REPRESENTATION THEOREM ON FINITE DIMENSIONAL PROBABILISTIC NORMED SPACES

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Abstract.In this paper, we gives the Riesz theorem on probabilistic normed spaces, studies the relations between convergence in probabilistic norm and convergence in coordinate, proves necessary and sufficient condition of the probabilistic normed spaces which are finite dimensional.

1 Introduction

In 1942, Menger (1) published the first paper in which was called statistical metrics. This paper introduced the idea of replacing the distance $d(p, q)$ between two point in a metric space by a probabilistic distribution function $F_{p,q}$. Serstnev (3) introduced the idea of probabilistic normed spaces. In this space the norm of an element is replaced by a distribution of norm.

In this paper, we introduces the concept of unit sphere, discusses the Riesz theorem on probabilistic normed spaces, studies the relations between convergence in probabilistic norm and convergence in coordinate, proves characteristic theorem of finite demensional probabilistic normed spaces.

Throughout this paper, we denote by D the set of distribution functions defined on R, i.e., $F \in D$ if F is nondecreasing left-continuous with $\sup_{t\in R} F(t) = 1$ and $\inf_{t\in R} F(t) = 0$.

Definition 1.1. A probabilistic normed space (shortly, PN-space) is an ordered pair (E, F), where E is a real linear space and F is a mapping from E into D (we denote $F(x)$ by F_x) satisfying the following conditions:

(PN-1)$F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$

(PN-2)$F_x(0) = 0$;

(PN-3)$F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for any $\alpha \in R, \alpha \neq 0$

(PN-4)if $F_x(t_1) = 1, F_y(t_2) = 1$, then $F_{x+y}(t_1 + t_2) = 1$.

Definition 1.2. A Menger PN space is a PN-space that satisfies (PN-5),

(PN-5)$F_{x+y}(t_1 + t_2) \geq \triangle(F_x(t_1), F_y(t_2))$ for all $x, y \in E, t_1, t_2 \in R^+ = [0, +\infty)$ where $\triangle$ is a 2-place function on the unit square satisfying:

(1)$T(0, 0) = 0$ and $T(a, 1) = a$

(2)$T(a, b) = T(b, a)$

(3)if $a \leq c$ and $b \leq d$, then $T(a, b) \leq T(c, d)$

(4)$T(T(a, b), c) = T(a, T(b, c))$

T is called a t-norm.

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2 Riesz theorem on PN-space

In this section, we shall discuss Riesz theorem on PN-space.

Lemma 2.1. Let (E,F) be a PN-space and A be a genuine subset of E. By the definition of $\sup_{y\in A} F_{x_1-y}(t)$, there exists $y_1 \in A$ such that for any $\epsilon > 0$ and $x_1 \in E\backslash A$

$$\sup_{y\in A} F_{x_1-y}(t) - \epsilon < F_{x_1-y_1}(t) \le \sup_{y\in A} F_{x_1-y}(t) \tag{2.1}$$

Suppose $p = \inf\{t > 0; F_{x_1-y_1}(t) > 1-\lambda\}$

$$p_1 = \inf\{t > 0; \sup_{y\in A} F_{x_1-y}(t) > 1-\lambda\} \tag{2.2}$$

$$p_2 = \inf\{t > 0; \sup_{y\in A} F_{x_1-y}(t) - \epsilon > 1-\lambda\} \tag{2.3}$$

where $\epsilon > 0$ and $\lambda \in (0,1)$. Then $p_2 \ge p \ge p_1$.

Proof. Suppose $t_0 \in \{t > 0; F_{x_1-y_1}(t) > 1-\lambda\}$, by (2.1) we have $t_0 \in \{t > 0; \sup_{y\in A} F_{x_1-y}(t) > 1-\lambda\}$, then $p \ge p_1$, similarly have $p_2 \ge p$.

Lemma 2.2: Let (E,F) be a PN-space and A be a genuine subset of E. Then we have the following:

(1)$\sup_{y\in A} F_{x_1-y}(t)$ is a left-continuous function at t for any $x_1 \in E\backslash A$.

(2)suppose $\bar{P} = \inf\{t > 0 : \sup_{y\in A} F_{x_1-y}(t-\delta) > 1-\lambda\}$ and P_1, P_2 be defined by (2.2)(2.3), then $P_2 \ge \bar{P} = P_1 + \delta$.

Proof. (1)Since $F_{x_1-y}(t)$ is left-continuous at $t(t > 0)$, thus for any $\epsilon > 0$ there exists $\delta \in (0,t)$ such that $F_{x_1-y}(t-\delta) > F_{x_1-y}(t) - \epsilon$, by continuous of real number, for every $t > \delta$ there exists $I(t)$ such that

$$F_{x_1-y}(t-\delta) > I(t) > F_{x_1-y}(t) - \epsilon$$

we have

$$\sup_{y\in A} F_{x_1-y}(t-\delta) \ge F_{x_1-y}(t-\delta) > I(t) \ge \sup_{y\in A} F_{x_1-y}(t) - \epsilon$$

Then

$$\sup_{y\in A} F_{x_1-y}(t-\delta) > \sup_{y\in A} F_{x_1-y}(t) - \epsilon$$

(2) Obviously $P_2 \ge \bar{P}$. By the definition of $\bar{P}$, we have

$$\begin{aligned}
\overline{p} &= \inf\{t > 0; \sup_{y\in A} F_{x_1-y}(t-\delta) > 1-\lambda, t > \delta\} \\
&= \inf\{t+\delta; \sup_{y\in A} F_{x_1-y}(t) > 1-\lambda, t > 0\} \\
&= \inf\{t > 0; \sup_{y\in A} F_{x_1-y}(t) > 1-\lambda\} + \delta \\
&= p_1 + \delta,
\end{aligned}$$

This completes the proof.

Definition 2.1. Let (E,F) be a PN-space and A be a genuine subset of E.

(1)We define a unit sphere $N(1,\lambda)$ of E by

$$N(1,\lambda) = \{y \in E; F_y(1) > 1-\lambda, \lambda \in (0,1)\} \tag{2.4}$$

(2)We define $P_\lambda : E \to R^+$ by

$$P_\lambda(y) = \inf\{t > 0; F_y(t) > 1 - \lambda\} \tag{2.5}$$

for each $\lambda \in (0, 1)$. We say that $P_\lambda(y)$ is the quasi-norm of y.

(3) We define $F_{x-A}(t)$ by

$$F_{x-A}(t) = \sup_{y \in A} F_{x-y}(t) \tag{2.6}$$

for all $t \in R$. We say that $F_{x-A}(t)$ is the probabilistic distance from the point x to the set A.

(4) The set A in E is said sequentially compact, if any infinite set of A must there exists a convergence subsequence. The set A in E is said self-sequentially compact, if limit of every convergence sequence in A belong to A.

Corollary 2.1: Let A be a nonempty closed set of E, then

$$F_{x-A}(t) = 1, \ for\ all\ t > 0\ if\ and\ only\ if\ x \in A.$$

Theorem 1: Let (E, F) be a PN-space and A be a nonempty closed genuine subset of E. Then for any $y \in A$ there exists $x_0 \in E \setminus A$, and $\lambda_0 \in [0, 1]$ such that $x_0 \in N(1, \lambda)$ and $P_\lambda(x_0 - y) \geq 1$ for each $\lambda \in (\lambda_0, 1]$.

Proof. Since A is a nonempty closed genuine subset of E, by corollary 2.1 there exist $x_1 \in E \setminus A$, such that

$$F_{x_1 - A}(t) < 1$$

for all $t > 0$. Suppose $\sup_{t>0} F_{x_1-A}(t) = \sup_{t>0} \sup_{y \in A} F_{x_1-y}(t) = \delta, \delta \leq 1$. Let $\lambda_0 = 1 - \delta$, for each $\lambda \in (\lambda_0, 1]$, we have

$$\sup_{t>0} \sup_{y \in A} F_{x_1-y}(t) > 1 - \lambda$$

By the definition of sup, there exist $t_0 > 0$, such that

$$\sup_{y \in A} F_{x_1-y}(t) > 1 - \lambda$$

for any $t \geq t_0$. By Lemma 2.1 and definition of sup there exist $y_1 \in A$ such that

$$F_{x_1-y_1}(t) > \sup_{y \in A} F_{x_1-y}(t) - \epsilon.$$

for any $\epsilon > 0$, and all $t \geq t_0$. Taking $x_0 = \frac{x_1 - y_1}{p_2}$, by Lemma 2.1 and 2.2, we have

$$\begin{aligned} F_{x_0}(1) &= F_{\frac{1}{p_2}(x_1-y_1)}(1) = F_{x_1-y_1}(p_2) > \sup_{y \in A} F_{x_1-y}(p_2) - \epsilon \\ &\geq \sup_{y \in A} F_{x_1-y}(p_1 + \delta) - \epsilon > 1 - \lambda - \epsilon. \end{aligned}$$

by the left-continuity of $F_{x_0}(t)$ at $t = 1$, there exists $\delta_1 > 0$ such that

$$F_{x_0}(1 - \delta_1) > 1 - \lambda.$$

Thus $F_{x_0}(1) \geq F_{x_0}(1 - \delta_1) > 1 - \lambda$, therefore $x_0 \in N(1, \lambda)$, for any $\lambda \in (\lambda_0, 1]$.

Taking $\delta_2 = \frac{\delta}{p_2}$, by (2.3) we have

$$\begin{aligned} F_{x_0-y}(1-\delta_2) &= F_{\frac{1}{p_2}[x_1-(y_1+p_2y)]}(1-\delta_2) \\ &= F_{x_1-(y_1+p_2y)}(p_2-p_2\delta_2) \\ &\leq \sup_{y\in A} F_{x_1-y}(p_2-\delta) \\ &\leq 1-\lambda+\epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, by the left-continuity of $F_{x_0-y}(t)$ at $t=1$, we have

$$F_{x_0-y}(1) \leq 1-\lambda,$$

therefore $P_\lambda(x_0-y) \geq 1$, for any $y \in A$, and $x_0 \in N(1,\lambda)$. This completes the proof.

3 Finite dimensional characterization on PN-spaces.

Throughout this section, we always assume that $(E, F, \triangle)$ is a Menger PN-space, where the t-norm Δ satisfy

$$\sup_{0<t<1} \Delta(t,t) = 1$$

Definition 3.1. Let (E, F) be a PN-space

(1)The element $x_1, x_2, \cdots, x_n$ of E is linearly dependent, if there exists $k_1, k_2, \cdots, k_n$ not all zero, such that

$$F_{k_1x_1+k_2x_2+\cdots+k_nx_n}(t) = H(t),$$

if finite set $x_1, x_2, \cdots, x_n$ is not linearly dependent, it is called linearly independent.

(2)The element $x_1, x_2, \cdots, x_n$ of E is called a basis of E, if $x_1, x_2, \cdots, x_n$ are linearly independent and if any element of E is a linear combination of the element $x_1, x_2, \cdots, x_n$.

The E is called n-dimensional, if E has a basis of n elements.

Lemma 3.1. Let $(E, F, \triangle)$ be a Menger PN-space, and the t-norm Δ satisfy

$$\sup_{t<1} \Delta(t,t) = 1$$

Then we have the following

(1) For any $x \in E$ and $k \in R$,

$$P_\lambda(kx) = |k| P_\lambda(x)$$

(2) For any $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that for any $x, y \in E$:

$$P_\mu(x+y) \leq P_\lambda(x) + P_\lambda(y) \tag{3.1}$$

(3) For any $\mu \in (0,1)$, there exists $\lambda \in (0,1)$, and $k_1, k_2 \in R$, such that for any $x_1, x_2 \in E$:

$$P_\mu(k_1x_1+k_2x_2) \leq |k_1|P_\lambda(x_1) + |k_2|P_\lambda(x_2) \tag{3.2}$$

(4) For any $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that for any $x_1, x_2, \cdots, x_n \in E$ and $k_1, k_2, \cdots, k_n \in R$.

$$P_\mu(\sum_{i=1}^n k_ix_i) \leq \sum_{i=1}^n (|k_i|P_\lambda(x_i) \leq \max_{1\leq i\leq n} P_\lambda(x_i) \sum_{i=1}^n |k_i| \tag{3.3}$$

Proof. (1) By the definition of $P_\lambda(x)$, it is easy to prove.

(2) Since $\sup_{0<t<1}\Delta(t,t)=1$, for any $\mu\in(0,1)$ there exists $\lambda>0$ such that

$$\Delta(1-\lambda,1-\lambda)>1-\mu$$

By the Menger triangle inequality, we have

$$\begin{aligned}F_{x+y}(P_\lambda(x)+P_\lambda(y)+2\epsilon) &\geq \Delta(F_x(P_\lambda(x)+\epsilon),F_y(P_\lambda(y)+\epsilon))\\ &\geq \Delta(1-\lambda,1-\lambda)\\ &> 1-\mu\end{aligned}$$

for every $\epsilon>0$, which implies that

$$P_\mu(x+y)\leq P_\lambda(x)+P_\lambda(y)+2\epsilon$$

Letting $\epsilon\to 0$, we have

$$P_\mu(x+y)\leq P_\lambda(x)+P_\lambda(y) \tag{3.4}$$

The conclusions (3) and (4) follow from the (1) and (2). This completes the proof.

Lemma 3.2. Let (E,F,Δ) be a Menger PN-space, and the t-norm Δ satisfy

$$\sup_{0<t<1}\Delta(t,t)=1$$

Then the following conclusions are equivalent:

(1) x in E covergence in probabilistic norm to a point x_0. i.e., $F_{x-x_0}(t)\to H(t)$(as $x\to x_0$)

(2) for each $\lambda\in(0,1), P_\lambda(x-x_0)\to 0$(as $x\to x_0$)

Proof: (1) and (2) are equivalent, it follows from the following

$$F_{x-x_0}(\epsilon)>1-\lambda\Leftrightarrow P_\lambda(x-x_0)<\epsilon$$

By the same way, we can prove the following:

Lemma 3.3. Let (E,F,Δ) be a Merger PN-space, and the t-norm Δ satisfy

$$\sup_{0<t<1}\Delta(t,t)=1$$

Then the following conclusions are equivalent:

(1) $\{x_n\}$ is a Cauchy sequence of E;

(2) $P_\lambda(x_n-x_m)\to 0$(as $n,m\to\infty$)

Lemma 3.4. Let (E,F,Δ) be a Menger PN-space, and the t-norm Δ satisfy

$$\sup_{0<t<1}\Delta(t,t)=1$$

For any $x_1,x_2,\cdots,x_n\in E$, and $k_1,k_2,\cdots,k_n$ not all zero, then there exists $m>0$ such that

$$m\sum_{i=1}^{n}|k_i|\leq P_\mu(\sum_{i=1}^{n}k_ix_i) \tag{3.5}$$

where $P_M(x)$ be defined by (2.5).

Proof. Let $P_\mu(\sum_{i=1}^n k_i x_i) = \overline{P}_\mu, \overline{P}_\mu > 0$ and $M = \max_{1\le i\le n} P_\lambda(x_i)$, by Lemma 3.1 (1)(4), we have

$$1 = \frac{1}{\overline{P}_\mu} P_\mu(\sum_{i=1}^n k_i x_i) = P_\mu(\sum_{i=1}^n \frac{k_i}{\overline{P}_\mu} x_i) \le \sum_{i=1}^n \frac{|k_i|}{\overline{P}_\mu} P_\lambda(x_i) \le M \sum_{i=1}^n \frac{|k_i|}{\overline{P}_\mu}$$

Let $K = \max_{1\le i\le n} \frac{|k_i|}{\overline{P}_\mu}, K > 0$, then we have

$$M \sum_{i=1}^n \frac{|k_i|}{\overline{P}_\mu} \le MKn,$$

which shows that $\frac{1}{nK} \sum_{i=1}^n |k_i| \le \overline{P}_\mu = P_\mu(\sum_{i=1}^n k_i x_i)$, taking $m = \frac{1}{nk}, m > 0$. This completes the proof of conclusion.

Theorem 2. Let (E_n, F, Δ) be a n-dimensional Menge PN-space, and the t-norm Δ satisfy

$$\sup_{0<t<1} \Delta(t,t) = 1$$

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of E_n. For any $x \in E_n$ and $x_0 \in E_n$, assume $x = \sum_{i=1}^n k_i e_i$ and $x_0 = \sum_{i=1}^n k_i^0 e_i$. Then x converges in probabilistic norm to x_0 if and only if $k_i \to k_i^0 (i = 1, 2, \cdots, n)$.

Proof. Since x converges in probabilistic norm to x_0, $F_{x-x_0}(t) \to H(t)$, by Lemma 3.2(2), for $\mu \in (0,1)$ we have $P_\mu(x - x_0) \to 0$, and by Lemma 3.4 there exists $m > 0$ such that $P_\mu(\sum_{i=1}^n (k_i - k_i^0) e_i) \ge m \sum_{i=1}^n |k_i - k_i^0| > 0$, therefore for every $i : |k_i - k_i^0| \to 0$, i.e., $k_i \to k_i^0 (i = 1, 2, \cdots, n)$.

Conversely, if $k_i \to k_i^0 (i = 1, 2, \cdots, n)$ by Lemma 3.1(4), we have

$$0 \le P_\mu(\sum_{i=1}^n (k_i - k_i^0) e_i) \le \max_{1\le i\le n} P_\lambda(e_i) \sum_{i=1}^n |k_i - k_i^0|$$

This implies that $P_\mu(\sum_{i=1}^n (k_i - k_i^0) e_i) \to 0$ or $P_\mu(x - x_0) \to 0$, and by Lemma 3.2, x converges in probabilistic norm to x_0.

Remark 1: In Menger PN-space (E, F, Δ), any finite dimensional linear subspace must are closed.

Theorem 3. Let (E, F, Δ) be a Meger PN-space and the t-norm Δ satisfy

$$\sup_{0<t<1} \Delta(t,t) = 1,$$

Then (E, F, Δ) is finite dimensional if and only if the unit sphere $N(1, \lambda) = \{x; F_x(1) > 1 - \lambda, x \in E\}$ of E is a self-sequentially compact.

Proof. Suppose (E, F, Δ) is finite dimensional, since bounded closed set of real number is a self-sequentially compact, by Theorem 2, for any a sequence $\{x_n\}$ in $N(1, \lambda)$ there exists convergent subsequence $\{x_{n_k}\}$ of which limit x_0 belong to $N(1, \lambda)$, then $N(1, \lambda)$ is a self-sequentially compact.

Conversely, suppose $N(1, \lambda)$ is a self-sequentially compact, but (E, F, Δ) is not finite dimensional. We choose x_1 in $N(1, \lambda), x_1 \ne \theta$, for any $k_1 \in R$, let $E_1 = \{k_1 x_1; x_1 \in$

$N(1,\lambda), k_1 \in R\}$, by Remark 1, E_1 is a linear closed genuine subset. By Theorem 1 for each $\lambda \in (\lambda_0, 1]$ there exist $x_2 \in E \setminus E_1$, and $x_2 \in N(1,\lambda)$, such that

$$P_\lambda(x_2 - x_1) \geq 1.$$

In this case, x_1 and x_2 are linear independent, in fact, if x_1 and x_2 are dependent, then there exists $k_1, k_2 \in R$, might as well assume $k_2 \neq 0$ such that $F_{k_1x_1+k_2x_2}(t) = H(t)$. Therefore, we have $k_1x_1 + k_2x_2 = \theta, x_2 = -\frac{k_1}{k_2}x_1 \in E_1$, which is a contradiction.

Let $E_2 = \{k_1x_1 + k_2x_2; x_1 \in E, x_2 \in E \setminus E_1, k_1, k_2 \in R\}$ by Theorem 1 there exists $x_3 \in E \setminus E_2, x_3 \in N(1,\lambda)$, such that

$$P_\lambda(x_3 - y) \geq 1$$

where $y \in E_2$. In particular, we choose $y = x_1$ and x_2, we have $P_\lambda(x_3 - x_2) \geq 1$ and $P_\lambda(x_3 - x_2) \geq 1$. By the same way, we can choose $\{x_n\} \in N(1,\lambda)$ such that

$$P_\lambda(x_n - x_m) \geq 1$$

where $n \neq m$. By Lemma 3.3, $\{x_n\}$ there exists no any convergent subsequence in E, which is a contradiction. This completes the proof.

Theorem 4. Let (E, F, Δ) be a finite dimensional Menger PN-space, where the t-norm Δ satisfy

$$\sup_{0<t<1} \Delta(t,t) = 1.$$

Let A be a closed genuine subset of E. Then for each $\lambda \in (\lambda_0, 1]$ there exists a element $x_0 \in N(1,\lambda)$ such that

$$\inf_{y\in A} P_\lambda(x_0 - y) = 1$$

Proof. Let a sequence $z_n \in E\backslash A$. By corollary 2.1 there exists $y_n \in A$ and $t_0 > 0$ such that

$$1 > F_{z_n - y_n}(t) > F_{z_n - A}(t) - \epsilon$$

for any $\epsilon > 0$ and all $t \geq t_0$. By Theorem 1 for $y_n \in A$ there exists $x_n = \frac{z_n - y_n}{P_2} \in E\backslash A$ and $\lambda_0 \in [0,1]$ such that

$$x_n \in N(1,\lambda) \textit{ and } P_\lambda(x_n - y) \geq 1 \tag{3.6}$$

for each $\lambda \in (\lambda_0, 1]$ and any $y \in A, P_2$ be defined by (2.3). Assume E is a finite demensional, by Theorem 3 the $N(1,\lambda)$ is self-sequentially compact, then there exists $x_0 \in N(1,\lambda)$ such that

$$F_{x_n - x_0}(t) \to H(t) \quad (n \to \infty) \tag{3.7}$$

for all $t > 0$. By Lemma 3.2, we have

$$P_\lambda(x_n - x_0) \to 0 \quad (n \to \infty)$$

Since $x_0 \in N(1,\lambda) : F_{x_0}(1) > 1 - \lambda$, hence $P_\lambda(x_0) \leq 1$.By null element $\theta \in A$, we have

$$1 \geq P_\lambda(x_0) = P_\lambda(x_0 - \theta) \geq \inf_{y\in A} P_\lambda(x_0 - y) \tag{3.8}$$

Next, we prove that $\inf_{y\in A} P_\lambda(x_0 - y) \geq 1$. By (3.6), we have

$$F_{x_n - y}(1) \leq 1 - \lambda.$$

Assume $F_{x_0-y}(1) > 1-\lambda$, since t-norm Δ satisfy $\sup_{0<t<1}\Delta(t,t)=1$, for $\lambda>0$ there exists $\lambda_1>0, \lambda_1 \le \lambda$ such that1

$$\Delta(1-\lambda_1, 1-\lambda_1) > 1-\lambda.$$

Since $F_{x_0-y}(t)$ is left-continuous at $t=1$, there exists $\delta>0$ such that

$$F_{x_0-y}(1-\delta) > 1-\lambda_1 \ge 1-\lambda.$$

By (3.7) for (δ, λ_1) there exists $N>0$ such that

$$F_{x_n-x_0}(\delta) > 1-\lambda_1$$

for all $n>N$. By the Menger triangle inequality, we have

$$1-\lambda \ge F_{x_n-y}(1) \ge \Delta(F_{x_n-x_0}(\delta), F_{x_0-y}(1-\delta)) \ge \Delta(1-\lambda_1, 1-\lambda_1) > 1-\lambda$$

which is a contradiction. Then for any $y \in A$ we have

$$F_{x_0-y}(1) \le 1-\lambda$$

which implies $\inf_{y\in A} P_\lambda(x_0-y) \ge 1$. This completes the proof.

By Theorem 1,2,3, easily prove the following corollary.

Corollary 4.1. Let (E,F,Δ) be a Menger PN-space, where the t-norm Δ satisfy

$$\sup_{0<t<1}\Delta(t,t)=1$$

if (E,F,Δ) be finite dimensional, then any bounded closed subset of (E,F,Δ) is self-sequentially compact. Conversely, only if some sphere $N(t,\lambda)=\{x\in E; F_x(t)>1-\lambda\}$ be self-sequentially compact, then E is finite dimensional.

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