BINARY DIGITS EXPANSION OF NUMBERS: HAUSDORFF DIMENSIONS OF INTERSECTIONS OF LEVEL SETS OF AVERAGES' UPPER AND LOWER LIMITS

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Abstract. The problem of averaging of binary digits of numbers is considered and the sequence of the averages calculated on the first digits is taken into account for every \( t \in [0,1] \). The Hausdorff dimensions of intersections of level sets of upper and lower limits of such sequences are computed.

1 Introduction

In this paper we consider the classic problem of averaging the binary digits of numbers in \([0,1]\) and of studying the (Hausdorff) dimensions of some sets related to these averages.

Let us more precisely consider \( t \in [0,1] \), the sequence \( x(t) = (x_n(t))_n \) of its binary digits (cf. (14) for the precise definition) and the sequence of their averages \( y(t) = (y_n(t))_n \) given by

\[
y_n(t) = \frac{1}{n} \sum_{k=1}^{n} x_k(t), \quad \forall n \in \mathbb{N}
\]

Then it is possible to consider the two (always existing) quantities

\[
\lim\inf_{n \to +\infty} y_n(t), \quad \lim\sup_{n \to +\infty} y_n(t)
\]

and the quantity, when it exists:

\[
\lim_{n \to +\infty} y_n(t).
\]

Let us set

\[
F^\alpha = \left\{ t \in [0,1] : \lim_{n\to+\infty} y_n(t) = \alpha \right\},
\]

\[
R^\alpha = \left\{ t \in [0,1] : \limsup_{n} y_n(t) \leq \alpha \right\}, \quad R_\alpha = \left\{ t \in [0,1] : \liminf_{n} y_n(t) \geq \alpha \right\},
\]

\[
S^\alpha = \left\{ t \in [0,1] : \limsup_{n} y_n(t) \geq \alpha \right\}, \quad S_\alpha = \left\{ t \in [0,1] : \liminf_{n} y_n(t) \leq \alpha \right\}.
\]

There are some classic results about the Hausdorff dimension of these sets we will recall. To this aim let us define the function \( d(t) \) as follows

\[
d(t) = \begin{cases} 
-t \log_2(t) + (1-t) \log_2(1-t) & \forall t \in (0,1) \\
0 & \text{if } t = 0,1.
\end{cases}
\]

and denote by \( \dim_H \) the Hausdorff dimension (cf. (10) \( \div \) (12) for the definition).

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In [HL] was proved (very well known result) that $F^{1/2}$ contains almost every $t$ in $[0,1]$ (and therefore $\dim_H(F^{1/2}) = 1$).

In [Bs] the Hausdorff dimensions of the sets $R^\alpha$ and $R_\alpha$ was computed; $d(\alpha)$ was proved to be equal to $\dim_H(R^\alpha)$ if $0 \leq \alpha < 1/2$ and to $\dim_H(R_\alpha)$ if $1/2 < \alpha \leq 1$ (in the other cases the sets trivially contains $F^{1/2}$).

In [K] the Hausdorff dimensions of the sets $S^\alpha$ and $S_\alpha$ was computed; $d(\alpha)$ was proved to be equal to $\dim_H(S^\alpha)$ if $0 \leq \alpha < 1/2$ and to $\dim_H(S_\alpha)$ if $1/2 < \alpha \leq 1$ (in the other cases the sets trivially contains $F^{1/2}$).

In [E] was proved that the Hausdorff dimension of the set $F^\alpha$ is equal to $d(\alpha)$ for every $0 \leq \alpha \leq 1$.

Let us now define the sets

\[ G^\alpha = \left\{ t \in [0,1] : \limsup_n y_n(t) = \alpha \right\}, \quad G_\alpha = \left\{ t \in [0,1] : \liminf_n y_n(t) = \alpha \right\}. \]

Taking into account the recalled results it easily follows (cf. Proposition 1) that

\[ \dim_H(G^\alpha) = \dim_H(G_\alpha) = d(\alpha) \quad \forall \alpha \in [0,1]. \]

Then we analyze the Hausdorff dimension of

\[ G_\beta \supseteq G^\alpha \cap G_\beta; \]

by (7) and (8) we obviously have

\[ \dim_H(G_\beta) \leq \min\{d(\alpha), d(\beta)\}. \]

Our result consists of proving the reverse inequality in (9), so that the equality

\[ \dim_H(G_\beta) = \min\{d(\alpha), d(\beta)\} \]

holds (cf. Theorem 6).

As last remark we observe that our proof is inspired by some fractal techniques (see [F2], p. 55).

Eventually we recall that the Hausdorff dimension is a very efficacious instrument to treat problems of Diophantine approximations. For this subject in addition to the above references see for example [S], [F1, section 8.5] and the more recent papers [DDY] and [DD].

2 Notations and preliminary results Let us denote by $\mathbb{N} = \{1,2,3,...\}$ and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a finite subset $M \subset \mathbb{N}$ we will denote by card ($M$) the number of its elements. Given a subset $E \subseteq \mathbb{R}$ we will denote by diam ($E$) = sup $\{|x-y| : x,y \in E\}$ its diameter and if in addition $E$ is Lebesgue measurable, we denote by $|E|$ its Lebesgue measure.

Let $\delta > 0$ and $s > 0$ real numbers and let us pose, for every $E \subset \mathbb{R}$,

\[ \mathcal{H}^s_\delta (E) = \inf \sum_{n=1}^{\infty} \text{diam}^s(B_n) \]

where the family $\{B_n\}_{n \in \mathbb{N}}$ is a countable covering of $E$ with open balls such that $\text{diam}(B_n) < \delta$, $\forall n \in \mathbb{N}$ and the infimum is taken on this kind of families. The $s$-dimensional Hausdorff outer measure of $E$ is given as usual by

\[ \mathcal{H}^s(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta (E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta (E), \]
while Hausdorff dimension of $E$ is given by

$$\dim_H(E) = \inf \{ s \in \mathbb{R} : \mathcal{H}^s(E) = 0 \}.$$  

Moreover it can be easily proved that (see [F1, p. 7])

$$\mathcal{H}^s(E) = 0 \text{ if } s > \dim_H(E); \quad \mathcal{H}^s(E) = +\infty \text{ if } s < \dim_H(E)$$

Let us observe that slightly different definitions of $s$-dimensional Hausdorff outer measure can be given, all of them leading to the same result in the definition (12).

Given $t \in \mathbb{R}$, we will denote by $[t]$ the integer part of $t$, i.e. $[t] = \max \{ m \in \mathbb{Z} : m \leq t \}$ and by $I$ the interval $[0, 1]$.

Let $t \in I$. We define the sequence $x(t) = \{ x_n(t) \}_n$ in the following way

$$x_n(t) = 2^n t - 2^{n-1} t \quad \forall n \in \mathbb{N}.$$  

Such sequence is the one of the binary digits of $t$ (the rational numbers of the form $t = \frac{k}{2^m}$ can be expressed in two ways as binary numbers: e.g. $\frac{1}{7} = 0.12$ and also $\frac{1}{7} = 0.012\ldots$; the sequence defined corresponds in this case to the representation with a finite number of digits equal to 1).

For a fixed $n \in \mathbb{N}$, $x_n(t)$ is a step function assuming only values 0 and 1 and it holds

$$x_n(t) = \frac{1}{2} \left( \chi_{[0,1)} + \sum_{j=0}^{2^n-1} (-1)^{j+1} \chi_{[2^{-j}, 2^{-j+1})}(t) \right) \quad \forall t \in I,$$

where for a set $A$, the function $\chi_A$ is the characteristic function of $A$.

Now let $y(t) = (y_n(t))_n$ the sequence defined by (1); $y_n(t)$ is a step function constant on every interval $[\frac{j}{2^n}, \frac{j+1}{2^n})$, $j = 0, 1, ..., 2^n - 1$, and takes only values $\frac{k}{n}$, $k = 0, 1, ..., n$.

Moreover

$$\left| \left\{ t : y_n(t) = \frac{k}{n} \right\} \right| = \binom{n}{k} 2^{-n},$$

where $\binom{n}{k}$ is the binomial coefficient of $n$ over $k$.

Obvious relations among the sets defined in the introduction are

$$F^\alpha = G^\alpha_\alpha, \quad G^\alpha_\beta = G^\alpha \cap G_\beta, \quad G^\alpha = \cup_{0 \leq \beta \leq \alpha} G^\alpha_\beta, \quad G_\alpha = \cup_{0 \leq \beta \leq 1} G^\alpha_\beta,$$

$$R^\alpha = \cup_{0 \leq \beta \leq \alpha} G^\beta, \quad R_\alpha = \cup_{0 \leq \beta \leq 1} G_\beta, \quad S^\alpha = \cup_{0 \leq \beta \leq 1} G^\beta, \quad S_\alpha = \cup_{0 \leq \beta \leq \alpha} G_\beta$$

for every $\alpha$ and $\beta$ in $[0, 1]$.

Therefore obvious relations among the Hausdorff dimensions of such sets are

$$\dim_H(F^\alpha) \leq \dim_H(G^\alpha) \leq \min \{ \dim_H(S^\alpha) \text{,} \dim_H(R^\alpha) \},$$

$$\dim_H(F^\alpha) \leq \dim_H(G_\alpha) \leq \min \{ \dim_H(S_\alpha) \text{,} \dim_H(R_\alpha) \},$$

$$\dim_H(G^\alpha_\beta) \leq \min \{ \dim_H(G^\alpha) \text{,} \dim_H(G_\beta) \}$$

for every $\alpha$ and $\beta$ in $I$.

Then, using the results recalled in the introduction, we obtain
Proposition 1 Let $\alpha, \beta \in [0,1]$ and let $F^\alpha$, $G_\alpha$, $G^\alpha$, $R_\alpha$, $R^\alpha$, $S_\alpha$, $S^\alpha$, $G_\beta^\alpha$ be defined by (4), (6) and (8). Then

$$
d(\alpha) = \dim_H(F^\alpha) = \dim_H(G^\alpha) = \min \{\dim_H(S^\alpha), \dim_H(R^\alpha)\},$$

(18)

$$
d(\alpha) = \dim_H(F^\alpha) = \dim_H(G_\alpha) = \min \{\dim_H(S_\alpha), \dim_H(R_\alpha)\},$$

$$
dim_H(G_\beta^\alpha) \leq \min \{\dim_H(G^\alpha), \dim_H(G_\beta)\} = \min \{d(\alpha), d(\beta)\}.
$$

For sake of completeness we give the proof of the following technical lemma.

Lemma 2 Let $m, n$ be natural numbers such that $n \geq 1$, $0 \leq m \leq n$; let $d$ the function defined by (5). Then

$$
n d \left( \frac{m}{n} \right) - \frac{1}{2} \log_2(n) - 1 \leq \log_2 \left( \frac{n}{m} \right) \leq n d \left( \frac{m}{n} \right).
$$

Proof. The thesis is trivial if $m = 0$ or $m = n$; then we can assume $0 < m < n$.

By the inequalities (cf. [Bu])

$$
n^n \sqrt{2\pi n e^{-n + \frac{1}{12n}}} < n! < n^n \sqrt{2\pi n e^{-n + \frac{1}{12n}}},
$$

we have

$$\binom{n}{m} \leq \frac{n^n \sqrt{2\pi n e^{-n + \frac{1}{12n}}}}{m^n \sqrt{2\pi (n-m)e^{-(n-m)}} \sqrt{2\pi (n-m)}} = \frac{n^n e^{\frac{n}{12n}}}{m^n (n-m)^{n-m} e^{-(n-m)}} \leq \frac{n^n}{m^n (n-m)^{n-m} e^{-(n-m)}} = \frac{(\frac{m}{n})^m n^m (1 - \frac{m}{n})^{n-m}}{n^n} = \left( \frac{m}{n} \right)^m (1 - \frac{m}{n})^{n-m};
$$

and

$$\binom{n}{m} \geq \frac{n^n \sqrt{2\pi n e^{-n}}}{m^n \sqrt{2\pi (n-m)e^{-(n-m)}} \sqrt{2\pi (n-m)}} = \frac{n^n e^{\frac{n}{12n}}}{m^n (n-m)^{n-m} e^{-(n-m)}} \geq \frac{n^n}{m^n (n-m)^{n-m} e^{-(n-m)}} = \left( \frac{m}{n} \right)^m (1 - \frac{m}{n})^{n-m} e^{-(n-m)}}
$$

taking the log$_2$ in (20) and (21) we obtain

$$
n d \left( \frac{m}{n} \right) - \frac{1}{2} \log_2(n) - 1 \leq \log_2 \left( \frac{n}{m} \right) \leq n d \left( \frac{m}{n} \right)
$$

and the thesis is proved. $\square$

3 Main result. We firstly give a simple construction of a generalization of Cantor like subsets of $[0,1]$ (see also on this subject the bibliographical remarks contained in [F1, section 1.5]).
Definition 3 Let us consider a sequence \( \{k_h\} \subseteq \mathbb{N} \) and \( r \in \mathbb{N} \) such that

\[
1 \leq k_h < r \quad \forall h \in \mathbb{N}.
\]

Furthermore, for every \( h \in k_h \) we consider a \( k_h \)-tuple of integers between 0 and \( r - 1 \)

\[
0 \leq p_h^1 < p_h^2 < \ldots < p_h^{k_h} < r.
\]

Let us denote

\[
P_h = \left(p_h^1, p_h^2, \ldots, p_h^{k_h}\right) \quad \text{and} \quad P_r = (P_h)_h.
\]

Let us, for short, denote \([0, 1]\) by \( I \) and build the following sequence of sets \( \{C_h\}_h \)

\[
C_0 = I, \quad C_1 = \bigcup_{i_1=1}^{k_1} \left[ \frac{p_{i_1}^1}{r} + \frac{1}{r} \right], \quad C_2 = \bigcup_{i_1=1}^{k_1} \bigcup_{i_2=1}^{k_2} \left[ \frac{p_{i_1}^1}{r} + \frac{1}{r} \left( \frac{p_{i_2}^2}{r} + \frac{1}{r} I \right) \right], \quad \ldots
\]

(22) \[ C_h = \bigcup_{i_1=1}^{k_1} \bigcup_{i_2=1}^{k_2} \ldots \bigcup_{i_h=1}^{k_h} \left[ \frac{p_{i_1}^1}{r} + \frac{1}{r} \left( \frac{p_{i_2}^2}{r} + \frac{1}{r} \left( \ldots \left( \frac{p_{i_h}^h}{r} + \frac{1}{r} I \right) \right) \right) \right], \quad \ldots
\]

and define

(23) \[ C = C(P_r) = \cap_{h=0}^{+\infty} C_h. \]

In other words \( C \) is a set obtained in a way similar to the Cantor set.

Every \( C_h \) is an essential disjoint union of \( k_1 k_2 \ldots k_h \) intervals of length \( r^{-h} \); you obtain \( C_{h+1} \) from \( C_h \) performing the following steps:

a) divide \( I \) in \( r \) intervals;

b) choose \( k_{h+1} \) intervals among them according to (order) numbers \( p_{h+1}^1, \ldots, p_{h+1}^{k_{h+1}} \);

c) scale down the set obtained in b) to the length of the intervals of \( C_h \);

d) replace every interval of \( C_h \) with the set obtained in c), translated by the left endpoint of the interval.

Lemma 4 Let \( C \) be the set given in definition 3. Then

(24) \[ \dim_H(C) = \liminf_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log r}. \]

Proof. We first prove the inequality

(25) \[ \dim_H(C) \leq \liminf_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log r}. \]

Let us pose \( \lambda = \liminf_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log r} \). Let \( \varepsilon > 0; \) let \( \{h_j\}_j \) an indexes subsequence and \( j_0 \)

such that \( \frac{\log(k_1 k_2 \ldots k_{h_j})}{h_j \log r} < \lambda + \varepsilon \) for every \( j > j_0 \).

Being \( k_1 k_2 \ldots k_{h_j} \left( r^{h_j} \right)^{-\frac{\log(k_1 k_2 \ldots k_{h_j})}{h_j \log r}} = 1 \) we get \( k_1 k_2 \ldots k_{h_j} \left( r^{h_j} \right)^{-(\lambda+\varepsilon)} < 1 \) for every \( j > j_0 \).
Since $C_{h_j}$ is an essential disjoint union of $k_1 k_2 \cdots k_{h_j}$ intervals of length $r^{-h_j}$, fixed $\delta_j > r^{-h_j}$, $C_{h_j}$ can be covered with open intervals $B^1_{h_j}, B^2_{h_j}, \ldots, B^{h_j}_{h_j}$ of diameter $\delta_j$ such that

\begin{equation}
\mathcal{H}^{\lambda+\varepsilon}(C) \leq 1 \quad \forall j > j_0.
\end{equation}

By (26), taking the sequence $\delta_j$ decreasing to 0, we obtain $\mathcal{H}^{\lambda+\varepsilon}(C) \leq 1$. By (13) we get $\dim_H(C) \leq \lambda + \varepsilon$ and, by the arbitrarity of $\varepsilon > 0$, the inequality (25).

Let now prove the opposite inequality.

If $\lambda = 0$ the thesis is obvious being non negative the Hausdorff dimension.

Otherwise, let $\varepsilon > 0$, then there exists $h_\varepsilon \in \mathbb{N}$ such that

\begin{equation}
h \geq h_\varepsilon \implies \frac{\log(k_1 k_2 \cdots k_{h_\varepsilon})}{h \log r} > \lambda - \varepsilon.
\end{equation}

Let $\delta > 0$ be such that

\begin{equation}
\delta < \frac{1}{r^{h_\varepsilon}};
\end{equation}

let $\{B_j\}_j$ a countable covering of $C$ with open balls such that $\text{diam} (B_j) < \delta$ for every $j \in \mathbb{N}$. By the compactness of $C$ we can assume that exists $\nu \in \mathbb{N}$ such that $\{B_j\}_{1 \leq j \leq \nu}$ is still a covering of $C$. For every $1 \leq j \leq \nu$ there exists $h_j \geq h_\varepsilon$ such that

\begin{equation}
\frac{1}{r^{h_j}} \leq \text{diam} (B_j) < \frac{1}{r^{h_j-1}}.
\end{equation}

Let $m = \max \{ h_j : 1 \leq j \leq \nu \}$ and observe that $C$ is contained in $C_m$ that in turn is the essential disjoint union of $k_1 k_2 \cdots k_m$ intervals of length $r^{-m}$, $C_m = C_1^m \cup C_2^m \cup \cdots C_m^m$.

Let us define

\begin{equation}
\mu_j = \frac{\text{card} \{ i = 1, \ldots, k_1 k_2 \cdots k_m : B_j \cap C_i^m \} }{k_1 k_2 \cdots k_m}.
\end{equation}

Since for every $i = 1, \ldots, k_1 k_2 \cdots k_m$ the interval $C_i^m$ contains points of $C$ and $\{B_j\}_{1 \leq j \leq \nu}$ is a covering of $C$ we have

\begin{equation}
\sum_{j=1}^{\nu} \mu_j \geq 1.
\end{equation}

If we divide $[0,1]$ in $r^{h_j-1}$ intervals, $B_j$ can have nonempty intersection with at most two such intervals, and each of these intervals contains $k_{h_j}, k_{h_j+1} \cdots k_m$ intervals of $C_m$.

By (28) and (27) we have

\begin{equation}
\mu_j \leq \frac{2k_{h_j} k_{h_j+1} \cdots k_m}{k_1 k_2 \cdots k_m} \leq \frac{2}{k_1 k_2 \cdots k_{h_j-1}} \leq \frac{2r}{k_1 k_2 \cdots k_{h_j}} = 2r \left( \frac{1}{r^{h_j}} \right)^{\lambda \log \frac{k_1 k_2 \cdots k_{h_j}}{h_j \log r}} \leq 2r (\text{diam}(B_j))^{\lambda-\varepsilon}.
\end{equation}

Then (30) and (29) give

\begin{equation}
\sum_{j=1}^{\nu} (\text{diam}(B_j))^{\lambda-\varepsilon} \geq \frac{1}{2r} \sum_{j=1}^{\nu} \mu_j = \frac{1}{2r} > 0;
\end{equation}
then we obtain $H^{\lambda - \varepsilon}(C) \geq \frac{1}{2^r} > 0$ for every $\delta > 0$, so by (13), $\dim_H(C) \geq \lambda - \varepsilon$ and, since $\varepsilon > 0$ is arbitrary

$$\dim_H(C) \geq \liminf_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log r}.$$  

By inequalities (25) and (31) we have the thesis. □

**Lemma 5** Let $q, p_1, p_2 \in \mathbb{N}$, $p_1 < q$, $p_2 < q$, consider a strictly increasing sequence of numbers $\{m_i\} \subset \mathbb{N}_0$ such that $m_0 = 0$ and let us define

$$t \in C \iff \begin{cases} \sum_{j=1}^{q} x_{hq+j}(t) = p_1 & m_{2i} \leq h < m_{2i+1} \\ \sum_{j=1}^{q} x_{hq+j}(t) = p_2 & m_{2i+1} \leq h < m_{2i+2} \end{cases} \quad \forall i \in \mathbb{N}_0.$$  

Then

$$\dim_H(C) \geq \min \left\{ d\left(\frac{p_1}{q}\right), d\left(\frac{p_2}{q}\right) \right\} - \frac{1}{2q} \log_2(q) - \frac{1}{q}.$$  

**Proof.** Let $t_0 \in C$ and let us observe that, taking $C_0 = [0, 1]$ and

$$C_h = \left\{ t \in [0, 1] : \sum_{j=1}^{q} x_{hq+j}(t) = \sum_{j=1}^{q} x_{hq+j}(t_0), \quad 0 \leq l \leq h - 1 \right\}$$  

for every $h \in \mathbb{N}$.

The sets $C_h$ are constructed as in (22) and $C = \cap_{h=1}^{\infty} C_h$ like in (23), where $r = 2^q$ and $k_h$ assume only the values $\left(\frac{q}{p_1}\right)$ or $\left(\frac{q}{p_2}\right)$. Obviously $k_h \geq \min \left\{ \left(\frac{q}{p_1}\right), \left(\frac{q}{p_2}\right) \right\}$ for every $h \in \mathbb{N}$.

Therefore, by Lemma 4, we have

$$\dim_H(C) \geq \liminfty_h \frac{\log(k_1 k_2 \ldots k_h)}{h \log 2^q} \geq \log_2 \left( \min \left\{ \left(\frac{q}{p_1}\right), \left(\frac{q}{p_2}\right) \right\} \right).$$  

By Lemma 2 we get

$$q d\left(\frac{p_i}{q}\right) - \frac{1}{2} \log_2(q) - 1 \leq \log_2 \left(\frac{q}{p_i}\right) \leq q d\left(\frac{p_i}{q}\right) \quad i = 1, 2.$$  

Then (35) and (36) give the thesis. □

**Theorem 6** Let $G^\alpha_\beta$ be the set defined in (8). Then

$$\dim_H(G^\alpha_\beta) = \min \left\{ d(\alpha), d(\beta) \right\}.$$  

**Proof.** By Proposition 1 we only have to prove

$$\dim_H(G^\alpha_\beta) \geq \min \left\{ d(\alpha), d(\beta) \right\}.$$
If \( \alpha = \beta \), (37) becomes

\[
\dim_H (F^\alpha) \geq d (\alpha),
\]

and it holds true (cf. again Proposition 1), while if \( \alpha = 1 \) or \( \beta = 0 \) the thesis is trivial.

Assume now that \( 0 < \beta < \alpha < 1 \).

Let \( 0 < \varepsilon < \min \{ \beta, 1 - \alpha \} \). Then there exists \( \eta \in \mathbb{N} \) such that for every \( q \geq \eta \) we have

\[
\frac{1}{2} \log_{\frac{1}{q}} + \frac{1}{q} < \varepsilon.
\]

Let us observe that there exist \( p_1, p_2, q \in \mathbb{N} \), with \( q \geq \eta \), such that

\[
0 < \beta - \varepsilon < \frac{p_1}{q} < \beta < \frac{p_1 + 1}{q} < \frac{p_2 - 1}{q} < \alpha < \frac{p_2}{q} < \alpha + \varepsilon < 1
\]

Let us observe that for every choice of the sequence \( (39) \)

\[
\{ j : j < m \text{ and } d(p_j) > d(\beta) - \varepsilon, \quad d(p_{j+1}) > d(\alpha) - \varepsilon \}
\]

Let us take \( C \) defined as in Lemma (5). By (38), (33) becomes

(39) \[
\dim_H (C) \geq \min \left\{ d \left( \frac{p_1}{q} \right), d \left( \frac{p_2}{q} \right) \right\} - \varepsilon \geq \min \{ d(\alpha), d(\beta) \} - 2\varepsilon
\]

for every choice of the sequence \( \{ m_i \} \) in (32).

Let us now show that, for a suitable choice of the sequence \( \{ m_i \} \) in (32) we have

(40) \[
C \subseteq G^{\alpha}_{\beta}.
\]

We take \( m_0 = 0 \) and, for every \( i \in \mathbb{N} \), by induction we assume to already have defined \( m_1, \ldots, m_{2i} \).

Then we denote by \( r_i = \sum_{h=1}^{i} (m_{2h-1} - m_{2h-2}) \) and \( s_i = \sum_{h=1}^{i} (m_{2h} - m_{2h-1}) \) and define

\[
\begin{cases}
    m_{2i+1} = \min \left\{ j : j > m_{2i} \text{ and } \frac{pr_1 + ps_1 + p_j (j - m_{2i})}{j} < \beta \right\} \\
    m_{2i+2} = \min \left\{ j : j > m_{2i+1} \text{ and } \frac{pr_1 + ps_2 + p_j (j - m_{2i+1})}{j} > \alpha \right\}
\end{cases}
\]

Let us observe that

\[
y_{(j+1)q} = \frac{y_{jq} (jq) + ps}{(j + 1)q} = \frac{jq}{(j+1)q} y_{jq} + \frac{q}{(j+1)q} \frac{ps}{q}
\]

where

\[
\begin{align*}
    s &= 1, \quad \text{if } m_{2i} \leq j < m_{2i+1}, \\
    s &= 2, \quad \text{if } m_{2i+1} \leq j < m_{2i+2}.
\end{align*}
\]

So \( y_{(j+1)q} \) is a convex combination of \( y_{jq} \) and \( \frac{p_1}{q} \) if \( m_{2i} \leq j < m_{2i+1} \), and of \( y_{jq} \) and \( \frac{p_2}{q} \) if \( m_{2i+1} \leq j < m_{2i+2} \).

By recalling that \( \frac{p_1}{q} < \beta \) and \( \frac{p_2}{q} > \alpha \), \( y_{m_{2i}q} > \alpha \) and \( y_{m_{2i+1}q} < \beta \), respectively beginning from \( j = m_{2i} \) and \( j = m_{2i+1} \), we obtain

(42) \[
\begin{align*}
    y_{jq} (t) &> y_{(j+1)q} (t) \quad \text{if } m_{2i} \leq j < m_{2i+1} \\
    y_{jq} (t) &< y_{(j+1)q} (t) \quad \text{if } m_{2i+1} \leq j \leq m_{2i+2},
\end{align*}
\]
so we have

\begin{align}
\text{(43)} \quad y_{m_{2i+1}q}(t) & \leq y_{jq}(t) \leq y_{m_{2i}q}(t) \quad m_{2i} \leq j \leq m_{2i+1} \\
& \text{and } y_{m_{2i+1}q}(t) \leq y_{jq}(t) \leq y_{m_{2i+2}q}(t) \quad m_{2i+1} \leq j \leq m_{2i+2}.
\end{align}

On the other side, by the definition of the sequence \(\{m_i\}_i\) (cf. (41)), we have

\[y_{m_{2i-1}q}(t) \leq \alpha < y_{m_{2i}q}(t).\]

But

\[y_{m_{2i}q}(t) \leq \frac{y_{(m_{2i-1})q}(t) \cdot (m_{2i} - 1)q + q}{m_{2i}q}\]

and so

\begin{align}
\text{(44)} \quad \alpha < y_{m_{2i}q}(t) \leq \frac{\alpha (m_{2i} - 1) + 1}{m_{2i}}.
\end{align}

In a similar way we obtain

\begin{align}
\text{(45)} \quad \beta \frac{(m_{2i+1} - 1)}{m_{2i+1}} \leq y_{m_{2i+1}q}(t) < \beta.
\end{align}

By (43), (44) and (45), we easily obtain

\begin{align}
\text{(46)} \quad \limsup_j y_{jq}(t) = \alpha \\
& \liminf_j y_{jq}(t) = \beta
\end{align}

Eventually

\begin{align}
\text{(47)} \quad \frac{[n/q]q \ y_{[n/q]q}(t)}{n} \leq y_n(t) \leq \frac{[n/q]q \ y_{[n/q]q}(t) + (n - [n/q]q)q}{n}.
\end{align}

By (46) and (47) we have (40).

By (40) and (39) we get

\[\dim_H(G^\alpha) \geq \min \{d(\alpha), d(\beta)\} - \varepsilon\]

and, by the arbitrariness of \(\varepsilon\), we obtain the thesis. \(\square\)

**References**


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