyset, \\ \displaystyle\sum_{i \in K} x_i & \text{for} \quad K \neq \emptyset. \end{cases}$$

We leave to the reader to verify that similarly to the previous two sections, the following two results are immediate consequences of the definitions of γ_w.

Theorem 5. If w is a fuzzy extension of a game $v \in V(I)$, then $\gamma_w(x) = 1$ for each $x \in C(v)$.

Theorem 6. If w_1 and w_2 are fuzzy games from $W(I)$ such that $\mu_K^{w_1}(x) \geq \mu_K^{w_2}(x)$ for each $K \subset I$ and each $x \in \mathbb{R}^n$, then $\gamma_{w_1}(x) \geq \gamma_{w_2}(x)$ for each $x \in \mathbb{R}^n$.

As an illustration of how this approach facilitates to use the well developed apparatus of the conventional game theory, we present the following result.

Theorem 7. If w is a fuzzy game from $W(I)$, then

$$\delta(w) \leq \sup_{x \in \mathbb{R}^n} \gamma_w(x).$$

Proof. Since every convex game $v \in V(I)$ has a nonempty core, we have

$$\begin{aligned} \delta(w) &= \sup\{\rho_w(v) : v \text{ is convex}\} \\ &\leq \sup\{\rho_w(v) : C(v) \neq \emptyset\} \\ &= \sup_{x \in \mathbb{R}^n} [\sup\{\rho_w(v) : x \in C(v)\}] \\ &= \sup_{x \in \mathbb{R}^n} \gamma_w(x). \end{aligned}$$

□

Shapley Value We again follow the principle that the solution concept of a fuzzy game is to be fuzzy. In contrast to the definition in Section 4, we now define the Shapley value of a fuzzy game $w \in W(I)$ to be the vector $(t_1(w), t_2(w), \ldots, t_n(w))$ of fuzzy quantities the membership function τ_i^w of which are defined by

$$\tau_i^w(x) = \sup \rho_w(v), \quad 1 \leq i \leq n,$$

where the supremum is taken over all games $v \in V(I)$ such that $x = T_i(v)$. We can see that even in this case the method (and paradigm) used in the previous paragraphs for the core, convexity and superadditivity is applied. Namely, instead of consequent copying the deterministic procedure with fuzzy (instead of crisp) quantities, we percept the Shapley value as a fuzzy set of imputations whose membership function is derived from the possibility of realization of such crisp TU game for which exactly the relevant imputation is its Shapley value.

As a straightforward consequence of the definition of τ_i and Lemma 1, we obtain the following two results.

Theorem 8. If w is a fuzzy extension of a game $v \in V(I)$, then $\tau_i^w(T_i(v)) = 1$ for each $i \in I$.

Theorem 9. If w_1 and w_2 are fuzzy games with player set I such that $\mu_K^{w_1}(x) \geq \mu_K^{w_2}(x)$ for each coalition K and each $x \in \mathbb{R}$, then $\tau_i^{w_1}(x) \geq \tau_i^{w_2}(x)$ for each $x \in \mathbb{R}$ and each $i \in I$.

The following results show that the Shapley value of fuzzy games reflects in a natural way the basic requirements from the definition of Shapley value of deterministic games.

First we observe that the components of Shapley value of a fuzzy game are not influenced by "re-naming" the players according to some permutation exactly in the same way as in the deterministic case. Second, we have the following result.

Theorem 10. If w is a fuzzy extension of $v \in V(I)$, then there exists a point $x = (x_1, x_2, \ldots, x_n)$ from $\mathbb{R}^n$ such that $\tau_i(x_i) = 1$ for each $i \in I$, and

$$\sum_{i \in I} x_i = v(I).$$

Proof. For example, the point with $x_i = T_i(v)$ has the required properties. □

Third, in order to see how the additivity of the conventional Shapley value is reflected in its fuzzy counterpart, we define the sum $w_1 + w_2$ of fuzzy games w_1, $w_2 \in W(I)$ in the most natural way, that is,

$$(w_1 \oplus w_2)(K) = w_1(K) \oplus w_2(K).$$

Since $\langle 0 \rangle + \langle 0 \rangle = \langle 0 \rangle$, $w_1 + w_2$ is also a game from $W(I)$.

Theorem 11. If w_1 and w_2 are fuzzy extensions of a game $v \in V(I)$, then there exist x, $y \in \mathbb{R}^n$ such that, for each $i \in I$,

$$\tau_i^{w_1}(x_i) = \tau_i^{w_2}(y_i) = \tau_i^{w_1+w_2}(x_i + y_i).$$

Proof. We know from Lemma 1 that $\mu_K^{w_1}(v(K)) = \mu_K^{w_2}(v(K)) = 1$ for all $K \subset I$. Then it follows from the definition of the addition of fuzzy quantities that $\mu_K^{w_1+w_2}(v(K)+v(K)) = 1$. Now it suffices to set both x and y equal to the Shapley value $T(v)$ of game v. □

Observe that this result admits the following generalization.

Theorem 12. If w_1 and w_2 are fuzzy extensions of v_1 and v_2 from $V(I)$, respectively, then there exist x and y from $\mathbb{R}^n$ such that, for each $i \in I$,

$$\tau_i^{w_1}(x_i) = \tau_i^{w_2}(y_i) = \tau_i^{w_1+w_2}(x_i + y_i).$$

Proof. The proof is analogous to that of the previous theorem. The only difference is that we set

$$x_i = T_i(v_1), \quad y_i = T_i(v_2) \quad \text{for each } i \in I.$$

□

Let us note also interesting ideas regarding the Shapley value presented in [11].

6. Conclusions The approach to cooperative games with fuzzy payoffs presented in previous sections represents an alternative to the fuzzification method considered in Mareš [4]. It seems that this alternative approach is advantageous at least in the sense that it offers the opportunity of utilizing effectively the tools and results of the conventional cooperative game theory for obtaining analogous results for cooperative games with fuzzy payoffs.

The fuzzification of TU games presented above opens a more general methodological question. The construction of fuzzy game w, used in Mareš [4] and other referred papers, as well as in the previous sections, can be called "construction from below". It means that the primary concepts of it are fuzzy payoffs $w(K)$ of particular coalitions, and the game w is their composition. This view on the fuzzification is preserved even in this paper – and the fuzzy class of (deterministic) games is defined via a single fuzzy game w composed from the fuzzy values $w(K)$.

There exists a possibility to approach to the fuzzification "from above". It means, to consider a fuzzy subclass of $V(I)$ as a primary source of the fuzziness. Such a fuzzy subclass $\mathcal{W}(I)$ of $V(I)$ can be used to define the fuzzy game w with values $\mu_K^w(x)$ derived from the memberships of games in $\mathcal{W}(K)$. More precisely, if $\mathcal{W}(K)$ is a fuzzy subset of $V(I)$ with membership function $\pi : V(I) \to [0,1]$, then it determines a single fuzzy game w, where for $x \in R$, $K \subset I$,

$$\mu_K^w(x) = \sup\{\pi(v) : v(K) = x\}.$$

This method can open new possibilities, even if it may be expected that some of the achievable results will be very near to those presented in this paper. Anyway, these considerations are out of the topic of this contribution, and they can be more precisely analyzed in the future.

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