BINARY DIGITS EXPANSION OF NUMBERS: HAUSDORFF DIMENSIONS OF INTERSECTIONS OF LEVEL SETS OF AVERAGES' UPPER AND LOWER LIMITS

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Abstract. The problem of averaging of binary digits of numbers is considered and the sequence of the averages calculated on the first digits is taken into account for every $t \in [0,1]$. The Hausdorff dimensions of intersections of level sets of upper and lower limits of such sequences are computed.

1 Introduction In this paper we consider the classic problem of averaging the binary digits of numbers in $[0,1]$ and of studying the (Hausdorff) dimensions of some sets related to these averages.

Let us more precisely consider $t \in [0,1]$, the sequence $x(t) = (x_n(t))_n$ of its binary digits (cf. (14) for the precise definition) and the sequence of their averages $y(t) = (y_n(t))_n$ given by

$$y_n(t) = \frac{1}{n}\sum_{k=1}^{n} x_k(t), \ \forall n \in \mathbf{N} \tag{1}$$

Then it is possible to consider the two (always existing) quantities

$$\liminf_{n\to+\infty} y_n(t), \qquad \limsup_{n\to+\infty} y_n(t) \tag{2}$$

and the quantity, when it exists:

$$\lim_{n\to+\infty} y_n(t). \tag{3}$$

Let us set

$$\begin{array}{ll} & F^{\alpha} \doteq \left\{t \in [0,1] : \lim_n y_n(t) = \alpha\right\}, \\ R^{\alpha} \doteq \{t \in [0,1] : \text{limsup}_n\, y_n(t) \le \alpha\}, & R_{\alpha} \doteq \{t \in [0,1] : \text{liminf}_n\, y_n(t) \ge \alpha\}, \\ S^{\alpha} \doteq \{t \in [0,1] : \text{limsup}_n\, y_n(t) \ge \alpha\}, & S_{\alpha} \doteq \{t \in [0,1] : \text{liminf}_n\, y_n(t) \le \alpha\}. \end{array} \tag{4}$$

There are some classic results about the Hausdorff dimension of these sets we will recall. To this aim let us define the function $d(t)$ as follows

$$d(t) \doteq \begin{cases} -\left(t\log_2(t) + (1-t)\log_2(1-t)\right), & \forall t \in (0,1) \\ 0, & \text{if } t = 0,1. \end{cases} \tag{5}$$

and denote by $\dim_H$ the Hausdorff dimension (cf. (10) $\div$ (12) for the definition).

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In [HL] was proved (very well known result) that $F^{1/2}$ contains almost every t in $[0,1]$ (and therefore $\dim_H \left(F^{1/2}\right) = 1$).

In [Bs] the Hausdorff dimensions of the sets R^α and R_α was computed; $d(\alpha)$ was proved to be equal to $\dim_H(R^\alpha)$ if $0 \le \alpha < 1/2$ and to $\dim_H(R_\alpha)$ if $1/2 < \alpha \le 1$ (in the other cases the sets trivially contains $F^{1/2}$).

In [K] the Hausdorff dimensions of the sets S^α and S_α was computed; $d(\alpha)$ was proved to be equal to $\dim_H(S_\alpha)$ if $0 \le \alpha < 1/2$ and to $\dim_H(S^\alpha)$ if $1/2 < \alpha \le 1$ (in the other cases the sets trivially contains $F^{1/2}$).

In [E] was proved that the Hausdorff dimension of the set F^α is equal to $d(\alpha)$ for every $0 \le \alpha \le 1$.

Let us now define the sets

$$G^\alpha \doteq \left\{t \in [0,1] : \limsup_n y_n(t) = \alpha\right\}, \qquad G_\alpha \doteq \left\{t \in [0,1] : \liminf_n y_n(t) = \alpha\right\}. \tag{6}$$

Taking into account the recalled results it easily follows (cf. Proposition 1) that

$$\dim_H(G^\alpha) = \dim_H(G_\alpha) = d(\alpha) \qquad \forall \alpha \in [0,1]. \tag{7}$$

Then we analyze the Hausdorff dimension of

$$G^\alpha_\beta \doteq G^\alpha \cap G_\beta\ ; \tag{8}$$

by (7) and (8) we obviously have

$$\dim_H\left(G^\alpha_\beta\right) \le \min\{d(\alpha), d(\beta)\}. \tag{9}$$

Our result consists of proving the reverse inequality in (9), so that the equality

$$\dim_H\left(G^\alpha_\beta\right) = \min\{d(\alpha), d(\beta)\}$$

holds (cf. Theorem 6).

As last remark we observe that our proof is inspired by some fractal techniques (see [F2], p. 55).

Eventually we recall that the Hausdorff dimension is a very efficacious instrument to treat problems of Diophantine approximations. For this subject in addition to the above references see for example [S], [F1, section 8.5] and the more recent papers [DDY] and [DD].

2 Notations and preliminary results Let us denote by $\mathbf{N} = \{1,2,3,...\}$ and by $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Given a finite subset $M \subset \mathbf{N}$ we will denote by $\mathrm{card}(M)$ the number of its elements. Given a subset $E \subseteq \mathbf{R}$ we will denote by $\mathrm{diam}(E) = \sup\{|x-y| : x, y \in E\}$ its diameter and if in addition E is Lebesgue measurable, we denote by $|E|$ its Lebesgue measure.

Let $\delta > 0$ and $s > 0$ real numbers and let us pose, for every $E \subset \mathbf{R}$,

$$\mathcal{H}^s_\delta(E) = \inf \sum_{n=1}^{\infty} diam^s(B_n) \tag{10}$$

where the family $\{B_n\}_{n\in\mathbf{N}}$ is a countable covering of E with open balls such that $diam(B_n) < \delta$, $\forall n \in \mathbf{N}$ and the infimum is taken on this kind of families. The s-dimensional Hausdorff outer measure of E is given as usual by

$$\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E) = \lim_{\delta\to0^+} \mathcal{H}^s_\delta(E), \tag{11}$$

while Hausdorff dimension of E is given by

$$\dim_H(E) = \inf\left\{s \in \mathbf{R} : \mathcal{H}^s(E) = 0\right\}. \tag{12}$$

Moreover it can be easily proved that (see [F1, p.7])

$$\mathcal{H}^s(E) = 0 \text{ if } s > \dim_H(E); \qquad \mathcal{H}^s(E) = +\infty \text{ if } s < \dim_H(E) \tag{13}$$

Let us observe that slightly different definitions of s-dimensional Hausdorff outer measure can be given, all of them leading to the same result in the definition (12).

Given $t \in \mathbf{R}$, we will denote by $[t]$ the integer part of t, i.e. $[t] = \max\{m \in \mathbf{Z} : m \leq t\}$ and by I the interval $[0,1]$.

Let $t \in I$. We define the sequence $x(t) = \{x_n(t)\}_n$ in the following way

$$x_n(t) = [2^n t] - 2\left[2^{n-1} t\right] \qquad \forall n \in \mathbf{N}. \tag{14}$$

Such sequence is the one of the binary digits of t (the rational numbers of the form $\frac{p}{2^m}$ can be expressed in two ways as binary numbers: e.g. $\frac{1}{2} = 0,1_2$ and also $\frac{1}{2} = 0,0\bar{1}_2$; the sequence defined corresponds in this case to the representation with a finite number of digits equal to 1).

For a fixed $n \in \mathbf{N}$, $x_n(t)$ is a step function assuming only values 0 and 1 and it holds

$$x_n(t) = \frac{1}{2}\left(\chi_{[0,1)} + \sum_{j=0}^{2^n-1} (-1)^{j+1} \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n})}(t)\right) \qquad \forall t \in I,$$

where for a set A, the function χ_A is the characteristic function of A.

Now let $y(t) = (y_n(t))_n$ the sequence defined by (1); $y_n(t)$ is a step function constant on every interval $[\frac{j}{2^n}, \frac{j+1}{2^n}), j = 0, 1, ..., 2^n - 1$, and takes only values $\dfrac{k}{n}$, $k = 0, 1, ..., n$.

Moreover

$$\left|\left\{t : y_n(t) = \frac{k}{n}\right\}\right| = \binom{n}{k} 2^{-n}, \tag{15}$$

where $\binom{n}{k}$ is the binomial coefficient of n over k.

Obvious relations among the sets defined in the introduction are

$$\begin{aligned} &F^\alpha = G^\alpha_\alpha, \quad G^\alpha_\beta = G^\alpha \cap G_\beta, \quad G^\alpha = \cup_{0\leq\beta\leq\alpha} G^\alpha_\beta, \quad G_\alpha = \cup_{\alpha\leq\beta\leq 1} G^\beta_\alpha, \\ &R^\alpha = \cup_{0\leq\beta\leq\alpha} G^\beta, \quad R_\alpha = \cup_{\alpha\leq\beta\leq 1} G_\beta, \quad S^\alpha = \cup_{\alpha\leq\beta\leq 1} G^\beta, \quad S_\alpha = \cup_{0\leq\beta\leq\alpha} G_\beta \end{aligned} \tag{16}$$

for every α and β in $[0,1]$.

Therefore obvious relations among the Hausdorff dimensions of such sets are

$$\begin{aligned} &\dim_H(F^\alpha) \leq \dim_H(G^\alpha) \leq \min\left\{\dim_H(S^\alpha), \dim_H(R^\alpha)\right\}, \\ &\dim_H(F^\alpha) \leq \dim_H(G_\alpha) \leq \min\left\{\dim_H(S_\alpha), \dim_H(R_\alpha)\right\}, \\ &\dim_H(G^\alpha_\beta) \leq \min\left\{\dim_H(G^\alpha), \dim_H(G_\beta)\right\} \end{aligned} \tag{17}$$

for every α and β in I.

Then, using the results recalled in the introduction, we obtain

Proposition 1 *Let $\alpha,\beta \in [0,1]$ and let F^{α}, G_{α}, G^{α}, R_{α}, R^{α}, S_{α}, S^{α}, G^{α}_{β} be defined by (4), (6) and (8). Then*

$$\begin{gathered} d(\alpha) = \dim_H(F^{\alpha}) = \dim_H(G^{\alpha}) = \min\{\dim_H(S^{\alpha}), \dim_H(R^{\alpha})\}, \\ d(\alpha) = \dim_H(F^{\alpha}) = \dim_H(G_{\alpha}) = \min\{\dim_H(S_{\alpha}), \dim_H(R_{\alpha})\}, \\ \dim_H(G^{\alpha}_{\beta}) \le \min\{\dim_H(G^{\alpha}), \dim_H(G_{\beta})\} = \min\{d(\alpha), d(\beta)\}. \end{gathered} \tag{18}$$

For sake of completeness we give the proof of the following technical lemma.

Lemma 2 *Let m, n be natural numbers such that $n \ge 1$, $0 \le m \le n$; let d the function defined by (5). Then*

$$n\,d(\frac{m}{n}) - \frac{1}{2}\log_2(n) - 1 \le \log_2\binom{n}{m} \le n\,d(\frac{m}{n}).$$

Proof. The thesis is trivial if $m = 0$ or $m = n$; then we can assume $0 < m < n$.
By the inequalities (cf. [Bu])

$$n^n\sqrt{2\pi n}e^{-n+\frac{1}{12n+\frac{1}{4}}} < n! < n^n\sqrt{2\pi n}e^{-n+\frac{1}{12n}}; \tag{19}$$

we have

$$\begin{aligned} \binom{n}{m} &\le \frac{n^n\sqrt{2\pi n}e^{-n+\frac{1}{12n}}}{m^m\sqrt{2\pi m}e^{-m}(n-m)^{n-m}e^{-(n-m)}\sqrt{2\pi\,(n-m)}} = \\ &= \frac{n^n}{m^m(n-m)^{n-m}}\frac{e^{\frac{1}{12n}}}{\sqrt{2\pi}}\sqrt{\frac{n}{m\,(n-m)}} \le \frac{n^n}{m^m(n-m)^{n-m}} = \\ &= \frac{n^n}{\left(\frac{m}{n}\right)^m n^m\left(1-\frac{m}{n}\right)^{n-m}n^{n-m}} = \frac{1}{\left(\frac{m}{n}\right)^m\left(1-\frac{m}{n}\right)^{n-m}}; \end{aligned} \tag{20}$$

and

$$\begin{aligned} \binom{n}{m} &\ge \frac{n^n\sqrt{2\pi n}e^{-n}}{m^m\sqrt{2\pi m}e^{-m}(n-m)^{n-m}e^{-(n-m)}e^{\frac{1}{12m}+\frac{1}{12(n-m)}}} = \\ &= \frac{n^n}{m^m(n-m)^{n-m}}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{m\,(n-m)}}\frac{1}{e^{\frac{1}{12m}+\frac{1}{12(n-m)}}} \ge \\ &\ge \frac{n^n}{m^m(n-m)^{n-m}}\frac{\sqrt{2}}{\sqrt{\pi n}}e^{-\frac{1}{6}} \ge \frac{n^n}{m^m(n-m)^{n-m}}\frac{1}{2\sqrt{n}}; \end{aligned} \tag{21}$$

taking the $\log_2$ in (20) and (21) we obtain

$$n\,d(\frac{m}{n}) - \frac{1}{2}\log_2(n) - 1 \le \log_2\binom{n}{m} \le n\,d(\frac{m}{n})$$

and the thesis is proved.□

3 Main result. We firstly give a simple construction of a generalization of Cantor like subsets of $[0,1]$ (see also on this subject the bibliographical remarks contained in [F1, section 1.5]).

Definition 3 *Let us consider a sequence* $\{k_h\}_h \subseteq \mathbf{N}$ *and* $r \in \mathbf{N}$ *such that*

$$1 \leq k_h < r \qquad \forall h \in \mathbf{N}.$$

Furthermore, for every $h \in k_h$ *we consider a* k_h*-tuple of integers between* 0 *and* $r-1$

$$0 \leq p_h^1 < p_h^2 < ... < p_h^{k_h} < r.$$

Let us denote

$$P_h = \left(p_h^1, p_h^2, ..., p_h^{k_h}\right) \quad \text{and} \quad \mathcal{P}_r = (P_h)_h \,.$$

Let us, for short, denote $[0,1]$ *by* I *and build the following sequence of sets* $\{C_h\}_h$

$$C_0 = I, \quad C_1 = \cup_{i_1=1}^{k_1} \left[\frac{p_1^{i_1}}{r} + \frac{1}{r} I\right], \quad C_2 = \cup_{i_1=1}^{k_1} \cup_{i_2=1}^{k_2} \left[\frac{p_1^{i_1}}{r} + \frac{1}{r}\left[\frac{p_2^{i_2}}{r} + \frac{1}{r} I\right]\right], \quad ...$$

$$C_h = \cup_{i_1=1}^{k_1} \cup_{i_2=1}^{k_2} ... \cup_{i_h=1}^{k_h} \left[\frac{p_1^{i_1}}{r} + \frac{1}{r}\left[\frac{p_2^{i_2}}{r} + \frac{1}{r}\left[... \left[\frac{p_h^{i_h}}{r} + \frac{1}{r} I\right] ...\right]\right]\right], \quad ... \tag{22}$$

and define

$$C = C\left(\mathcal{P}_r\right) = \cap_{h=0}^{+\infty} C_h. \tag{23}$$

In other words C is a set obtained in a way similar to the Cantor set.

Every C_h is an essential disjoint union of $k_1 k_2 \cdots k_h$ intervals of length r^{-h}; you obtain C_{h+1} from C_h performing the following steps:

a) divide I in r intervals;

b) choose k_{h+1} intervals among them according to (order) numbers $p_{h+1}^1, ..., p_{h+1}^{k_{h+1}}$;

c) scale down the set obtained in b) to the length of the intervals of C_h;

d) replace every interval of C_h with the set obtained in c), translated by the left endpoint of the interval.

Lemma 4 *Let* C *be the set given in definition 3. Then*

$$\dim_H(C) = \liminf_h \frac{\log(k_1 k_2 \cdots k_h)}{h \log r}. \tag{24}$$

Proof. We first prove the inequality

$$\dim_H(C) \leq \liminf_h \frac{\log(k_1 k_2 \cdots k_h)}{h \log r}. \tag{25}$$

Let us pose $\lambda = \liminf_h \frac{\log(k_1 k_2 \cdots k_h)}{h \log r}$. Let $\varepsilon > 0$; let $\{h_j\}_j$ an indexes subsequence and j_0 such that $\frac{\log(k_1 k_2 \cdots k_{h_j})}{h_j \log r} < \lambda + \varepsilon$ for every $j > j_0$.

Being $k_1 k_2 \cdots k_{h_j} \left(r^{h_j}\right)^{-\frac{\log(k_1 k_2 \cdots k_{h_j})}{h_j \log r}} = 1$ we get $k_1 k_2 \cdots k_{h_j} \left(r^{h_j}\right)^{-(\lambda+\varepsilon)} < 1$ for every $j > j_0$.

Since C_{h_j} is an essential disjoint union of $k_1 k_2 \cdots k_{h_j}$ intervals of length r^{-h_j}, fixed $\delta_j > r^{-h_j}$, C_{h_j} can be covered with open intervals $B_1^{\delta_j}, B_2^{\delta_j}, ..., B_{n_j}^{\delta_j}$ of diameter δ_j such that

$$\mathcal{H}^{\lambda+\varepsilon}_{\delta_j}(C) \leq 1 \qquad \forall j > j_0. \tag{26}$$

By (26), taking the sequence δ_j decreasing to 0, we obtain $\mathcal{H}^{\lambda+\varepsilon}(C) \leq 1$. By (13) we get $\dim_H(C) \leq \lambda + \varepsilon$ and, by the arbitrariness of $\varepsilon > 0$, the inequality (25).

Let now prove the opposite inequality.

If $\lambda = 0$ the thesis is obvious being non negative the Hausdorff dimension.

Otherwise, let $\varepsilon > 0$, then there exists $h_\varepsilon \in \mathbf{N}$ such that

$$h \geq h_\varepsilon \Longrightarrow \frac{\log(k_1 k_2 \cdots k_h)}{h \log r} > \lambda - \varepsilon. \tag{27}$$

Let $\delta > 0$ be such that

$$\delta < \frac{1}{r^{h_\varepsilon}};$$

let $\{B_j\}_j$ a countable covering of C with open balls such that $\operatorname{diam}(B_j) < \delta$ for every $j \in \mathbf{N}$. By the compactness of C we can assume that exists $\nu \in \mathbf{N}$ such that $\{B_j\}_{1 \leq j \leq \nu}$ is still a covering of C. For every $1 \leq j \leq \nu$ there exists $h_j \geq h_\varepsilon$ such that

$$\frac{1}{r^{h_j}} \leq \operatorname{diam}(B_j) < \frac{1}{r^{h_j - 1}}.$$

Let $m = \max\{h_j : 1 \leq j \leq \nu\}$ and observe that C is contained in C_m that in turn is the essential disjoint union of $k_1 k_2 \cdots k_m$ intervals of length r^{-m}, $C_m = C_m^1 \cup C_m^2 \cup ... \cup C_m^{k_1 k_2 \cdots k_m}$.

Let us define

$$\mu_j \doteq \frac{\operatorname{card}\left\{i = 1, ..., k_1 k_2 \cdots k_m : B_j \cap C_m^i\right\}}{k_1 k_2 \cdots k_m}. \tag{28}$$

Since for every $i = 1, ..., k_1 k_2 \cdots k_m$ the interval C_m^i contains points of C and $\{B_j\}_{1 \leq j \leq \nu}$ is a covering of C we have

$$\sum_{j=1}^{\nu} \mu_j \geq 1. \tag{29}$$

If we divide $[0, 1]$ in $r^{h_j - 1}$ intervals, B_j can have nonempty intersection with at most two such intervals, and each of these intervals contains $k_{h_j} k_{h_{j+1}} \cdots k_m$ intervals of C_m.

By (28) and (27) we have

$$\mu_j \leq \frac{2 k_{h_j} k_{h_{j+1}} \cdots k_m}{k_1 k_2 \cdots k_m} \leq \frac{2}{k_1 k_2 \cdots k_{h_j - 1}} \leq \tag{30}$$

$$\leq \frac{2r}{k_1 k_2 \cdots k_{h_j}} = 2r \left(\frac{1}{r^{h_j}}\right)^{\frac{\log(k_1 k_2 \cdots k_{h_j})}{h_j \log r}} \leq 2r \left(\operatorname{diam}(B_j)\right)^{\lambda - \varepsilon}.$$

Then (30) and (29) give

$$\sum_{j=1}^{\nu} \operatorname{diam}(B_j)^{\lambda - \varepsilon} \geq \frac{1}{2r} \sum_{j=1}^{\nu} \mu_j = \frac{1}{2r} > 0;$$

then we obtain $H_{\delta}^{\lambda-\varepsilon}(C) \geq \frac{1}{2r} > 0$ for every $\delta > 0$, so by (13), $\dim_H(C) \geq \lambda - \varepsilon$ and, since $\varepsilon > 0$ is arbitrary

$$\dim_H(C) \geq \liminf_h \frac{\log(k_1 k_2 \cdots k_h)}{h \log r}. \tag{31}$$

By inequalities (25) and (31) we have the thesis. □

Lemma 5 *Let $q, p_1, p_2 \in \mathbf{N}$, $p_1 < q$, $p_2 < q$, consider a strictly increasing sequence of numbers $\{m_i\}_i \subset \mathbf{N}_0$ such that $m_0 = 0$ and let us define*

$$t \in C \iff \begin{cases} \sum_{j=1}^{q} x_{hq+j}(t) = p_1 & m_{2i} \leq h < m_{2i+1} \\ \sum_{j=1}^{q} x_{hq+j}(t) = p_2 & m_{2i+1} \leq h < m_{2i+2} \end{cases} \quad \forall i \in \mathbf{N}_0. \tag{32}$$

Then

$$\dim_H(C) \geq \min\left\{d(\frac{p_1}{q}), d(\frac{p_2}{q})\right\} - \frac{1}{2q}\log_2(q) - \frac{1}{q}. \tag{33}$$

Proof. Let $t_0 \in C$ and let us observe that, taking $C_0 = [0,1]$ and

$$C_h \doteq \left\{ t \in [0,1] : \sum_{j=1}^{q} x_{lq+j}(t) = \sum_{j=1}^{q} x_{lq+j}(t_0), \quad 0 \leq l \leq h-1 \right\} \quad \text{for every } h \in \mathbf{N}. \tag{34}$$

The sets C_h are constructed as in (22) and $C = \cap_{h=1}^{\infty} C_h$ like in (23), where $r = 2^q$ and k_h assume only the values $\binom{q}{p_1}$ or $\binom{q}{p_2}$. Obviously $k_h \geq \min\left\{\binom{q}{p_1}, \binom{q}{p_2}\right\}$ for every $h \in \mathbf{N}$.

Therefore, by Lemma 4, we have

$$\dim_H(C) \geq \liminf_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log 2^q} \geq \frac{\log_2\left(\min\left\{\binom{q}{p_1}, \binom{q}{p_2}\right\}\right)}{q}. \tag{35}$$

By Lemma 2 we get

$$q\, d(\frac{p_i}{q}) - \frac{1}{2}\log_2(q) - 1 \leq \log_2 \binom{q}{p_i} \leq q\, d(\frac{p_i}{q}) \qquad i = 1, 2. \tag{36}$$

Then (35) and (36) give the thesis. □

Theorem 6 *Let G_{β}^{α} be the set defined in (8). Then*

$$\dim_H(G_{\beta}^{\alpha}) = \min\{d(\alpha), d(\beta)\}.$$

Proof. By Proposition 1 we only have to prove

$$\dim_H(G_{\beta}^{\alpha}) \geq \min\{d(\alpha), d(\beta)\} \tag{37}$$

If $\alpha = \beta$, (37) becomes

$$\dim_H (F^\alpha) \geq d(\alpha),$$

and it holds true (cf. again Proposition 1), while if $\alpha = 1$ or $\beta = 0$ the thesis is trivial.

Assume now that $0 < \beta < \alpha < 1$.

Let $0 < \varepsilon < \min\{\beta, 1-\alpha\}$. Then there exists $\overline{q} \in \mathbf{N}$ such that for every $q \geq \overline{q}$ we have $\frac{1}{2}\frac{\log_2 q}{q} + \frac{1}{q} < \varepsilon$.

Let us observe that there exist $p_1, p_2, q \in \mathbf{N}$, with $q \geq \overline{q}$, such that

$$\text{(38)} \qquad \begin{array}{l} 0 < \beta - \varepsilon < \dfrac{p_1}{q} < \beta < \dfrac{p_1+1}{q} < \dfrac{p_2-1}{q} < \alpha < \dfrac{p_2}{q} < \alpha + \varepsilon < 1 \\ d(\dfrac{p_1}{q}) > d(\beta) - \varepsilon, \qquad d(\dfrac{p_2}{q}) > d(\alpha) - \varepsilon. \end{array}$$

Let us take C defined as in Lemma (5). By (38), (33) becomes

$$\text{(39)} \qquad \dim_H(C) \geq \min\left\{d(\frac{p_1}{q}), d(\frac{p_2}{q})\right\} - \varepsilon \geq \min\{d(\alpha), d(\beta)\} - 2\varepsilon,$$

for every choice of the sequence $\{m_i\}_i$ in (32).

Let us now show that, for a suitable choice of the sequence $\{m_i\}_i$ in (32) we have

$$\text{(40)} \qquad C \subseteq G^\alpha_\beta.$$

We take $m_0 = 0$ and, for every $i \in \mathbf{N}$, by induction we assume to already have defined $m_1, ..., m_{2i}$.

Then we denote by $r_i = \sum_{h=1}^{i} (m_{2h-1} - m_{2h-2})$ and $s_i = \sum_{h=1}^{i} (m_{2h} - m_{2h-1})$ and define

$$\text{(41)} \qquad \begin{cases} m_{2i+1} = \min\left\{j : j > m_{2i} \text{ and } \dfrac{p_1 r_i + p_2 s_i + p_1 (j - m_{2i})}{jq} < \beta\right\} \\ m_{2i+2} = \min\left\{j : j > m_{2i+1} \text{ and } \dfrac{p_1 r_{i+1} + p_2 s_i + p_2 (j - m_{2i+1})}{jq} > \alpha\right\} \end{cases}.$$

Let us observe that

$$y_{(j+1)q} = \frac{y_{jq}(jq) + p_s}{(j+1)q} = \frac{jq}{(j+1)q} y_{jq} + \frac{q}{(j+1)q}\frac{p_s}{q}$$

where

$$\begin{array}{ll} s = 1, & \text{if } m_{2i} \leq j < m_{2i+1}, \\ s = 2, & \text{if } m_{2i+1} \leq j < m_{2i+2}. \end{array}$$

So $y_{(j+1)q}$ is a convex combination of y_{jq} and $\dfrac{p_1}{q}$ if $m_{2i} \leq j < m_{2i+1}$, and of y_{jq} and $\dfrac{p_2}{q}$ if $m_{2i+1} \leq j < m_{2i+2}$.

By recalling that $\dfrac{p_1}{q} < \beta$ and $\dfrac{p_2}{q} > \alpha$, $y_{m_{2i}q} > \alpha$ and $y_{m_{2i+1}q} < \beta$, respectively beginning from $j = m_{2i}$ and $j = m_{2i+1}$, we obtain

$$\text{(42)} \qquad \begin{array}{ll} y_{jq}(t) > y_{(j+1)q}(t) & m_{2i} \leq j < m_{2i+1} \\ y_{jq}(t) < y_{(j+1)q}(t) & m_{2i+1} \leq j \leq m_{2i+2}, \end{array}$$

so we have

$$(43)\qquad \begin{aligned} &y_{m_{2i+1}q}(t) \le y_{jq}(t) \le y_{m_{2i}q}(t) && m_{2i} \le j \le m_{2i+1} \\ &y_{m_{2i+1}q}(t) \le y_{jq}(t) \le y_{m_{2i+2}q}(t) && m_{2i+1} \le j \le m_{2i+2}\ , \end{aligned}$$

On the other side, by the definition of the sequence $\{m_i\}_i$ (cf. (41)), we have

$$y_{(m_{2i}-1)q}(t) \le \alpha < y_{m_{2i}q}(t)\,.$$

But

$$y_{m_{2i}q}(t) \le \frac{y_{(m_{2i}-1)q}(t)\cdot(m_{2i}-1)\,q+q}{m_{2i}q}$$

and so

$$(44)\qquad \alpha < y_{m_{2i}q}(t) \le \frac{\alpha\,(m_{2i}-1)+1}{m_{2i}}.$$

In a similar way we obtain

$$(45)\qquad \frac{\beta\,(m_{2i+1}-1)}{m_{2i+1}} \le y_{m_{2i+1}q}(t) < \beta.$$

By (43), (44) and (45), we easily obtain

$$(46)\qquad \begin{aligned} \limsup_j\, y_{jq}(t) &= \alpha \\ \liminf_j\, y_{jq}(t) &= \beta \end{aligned}$$

Eventually

$$(47)\qquad \frac{[n/q]\,q\ y_{[n/q]q}(t)}{n} \le y_n(t) \le \frac{[n/q]\,q\ y_{[n/q]q}(t)+(n-[n/q]\,q)\,q}{n}.$$

By (46) and (47) we have (40).

By (40) and (39) we get

$$\dim_H(G^{\alpha}_{\beta}) \ge \min\{d(\alpha)\,, d(\beta)\} - \varepsilon$$

and, by the arbitrariness of ε, we obtain the thesis. □

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