WEAK CONVERGENCE OF MEASURES ON THE UNION OF HAUSDORFF SPACES

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Abstract. S. Nakanishi gave some interesting results on the weak convergence of measures on the union X of metric spaces (X_α, d_α) $(\alpha \in \Sigma)$, endowed with the finest topology for which all the canonical injections of X_α to X are continiuous. Our aim in this note is to extend her results to the case where each component space X_α is simply a Hausdorff space.

1. Introduction. Let X be the union of metric spaces (X_α, d_α) $(\alpha \in \Sigma)$ satisfying separation axiom (1.1) and metric condition (1.4) in [2], and endowed with the finest topology for which all the canonical injections of X_α to X are continuous. S. Nakanishi gave a criterion for relative compactness of a family $\mathcal{M}$ of measures defined on $\beta(X)$ (the σ-algebra of Borel sets in the topological space X). Our aim in this note is to extend her results to the case where each component space X_α is simply Hausdorff.

To do so we need to replace the word "sequence" in the separation axiom (1.1) in [2] by "net" and to replace the metric condition (1.4) in [2] by the condition that

if $\alpha \leq \beta$ $(\alpha,\ \beta \in \Sigma)$, then the topology of X_α is stronger than the relative topology induced by X_β.

S. Nakanishi [2] treated real valued countably additive Borel measures. In our case τ-smooth measures are treated.

In this note compactness means net compactness. To prove our result Topøse [3], Part II, Theorem 9.2 plays an important role. In §2, we shall state some results concerning the measures needed in this note. In §3, we shall state the topological properties of the union of Hausdorff spaces. In §4, we shall state the main result in this note. In particular, in the case where $\{S_n\}$ is an increasing sequence of compact Hausdorff spaces such that if $m \leq n$, then the topology of S_m is stronger than the relative topology induced by S_n (note that in this case the sequence $\{S_n\}$ satisfies the separation axiom (1.1) in §3), we shall show that the following result holds: Let S be the union of $\{S_n\}$ and let S be endowed with the finest topology for which all the canonical injections of S_n into S are continuous. Then a non-empty set $\mathcal{M}$ of τ-smooth measures with $\sup\{\mu(S) : \mu \in \mathcal{M}\} < \infty$ is relatively compact if and only if it satisfies the condition

$$\inf_n \sup\{\mu(S - S_n) : \mu \in \mathcal{M}\} = 0.$$

2. Preliminaries. Let X be a set and $O(X)$ a topology of X. By $(X, O(X))$ we denote the topological space X endowed with the topology $O(X)$ and by $\beta(X, O(X))$ we denote the family of its Borel sets and so on. If no confusion happens, then we simply write them as X, $\beta(X)$ and so on.

Let X be a topological space (not necessarily a Hausdorff space). A net (x_α) on X is

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said to be compact if every subnet has a further subnet which converges (see [1] for the definition of "the convergence of a net"). A subset A of X is called *net compact* if every net on A has a convergent subnet. In case X is a regular topological space, a subset A of X is net compact if and only if A is relatively compact.

By $\mathcal{F}(X)$ we denote the set of all closed subsets of X. By $m_+(X)$ we denote the space of all totally finite measures defined on $\beta(X)$.

Let μ be in $m_+(X)$. μ is said to be τ-smooth if

$$\mu(\cap_{F\in\mathcal{F}}F) = \inf_{F\in\mathcal{F}} \mu(F)$$

holds for any subclass $\mathcal{F}$ of closed sets filtering downwards (see Topsøe [3], p. XII, P13). By $m_\tau(X)$ we denote the set of all τ-smooth measures on X.

Definition 2.1. *Let X be a topological space. The weak topology on $m_+(X)$ is defined as the weakest topology on $m_+(X)$ for which every map $\mu \mapsto \mu(f)$, where f is a bounded real valued upper semi-continuous function, is upper semi-continuous.*

Under this definition the same conditions (i) – (v) in Topsøe [3], Part II, Theorem 8.1 still hold. The topology on $m_\tau(X)$ induced by $m_+(X)$ with the weak topology is called the weak topology, too.

Definition 2.2.

(1) *Let (μ_α) be a net on $m_+(X)$. (μ_α) is said to be τ-smooth if the relation*

$$\inf_{F\in\mathcal{F}} \limsup_{\alpha} \mu_\alpha(F) = 0$$

holds for every subclass $\mathcal{F}$ of $\mathcal{F}(X)$ filtering downwards to the empty set.

(2) *Let A be a subset of $m_+(X)$. A is said to be τ-smooth if the relation*

$$\inf_{F\in\mathcal{F}} \sup_{\mu\in A} \mu(F) = 0$$

holds for every subclass $\mathcal{F}$ of $\mathcal{F}(X)$ filtering downwards to the empty set.

Theorem 2.3(Topsøe [3], p. 43, Theorem 9.2). *Let X be a regular space and consider $m_\tau(X)$ with the weak topology. Then,*

(1) *a net (μ_α) on $m_\tau(X)$ with $\limsup_\alpha \mu_\alpha(X) < \infty$ is compact if and only if it is τ-smooth.*

(2) *a subset $\mathcal{M}$ of $m_\tau(X)$ with $\sup\{\mu(X) : \mu \in \mathcal{M}\} < \infty$ is relatively compact if and only if it is τ-smooth.*

3. Topology for the union of Hausdorff spaces. Let $\{X_\alpha\}_{\alpha\in\Sigma}$ be a family of Hausdorff spaces and X the union of X_α $(\alpha \in \Sigma)$. By $O_r(X)$ we denote the finest topology on X for which, for any α in Σ, the canonical injection of X_α into X is continuous. In other words,

$$O_r(X) = \{O \subset X : \text{for any } \alpha \text{ in } \Sigma,\ O \cap X_\alpha \text{ is an open subset of } X_\alpha\}.$$

By $(X,\ O_r(X))$ we denote the topological space X endowed with this topology. Put

$$F_r(X) = \{X - O : O \in O_r(X)\}.$$

We impose the following condition (1.1) on the family $\{X_\alpha\}_{\alpha\in\Sigma}$. It is a generalization of the *axiom* (1.1) in S. Nakanishi [2].

(1.1)(*Separation axiom*)
If (x_i) is a net on $X_\alpha \cap X_\beta$ such that simultaneously $\lim_i x_i = x$ in X_α and $\lim_i x_i = y$ in X_β, then $x = y$.

By $\mathcal{K}(X_\alpha)$ we denote the set of all compact subsets of X_α.

Proposition 3.1. *If X satisfies the separation axiom* (1.1), *then $\mathcal{K}(X_\alpha)$ is included in $F_r(X)$.*

Proof. Let C be in $\mathcal{K}(X_\alpha)$ and $C \cap X_\beta \neq \emptyset$. It is sufficient to show that $C \cap X_\beta$ is a closed subset of X_β. Let (x_i) be a net on $C \cap X_\beta$ converging to a point x in X_β. Since C is a compact subset of X_α, the net (x_i) has a subnet (x_{i_j}) converging to a point y in C. Since the net (x_{i_j}) converges to $x \in X_\beta$, by separation axiom (1.1), we have $x = y$. This implies that $C \cap X_\beta$ is a closed subset of X_β. Thus Proposition 3.1 has been proved.

In the sequel we impose the following conditions $(1.2) - (1.4)$ on the family $\{X_\alpha\}_{\alpha\in\Sigma}$ of Hausdorff spaces:
(1.2) Σ is quasi-ordered set by a relation "$\leq$"(see [2], (1.2)).
(1.3) if $\alpha \leq \beta$ $(\alpha,\ \beta \in \Sigma)$, then $X_\alpha \subset X_\beta$.
(1.4) if $\alpha \leq \beta$, then the topology of X_α is stronger than the relative topology induced by X_β.
Furthermore we consider a sequence $\{S_n\}_{n=1,2,\cdots}$ of sets such that:
(1.5) $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$,
and each S_n is contained in some X_{α_n} in such a way that
(1.6) if $m \leq n$, then $\alpha_m \leq \alpha_n$,
(1.7) for every n, S_n belongs to $\mathcal{K}(X_{\alpha_n})$.
Let each S_n be endowed with the relative topology induced by X_{α_n}. Put $S = \cup_{n=1}^{\infty} S_n$. By $(S,\ O_r(S))$ we denote the topological space consisting of the set S and the topology $O_r(S)$ determined by the sequence $\{S_n\}$.

In the following, we suppose that X satisfies the conditions $(1.1) - (1.4)$, and let S be defined as above.

Proposition 3.2. $(S,\ O_r(S))$ *is a normal space.*

Proof. First we shall show that $(S,\ O_r(S))$ satisfies the axiom of T_1-separation. A set consisting of only one point is a compact subset in some S_n. By applying Proposition 3.1 to the spaces $\{S_n\}$ and S we see that it is a closed subset in S. Hence S satisfies the axiom of T_1-separation. Next we shall show that $(S,\ O_r(S))$ satisfies the axiom of T_4-separation. Let A and B be disjoint closed subsets of $(S,\ O_r(S))$. For every positive integer n put $A_n = S_n \cap A$ and $B_n = S_n \cap B$. A_n and B_n are disjoint compact subsets of S_n. Since S_1 is a compact Hausdorff space, there exist two open subsets G_1 and H_1 of S_1 such that $A_1 \subset G_1$ and $B_1 \subset H_1$ and $(G_1)^{a_1} \cap (H_1)^{a_1} = \emptyset$, where for any subset A of S_n, $(A)^{a_n}$ denotes the closure of A in S_n. Put

$$A'_2 = (G_1)^{a_1} \cup A_2 \text{ and } B'_2 = (H_1)^{a_1} \cup B_2.$$

We have

$$A'_2 \cap B'_2 = ((G_1)^{a_1} \cap B_2) \cup (A_2 \cap (H_1)^{a_1}) = ((G_1)^{a_1} \cap B_1) \cup (A_1 \cap (H_1)^{a_1})$$

$$\subset ((G_1)^{a_1} \cap (H_1)^{a_1}) \cup ((G_1)^{a_1} \cap (H_1)^{a_1}) = \emptyset.$$

Since A'_2 and B'_2 are disjoint closed subsets of S_2 and S_2 is a compact Hausdorff space, there exist two open subsets G_2 and H_2 of S_2 such that $A'_2 \subset G_2$, $B'_2 \subset H_2$ and $(G_2)^{a_2} \cap (H_2)^{a_2} = \emptyset$. Repeating this process, we obtain two monotone increasing sequences $\{G_n\}$ and $\{H_n\}$ of sets such that, for every positive integer n, G_n and H_n are disjoint open subsets of S_n. Put

$$G = \cup_{n=1}^{\infty} G_n \text{ and } H = \cup_{n=1}^{\infty} H_n.$$

It is obvious that G and H are disjoint sets. Furthermore, G and H are open subsets of S. Indeed, for every positive integer m we have

$$G \cap S_m = \cup_{n=m}^{\infty} (G_n \cap S_m).$$

For positive integer n with $m \leq n$, S_m is homeomorphic to S_m endowed with the relative topology of S_n. Hence, for every positive integer n with $m \leq n$, $G_n \cap S_m$ is an open subset of S_m. Accordingly $G \cap S_m$ is an open subset of S_m. Hence G is an open subset of $(S,\ O_r(S))$. Similarly it is proved that H is an open subset of $(S,\ O_r(S))$. Therefore $(S,\ O_r(S))$ satisfies the axiom of T_4-separation. Thus Proposition 3.2 has been proved.

By $(S,\ O_r(X))$ we denote the topological space S endowed with the relative topology with respect to $(X,\ O_r(X))$.

Proposition 3.3.
(1) $S \in \beta(X,\ O_r(X))$,
(2) $\beta(S,\ O_r(S)) = \beta(S,\ O_r(X)) = \beta(X,\ O_r(X)) \cap S$.

The proof of Proposition 3.3 is similar to that of Lemma 5 in [2].

4. Weak convergence of measures on the union of Hausdorff spaces. Throughout this §, we suppose that X satisfies the conditions (1.1) $-$ (1.4) in §3 and let S be the same as in §3.

The following lemma is well known (see [2], p.XIII, P16).

Lemma 4.1. *If μ is in $m_\tau(S,\ O_r(S))$, then, for any F in $F_r(S)$,*

$$\inf\{\mu(G) : F \subset G \text{ and } G \in O_r(S)\} = \mu(F).$$

Theorem 4.2. *Let $\mathcal{M}$ be a non-empty subset of $m_\tau(S,\ O_r(S))$ with* $\sup\{\mu(S) : \mu \in \mathcal{M}\} < \infty$. *Then $\mathcal{M}$ is relatively compact with respect to the weak topology if and only if*

$$\inf_n \sup\{\mu(S - S_n) : \mu \in \mathcal{M}\} = 0.$$

Proof. “If part”: Let $\mathcal{F}$ be any subclass of $F_r(S)$ filtering downwards to the empty set.

Put

$$\mathcal{G} = \{G = S - F : F \in \mathcal{F}\}.$$

$\mathcal{G}$ is a subset of $O_r(S)$ filtering upwards to S. Since S is σ-compact, there exists an increasing sequence $\{G_n\}$ of sets in $\mathcal{G}$ with $S_n \subset G_n$ for every positive integer n. For any positive number ϵ there exists a positive integer n such that

$$\sup\{\mu(S - S_n) : \mu \in \mathcal{M}\} < \epsilon.$$

Put $F_n = S - G_n$. Then $F_n \in \mathcal{F}$ and

$$\inf_{F \in \mathcal{F}} \sup\{\mu(F) : \mu \in \mathcal{M}\} \leq \sup\{\mu(F_n) : \mu \in \mathcal{M}\} \leq \sup\{\mu(S - S_n) : \mu \in \mathcal{M}\} < \epsilon.$$

Since ϵ is arbitrary,

$$\inf_{F \in \mathcal{F}} \sup\{\mu(F) : \mu \in \mathcal{M}\} = 0.$$

This shows that $\mathcal{M}$ is τ-smooth. Since S is normal, by Theorem 2.3 in §2, $\mathcal{M}$ is relatively compact.

"Only if part": Suppose the contrary, that is,

$$\inf_n \sup\{\mu(S - S_n) : \mu \in \mathcal{M}\} = 2\alpha > 0.$$

For every positive integer n, there exists a measure μ_n in $\mathcal{M}$ such that $\mu_n(S - S_n) > \alpha$. Put $G_n = S - S_n$. G_n is open in S. For every positive integer n, by Lemma 4.1, there exists an F_n in $F_r(S)$ such that $F_n \subset G_n$ and $\mu_n(G_n - F_n) < \dfrac{\alpha}{2}$. Put $A_n = \cup_{n \leq i} F_i$ $(n = 1, 2, \cdots)$. Since, for every positive integer m, we have

$$A_n \cap S_m = (\cup_{n \leq i} F_i) \cap S_m = \cup_{n \leq i < m}(F_i \cap S_m),$$

$A_n \cap S_m$ is compact in S_m. Therefore, $\{A_n\}$ is a decreasing sequence of closed subsets of S with $\cap_{n=1}^{\infty} A_n = \emptyset$. Since $\mathcal{M}$ is relatively compact, $\{\mu_n\}$ is relatively compact. By §2, Theorem 2.3, we have

$$\inf_m \sup\{\mu_n(A_m) : n = 1, 2, \cdots\} = 0.$$

Accordingly there exists a positive integer m_0 such that $\sup\{\mu_n(A_{m_0}) : n = 1, 2, \cdots\} < \dfrac{\alpha}{2}$. Thus

$$\frac{\alpha}{2} \leq \mu_{m_0}(F_{m_0}) \leq \mu_{m_0}(A_{m_0}) \leq \sup\{\mu_n(A_{m_0}) : n = 1, 2, \dots\} < \frac{\alpha}{2}.$$

This is a contradiction. Thus Theorem 4.2 has been proved.

Lemma 4.3. *Let μ be in $m_\tau(X, O(X))$. If μ is concentrated on S, then the restriction ν of μ to $\beta(S,\ O_r(S))$ belongs to $m_\tau(S,\ O_r(S))$.*

Proof. By Proposition 3.3 in §3 we can consider the restriction ν of μ to $\beta(S,\ O_r(S))$. Let $\mathcal{F}$ be any subclass of $F_r(S)$ filtering downwards. Put $F = \cap\{F' : F' \in \mathcal{F}\}$. For any positive number ϵ there exists a positive integer n such that $\mu(X - S_n) = \nu(S - S_n) < \epsilon$. For any set F' in $\mathcal{F}$, $F' \cap S_n$ belongs to $\mathcal{K}(X_{\alpha_n})$ and, accordingly, $F' \cap S_n$ belongs to $F_r(X)$ by

Proposition 3.1. Put $\mathcal{F} \cap S_n = \{F' \cap S_n : F' \in \mathcal{F}\}$. $\mathcal{F} \cap S_n$ is a subclass of $F_r(X)$ filtering downwards to $F \cap S_n$. Since μ is a τ-smooth measure on X concentrated on S, we have

$$\inf\{\nu(F' \cap S_n) : F' \in \mathcal{F}\} = \inf\{\mu(F' \cap S_n) : F' \in \mathcal{F}\} = \mu(F \cap S_n) = \nu(F \cap S_n).$$

There exists an F_0' in $\mathcal{F}$ such that $\nu(F_0' \cap S_n) < \nu(F \cap S_n) + \epsilon$. Hence we have

$$\nu(F_0') = \nu(F_0' \cap S_n) + \nu((S - S_n) \cap F_0') < \nu(F_0' \cap S_n) + \epsilon < \nu(F) + 2\epsilon.$$

Since ϵ is arbitrary, we have $\inf\{\nu(F') : F' \in \mathcal{F}\} \leq \nu(F)$. Since the reverse inequality is obvious, the proof of Lemma 4.3 has been completed.

Theorem 4.4. *Let $\mathcal{M}$ be a non-empty subset of $m_\tau(X,\ O_r(X))$. In order that $\mathcal{M}$ is net compact in $m_\tau(X,\ O_r(X))$, it is sufficient that $\mathcal{M}$ satisfies the following two conditions.*
(1) $\sup\{\mu(X) : \mu \in \mathcal{M}\} < \infty$.
(2) *There exists an increasing sequence $\{S_n\}$ of sets such that*

$$\inf_n \sup\{\mu(X - S_n) : \mu \in \mathcal{M}\} = 0,$$

where, for every positive integer n, S_n is a compact subset of some X_{α_n} with $\alpha_m \leq \alpha_n$ for $m \leq n$.

Proof. Put $S = \cup_{n=1}^{\infty} S_n$. Since, for every n, S_n is endowed with the relative topology induced by X_{α_n} and since S is endowed with the topology determined by the sequence $\{S_n\}$, by Proposition 3.3 and the condition (2) every measure μ in $\mathcal{M}$ is concentrated on S. Let $\mathcal{M}_S$ be the set of all the restrictions of measures in $\mathcal{M}$ to $\beta(S,\ O_r(S))$. By Lemma 4.3 we have $\mathcal{M}_S \subset m_\tau(S,\ O_r(S))$. Since

$$\inf_n \sup\{\nu(S - S_n) : \nu \in \mathcal{M}\} = \inf_n \sup\{\mu(X - S_n) : \mu \in \mathcal{M}\} = 0,$$

by Theorem 4.2 $\mathcal{M}_S$ is relatively compact. Let $\{\mu_i\}$ be any net on $\mathcal{M}$ and let ν_i be the restriction of μ_i to $\beta(S,\ O_r(S))$ for every i. Since $\mathcal{M}_S$ is relatively compact in $m_\tau(S,\ O_r(S))$, there exists a subnet $\{\nu_{i_j}\}$ of $\{\nu_i\}$ converging weakly to some point ν_0 in $m_\tau(O,\ O_r(S))$. For any A in $\beta(X,\ O_r(X))$, put $\mu_0(A) = \nu_0(A \cap S)$. Since every μ in $\mathcal{M}$ is concentrated on S, we have

$$\mu_0(X) = \nu_0(S) = \lim_j \nu_{i_j}(S) = \lim_j \mu_{i_j}(X).$$

For any G in $O_r(X)$ the set $X_{\alpha_n} \cap G$ is open in X_{α_n} and, accordingly, $S_n \cap G$ is an open subset of S_n. Hence $G \cap S$ is an open subset of S. Since $\{\nu_{i_j}\}$ converges weakly to ν_0 in $m_\tau(S,\ O_r(S))$, we have

$$\mu_0(G) = \nu_0(G \cap S) \leq \liminf_j \nu_{i_j}(G \cap S) = \liminf_j \mu_{i_j}(G).$$

Hence $\{\mu_{i_j}\}$ converges weakly to μ_0. Since ν_0 belongs to $m_\tau(S,\ O_r(S))$, by definition of μ_0, μ_0 belongs to $m_\tau(X,\ O_r(X))$. Thus Theorem 4.4 has been proved.

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